Mechanical Engineering Mathematics ME3.6
Sheet 4 Fourier Transforms II

1. Find the Fourier transforms of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ in terms of the Dirac delta function.

2. A signal is given by $f(t) = \exp(-\alpha|t|)$, where $\alpha > 0$,
(i) Show that this signal has finite energy and delivers zero average power.
(ii) Show that its autocorrelation is

$$\rho_f(t) = \left(\frac{1}{\alpha} + |t|\right)\exp(-\alpha|t|).$$

(iii) Using this form for $\rho_f(t)$, calculate the energy spectrum $E(\omega)$.
(iv) Check your answer by finding $\hat{f}(\omega)$, and hence $E(\omega)$.

3. A finite energy impulse train is given by

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \delta(t - nT), \text{ with } \sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty.$$

Show that

$$\rho_f(t) = \sum_{s=-\infty}^{\infty} A_s \delta(t - sT) \text{ with } A_s = \sum_{p=-\infty}^{\infty} \alpha_p \alpha_{p+s}.$$

Hint: Use the convolution property of δ –functions: $\delta(t - nT) * \delta(t + pT) = \delta(t - (n - p)T)$.

4. Let

$$f(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

be a signal of finite average power, with period T. Writing $f(t)$ in Fourier series form we have

$$f(t) = \sum_{n=-\infty}^{\infty} A_n \exp(2\pi int/T); \; A_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-2\pi int/T)\,dt.$$

(i) Show that $A_n = 1/T$ for all n.
(ii) Deduce that

$$\sum_{k=-\infty}^{\infty} \exp(-i\omega kc) = \frac{2\pi}{c}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2k\pi}{c}).$$

5. (i) Determine the Fourier transform of

$$f(t) = \begin{cases} \exp(t), & |t| < a, \\ 0 & |t| > a. \end{cases}$$

(ii) Show that for small ω :

$$\hat{f}(\omega) \sim 2\sinh a + i\omega q(a),$$

where q is a quantity to be determined.
(iii) The energy spectrum of a time signal is known to be given by

$$E(\omega) = \begin{cases} |\omega|^{-\beta}, & |\omega| < \omega_1, \\ 0, & |\omega| > \omega_1, \end{cases}$$

where $0 < \beta < 1$. Show that the autocorrelation function $\rho_f(t) \propto t^{\beta-1}$ as $t \to \infty$.

6. (i) Use Fourier transforms to find a solution to the equation

$$-\frac{d^2f}{dt^2} + f = \cos(\alpha t).$$

(ii) Use the same method to solve

$$-\frac{d^2f}{dt^2} + \frac{df}{dt} + f = g(t),$$

where the function $g(x)$ is known to have the following Fourier transform:

$$\hat{g}(\omega) = \begin{cases} |\omega|(\omega^2 + i\omega + 1), & |\omega| < \omega_0, \\ 0, & |\omega| > \omega_0. \end{cases}$$

7. Consider an infinite beam of negligible mass lying along the x –axis.
The deflection of the beam is modelled by the equation

$$\frac{d^4y}{dx^4} + y = -p(x)$$

where the load p is given by

$$p(x) = \begin{cases} P_0/2L, & |x| < L, \\ 0 & |x| \geq L, \end{cases}$$

with P_0 and L constants.

(i) Express $y(x)$ in terms of the Fourier transform $\hat{p}(k)$ of the load, where k is the transform variable.

(ii) Calculate $\hat{p}(k)$.

(iii) Show that, in the limit $L \to 0$:

$$y(x) = -\frac{P_0}{\pi} \int_0^\infty \frac{\cos(kx)}{1 + k^4}\, dk$$

[You may assume $(\sin\alpha)/\alpha \to 1$ as $\alpha \to 0$].

(iv) Now consider the load

$$p(x) = \begin{cases} P_0/2, & |x| < L, \\ 0 & |x| \geq L. \end{cases}$$

Find the corresponding solution for $y(x)$ in the limit $L \to \infty$.
[You may assume that $(\sin\alpha x)/x \to \pi\delta(x)$ as $\alpha \to \infty$].

8. Let $f(x,t)$ be governed by the PDE:

$$\alpha\frac{\partial^2 f}{\partial t^2} + \frac{\partial f}{\partial t} - \beta\frac{\partial^2 f}{\partial x^2} = g(x,t),$$

where g is a given function of x and t with double Fourier transform $G(k,\omega)$.

(i) Express $f(x,t)$ in terms of $G(k,\omega)$.

(ii) Define the quantity

$$N(t) = \int_{-\infty}^{\infty} f(x,t)\,dx,$$

and show that

$$N(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(0,\omega)}{i\omega - \alpha\omega^2}\exp(i\omega t)\,d\omega.$$

(iii) Express the autocorrelation

$$\rho(t) = \int_{-\infty}^{\infty} N(u+t)N(u)\,du$$

in terms of $|G(0,\omega)|^2$.

(iv) Assume that $G(0,\omega) \propto |\omega|^{\sigma}$. Determine σ such that $\rho(t)$ becomes independent of t in the limit $\alpha \to 0$.