ME3.6 Sheet 4 Answers

1. $\text{FT}\{\cos\omega_0 t\} = \int_{-\infty}^{\infty} (\cos\omega_0 t)\, e^{-i\omega t}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} (e^{i\omega_0 t} + e^{-i\omega_0 t})\, e^{-i\omega t}\, dt$
$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\, dt = \underline{\pi\delta(\omega_0-\omega) + \pi\delta(\omega_0+\omega)}$.
(Using the result $\delta(\omega) = (1/2\pi)\int_{-\infty}^{\infty} e^{\pm i\omega t}\, dt$).
$\text{FT}\{\sin\omega_0 t\} = \int_{-\infty}^{\infty} (\sin\omega_0 t)\, e^{-i\omega t}\, dt = \frac{1}{2i}\int_{-\infty}^{\infty} (e^{i\omega_0 t} - e^{-i\omega_0 t})\, e^{-i\omega t}\, dt$
$= \frac{1}{2i}\int_{-\infty}^{\infty} e^{i(\omega_0-\omega)t} - e^{-i(\omega_0+\omega)t}\, dt = \underline{-i\pi\delta(\omega_0-\omega) + i\pi\delta(\omega_0+\omega)}$.

2. (**i**) Energy $= \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \exp(-2\alpha|t|)\, dt = E_0$, say $\underline{< \infty}$.
Average power $P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 dt = \lim_{T\to\infty} E_0/2T \underline{= 0}$.

(**ii**) $\rho_f(t) = \int_{-\infty}^{\infty} e^{-\alpha|t+u|}e^{-\alpha|u|} du = \int_{-\infty}^{0} e^{-\alpha|t+u|}e^{\alpha u} du + \int_{0}^{\infty} e^{-\alpha|t+u|}e^{-\alpha u} du$
$= I_1 + I_2$, say.
First consider $t > 0$. I_1 needs to be split.
$I_1 = \int_{-\infty}^{-t} e^{\alpha(t+u)}e^{\alpha u} du + \int_{-t}^{0} e^{-\alpha(t+u)}e^{\alpha u} du$
$= e^{\alpha t}\int_{-\infty}^{-t} e^{2\alpha u} du + e^{-\alpha t}\int_{-t}^{0} 1\, du = te^{-\alpha t} + (1/2\alpha)e^{-\alpha t}$.
$I_2 = \int_{0}^{\infty} e^{-\alpha(t+u)}e^{-\alpha u} du = (1/2\alpha)e^{-\alpha t}$.
$\rho_f(t) = I_1 + I_2 = \underline{(t + 1/\alpha)\exp(-\alpha t)\ \ (t > 0)}$.
Now consider $t < 0$. This time we need to split I_2.
$I_2 = \int_{0}^{-t} e^{\alpha(t+u)}e^{-\alpha u} du + \int_{-t}^{\infty} e^{-\alpha(t+u)}e^{-\alpha u} du$
$= e^{\alpha t}\int_{0}^{-t} 1\, du + e^{-\alpha t}\int_{-t}^{\infty} e^{-2\alpha u} du = -te^{\alpha t} + (1/2\alpha)e^{\alpha t}$.
$I_1 = \int_{-\infty}^{0} e^{\alpha(t+u)}e^{\alpha u} du = (1/2\alpha)e^{\alpha t}$.
$\rho_f(t) = I_1 + I_2 = \underline{(-t + 1/\alpha)\exp(\alpha t)\ \ (t < 0)}$.
Combining the 2 results:
$\underline{\rho_f(t) = (|t| + 1/\alpha)\exp(-\alpha|t|)}$, as required.

(**iii**) $E(\omega) = \text{FT}\{\rho_f\} = \int_{-\infty}^{\infty} (|t| + \alpha^{-1})e^{-\alpha|t|}\, e^{-i\omega t}\, dt$
$= \int_{-\infty}^{0} (\alpha^{-1} - t)e^{(\alpha - i\omega)t}\, dt + \int_{0}^{\infty} (\alpha^{-1} + t)e^{-(\alpha+i\omega)t}\, dt$
Substitute $s = -t$ in first integral.
Find that 1st integral is complex conjugate (c.c.) of second integral.
$\Rightarrow E(\omega) = \int_{0}^{\infty} (\alpha^{-1} + t)\exp(-(\alpha + i\omega)t)\, dt + c.c.$
$=$ (by parts) $= \cdots = \alpha^{-1}(\alpha + i\omega)^{-1} + (\alpha + i\omega)^{-2} + c.c.$
$= \cdots \underline{= 4\alpha^2/(\alpha^2+\omega^2)^2}$.

(**iv**) From Sheet 3 Q1(i) we have $\hat{f}(\omega) = 2\alpha/(\alpha^2+\omega^2)$.
Then $E(\omega) = \left|\hat{f}(\omega)\right|^2 \underline{= 4\alpha^2/(\alpha^2+\omega^2)^2}$ and so agrees with part (iii).

3. $\rho_f(t) = f(t) * f(-t)$
$= \int_{-\infty}^{\infty} \sum_n \alpha_n\delta(t - nT - u)\sum_p \alpha_p\delta(-u - pT)\, du$
$= \int_{-\infty}^{\infty} \sum_n \alpha_n\delta(t - nT - u)\sum_p \alpha_p\delta(u + pT)\, du$, since δ is even
$= \sum_n \alpha_n\sum_p \alpha_p\delta(t - nT) * \delta(t + pT)$
$= \sum_n\sum_p \alpha_n\alpha_p\delta(t - (n-p)T)$, using hint.
Using the substitution $s = n - p$ we obtain
$\rho_f(t) = \sum_s\sum_p \alpha_p\alpha_{s+p}\delta(t - sT)$, i.e.

$$\rho_f(t) = \sum_{s=-\infty}^{\infty} A_s \delta(t - sT) \text{ with } A_s = \sum_{p=-\infty}^{\infty} \alpha_p \alpha_{p+s}$$

as required. So the autocorrelation of an impulse train is another impulse train.

4. (**i**) We have

$$A_n = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty} \delta(t - mT)\exp(-2\pi int/T)\,dt$$
$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\int_{-T/2}^{T/2} \delta(t - mT)\exp(-2\pi int/T)\,dt$$

The integral is non-zero when $t = mT$, provided $-T/2 < mT < T/2$
$\Rightarrow m = 0$ and hence $t = 0$.
$\Rightarrow A_n = (1/T)[\exp(-2\pi int/T)]_{t=0} = \underline{1/T}$ for all n.

(ii) From (i):

$$\sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\exp(2\pi int/T).$$

Letting $c = 2\pi/T$ and $\omega = t$ we get

$$\underline{\sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi m}{c}\right) = \sum_{n=-\infty}^{\infty} \frac{c}{2\pi}\exp(in\omega c).}$$

Letting $m = k, n = -k$ we obtain the desired result.

5 (i) $\hat{f}(\omega) = \int_{-a}^{a} e^t e^{-i\omega t}\,dt = (1 - i\omega)^{-1}(\exp((1 - i\omega)a) - \exp(-(1 - i\omega)a))$
$= (1 - i\omega)^{-1} 2\sinh((1 - i\omega)a) = \underline{(1 - i\omega)^{-1}(2\sinh a\cos\omega a - 2i\sin\omega a\cosh a)}.$

(**ii**) If ω is small then $\cos\omega a \simeq 1, \sin\omega a \simeq \omega a, (1 - i\omega)^{-1} \simeq 1 + i\omega$.
Thus $\hat{f}(\omega) \sim 2(\sinh a - i\omega a\cosh a)(1 + i\omega)$,
and so $\underline{q(a) = 2\sinh a - 2a\cosh a}$.

(**iii**) We have $\rho_f(t) = (\mathrm{FT})^{-1}(E(\omega)) = (2\pi)^{-1}\int_{-\infty}^{\infty} E(\omega)\,e^{i\omega t}\,d\omega$
$= (2\pi)^{-1}\int_{-\omega_1}^{\omega_1} |\omega|^{-\beta} e^{i\omega t} d\omega = 2(2\pi)^{-1}\int_0^{\omega_1} \omega^{-\beta}\cos\omega t\,d\omega$,
(since the sin contribution is zero).
Now let $u = \omega t, du = t d\omega$:
$\Rightarrow \rho_f(t) = (1/\pi)\int_0^{\omega_1 t}(u/t)^{-\beta}(\cos u)\,t^{-1}du = t^{\beta-1}\pi^{-1}\int_0^{\omega_1 t} u^{-\beta}\cos u\,du.$
Thus $\underline{\rho_f(t) \to C(\beta)t^{\beta-1} \text{ as } t \to \infty}$, where $C(\beta) = \pi^{-1}\int_0^{\infty} u^{-\beta}\cos u\,du$.

6. (**i**) Taking FT $\Rightarrow -(i\omega)^2\hat{f}(\omega) + \hat{f}(\omega) = \mathrm{FT}\{\cos\alpha t\} = (Q1) = \pi\delta(\omega + \alpha) + \pi\delta(\omega - \alpha)$.
Therefore $\hat{f}(\omega) = \pi(\delta(\omega + \alpha) + \delta(\alpha - \omega))/(1 + \omega^2)$.
Taking inverse transform,
$f(t) = (2\pi)^{-1}\pi\int_{-\infty}^{\infty}(\delta(\omega + \alpha) + \delta(\alpha - \omega))\,e^{i\omega t}/(1 + \omega^2)\,d\omega$
$= (1/2)(\exp(-i\alpha t) + \exp(i\alpha t))/(1 + \alpha^2) = \underline{(\cos\alpha t)/(1 + \alpha^2)}$.

(**ii**) Taking FT $\Rightarrow -(i\omega)^2\hat{f} + i\omega\hat{f} + \hat{f} = |\omega|(\omega^2 + i\omega + 1)$, provided $|\omega| < \omega_0$
and zero otherwise $\Rightarrow \hat{f} = |\omega|$ for $|\omega| < \omega_0$ and $\hat{f} = 0$ otherwise.
Inverting: $f(t) = (2\pi)^{-1}\int_{-\omega_0}^{\omega_0}|\omega|e^{i\omega t}d\omega = \pi^{-1}\int_0^{\omega_0}\omega\cos\omega t\,d\omega$,

(since the sin contribution is zero).
Therefore, upon integration by parts:

$$\underline{f(t) = \frac{1}{\pi}\left(\frac{\omega_0 \sin\omega_0 t}{t} + \frac{\cos\omega_0 t}{t^2} - \frac{1}{t^2}\right).}$$

7 (**i**) Taking FT with transform variable k we obtain $((ik)^4 + 1)\hat{y} = -\hat{p}(k)$
$\Rightarrow \underline{\hat{y} = -\hat{p}(k)/(1+k^4)}.$
(**ii**) $\hat{p} = \int_{-L}^{L}(P_0/2L)e^{-ikx}dx = (P_0/2L)(-ik)^{-1}(e^{-ikL} - e^{ikL}) = \underline{(P_0/kL)\sin(kL)}.$
(**iii**) Inverting: $y(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} -(P_0/kL)\sin(kL)(1+k^4)^{-1}\exp(ikx)\,dk.$
As $L \to 0, \sin(kL)/kL \to 1$. Therefore
$y(x) \to -(P_0/2\pi)\int_{-\infty}^{\infty}(1+k^4)^{-1}\exp(ikx)\,dk = \underline{-(P_0/\pi)\int_0^{\infty}(1+k^4)^{-1}\cos kx\,dk},$
as required (since sin part of integral is zero).
(**iv**) In this case $\hat{p}(k) = \int_{-L}^{L}(P_0/2)e^{-ikx}dx = \cdots = (P_0/k)\sin kL.$
As $L \to \infty, (1/k)\sin kL \to \pi\delta(k),$ so we obtain
$y(x) \to -(P_0/2\pi)\int_{-\infty}^{\infty}\pi\delta(k)(1+k^4)^{-1}\exp(ikx)\,dk = \underline{-P_0/2}.$

8. Take double FT of equation
$\Rightarrow -\alpha\omega^2\hat{f} + i\omega\hat{f} + \beta k^2\hat{f} = G(k,\omega) \Rightarrow \hat{f} = G/(\beta k^2 - \alpha\omega^2 + i\omega).$
(**i**) Inverting:

$$\underline{f(x,t) = \frac{1}{(2\pi)^2}\int_{\omega=-\infty}^{\infty}\int_{k=-\infty}^{\infty}\frac{G(k,\omega)}{\beta k^2 - \alpha\omega^2 + i\omega}\exp(ikx + i\omega t)\,dk\,d\omega}$$

(**ii**) Using result in (i),

$$N(t) = \frac{1}{(2\pi)^2}\int_{\omega=-\infty}^{\infty}\int_{k=-\infty}^{\infty}\frac{G(k,\omega)}{\beta k^2 - \alpha\omega^2 + i\omega}\left[\int_{-\infty}^{\infty}\exp(ikx)\,dx\right]\exp(i\omega t)\,dk\,d\omega.$$

The integral in [ ] is equal to $2\pi\delta(k)$ and so the expression for $N(t)$ simplifies to:

$$\underline{N(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(0,\omega)}{-\alpha\omega^2 + i\omega}\exp(i\omega t)\,d\omega,}$$

as required.
(**iii**) Using expression for $N(t)$ from (ii):

$$\rho(t) = \frac{1}{(2\pi)^2}\int_{u=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{G(0,\omega_1)}{-\alpha\omega_1^2 + i\omega_1}\exp(i\omega_1 u)\,d\omega_1\int_{-\infty}^{\infty}\frac{G(0,\omega_2)}{-\alpha\omega_2^2 + i\omega_2}\exp(i\omega_2(t+u))\,d\omega_2\right]du$$
$$= \frac{1}{(2\pi)^2}\int_{\omega_1=-\infty}^{\infty}\int_{\omega_2=-\infty}^{\infty}\frac{G(0,\omega_1)G(0,\omega_2)}{(-\alpha\omega_1^2 + i\omega_1)(-\alpha\omega_2^2 + i\omega_2)}\exp(i\omega_2 t)\left[\int_{-\infty}^{\infty}\exp(i(\omega_1+\omega_2)u)\,du\right]d\omega_2 d\omega_1.$$

The integral in square brackets is equal to $2\pi\delta(\omega_1 + \omega_2)$.
Therefore using the sifting property, $\rho(t)$ reduces to

$$\rho(t) = \frac{1}{2\pi}\int_{\omega_1=-\infty}^{\infty}\frac{G(0,\omega_1)G(0,-\omega_1)}{(-\alpha\omega_1^2 + i\omega_1)(-\alpha\omega_1^2 - i\omega_1)}\exp(-i\omega_1 t)\,d\omega_1.$$

Now, since $g(x,t)$ is real it follows that $G(0,\omega) = (G(0,-\omega))^*$
($*$ represents complex conjugate).
Thus, $G(0,\omega_1)G(0,-\omega_1) = |G(0,\omega_1)|^2$ and we obtain

$$\rho(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|G(0,\omega)|^2 \exp(-i\omega t)}{(\alpha\omega^2)^2 + \omega^2}\, d\omega.$$

(**iv**) Given $G(0,\omega) \propto |\omega|^\sigma$

$$\Rightarrow \rho(t) \propto \int_{-\infty}^{\infty} \frac{|\omega|^{2\sigma} \exp(-i\omega t)}{\alpha^2\omega^4 + \omega^2}\, d\omega \sim \int_{-\infty}^{\infty} \frac{|\omega|^{2\sigma} \exp(-i\omega t)}{\omega^2}\, d\omega \text{ as } \alpha \to 0.$$

Make the substitution $u = \omega t, du = td\omega$

$$\Rightarrow \rho(t) \propto \int_{-\infty}^{\infty} \frac{|u|^{2\sigma}}{|t|^{2\sigma}} \frac{t^2}{u^2} \exp(-iu)\, \frac{du}{t} \propto t^{1-2\sigma} \int_{-\infty}^{\infty} \frac{|u|^{2\sigma}}{u^2} e^{-iu} du.$$

This expression is independent of t if $1 - 2\sigma = 0$, i.e. we require $\sigma = 1/2$.