Mechanical Engineering Mathematics ME3.6
Sheet 5 Nonlinear Maps

1. Consider the following maps:

$$\begin{aligned}
\text{(i)}\, x_{n+1} &= \alpha \ln x_n + 1, \ (0 < \alpha < 1);\\
\text{(ii)}\, x_{n+1} &= \cos x_n;\\
\text{(iii)}\, x_{n+1} &= \frac{1}{3}(x_n^2 + 1).
\end{aligned}$$

In each case determine graphically the number of fixed points of the map. Discuss the stability of each point by means of a cobweb diagram.

2. Consider the following three maps with $a > 0$:

$$\begin{aligned}
\text{(i)}\, x_{n+1} &= \frac{2a - x_n^3}{x_n^2};\\
\text{(ii)}\, x_{n+1} &= \frac{a + x_n^3}{2x_n^2};\\
\text{(iii)}\, x_{n+1} &= \frac{a + 2x_n^3}{3x_n^2}.
\end{aligned}$$

Show that each map has the same fixed point. In each case determine the stability by considering the derivative of the right-hand-side.

3. A map is given by

$$x_{n+1} = x_n^2 + \frac{1}{4} - a.$$

(i) Show that there are two fixed points provided $a > 0$, and show that one is unstable and one is stable for $a < a_1$, where a_1 is to be found.

(ii) Show that a period 2 solution is born at $a = a_1$ and that this is stable until $a = a_2$ where a_2 is to be determined.

4. The logistic map is given by

$$x_{n+1} = f(x_n), \text{ where } f(x_n) = \lambda x_n(1 - x_n).$$

We wish to examine the existence of a 3-cycle for this map, i.e. to see whether real numbers a, b, c exist such that

$$b = f(a), \ c = f(b), \ a = f(c).$$

(i) Show that a, b, c all satisfy $X = f^{(3)}(X)$. Show also that the cycle is stable provided

$$|f'(a)f'(b)f'(c)| < 1.$$

(ii) Explain why, in order to find a, b, c we can consider the simpler equation

$$(x - f^{(3)}(x))/(x - f(x)) = 0$$

instead.

(ii) (Optional) Show, with the help of a computer, that

$$\frac{x - f^{(3)}(x)}{x - f(x)} = \lambda^6 x^6 - (3\lambda^6 + \lambda^5)x^5 + (\lambda^4 + 4\lambda^5 + 3\lambda^6)x^4$$
$$- (\lambda^3 + 3\lambda^4 + 5\lambda^5 + \lambda^6)x^3 + (\lambda^2 + 3\lambda^3 + 3\lambda^4 + 2\lambda^5)x^2$$
$$- (\lambda + 2\lambda^2 + 2\lambda^3 + \lambda^4)x + (1 + \lambda + \lambda^2).$$

(iii) Use a calculator or a computer package to plot this polynomial with $0 \leq x \leq 1$ for different values of λ. If $\lambda < 1 + 2\sqrt{2}$, how many roots are there? Show graphically that three real roots are created at $\lambda = 1 + 2\sqrt{2}$.

5. Two students are asked to find the solution as $t \to \infty$ of

$$\frac{dx}{dt} = 100x(3 - 4x); \text{ with } x(0) = 1.$$

(i) One student achieves this by solving the equation exactly. Show that this approach yields the solution $x \to 3/4$ as $t \to \infty$.

(ii) The second student is unaware of the exact solution. Instead they decide to solve the equation numerically using Euler's Method. This involves setting $x_i = x(t_i)$, evaluating the right-hand-side of the differential equation at $t = t_i$ and approximating

$$\left.\frac{dx}{dt}\right|_{t=t_i} \simeq \frac{x_{i+1} - x_i}{\Delta t}.$$

The student uses a timestep $\Delta t = 0.01$. Show that the resulting difference equation that the student obtains is

$$x_{i+1} = 4x_i(1 - x_i),$$

and hence deduce that this method is doomed to failure.