# ME3.6 Sheet 3 Answers

**1(i)** $\text{FT}\{\exp(-a|t|\} = \int_{-\infty}^{\infty} \exp(-a|t|)e^{-i\omega t}\,dt$
$= \int_{-\infty}^{0} e^{(a-i\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+i\omega)t}\,dt = \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \underline{\frac{2a}{a^2+\omega^2}}$

**(ii)** $\text{FT}\{sgn(t)\exp(-a|t|\} = \int_{-\infty}^{0}(-1)e^{at}e^{-i\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-i\omega t}\,dt = -\frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \underline{-\frac{2i\omega}{a^2+\omega^2}}$.

**(iii)** We know from (i) that if $f(t) = \exp(-a|t|)$, then $\hat{f}(\omega) = 2a/(a^2+\omega^2)$
$\Rightarrow \hat{f}(t) = 2a/(a^2+t^2)$.
By symmetry formula $\text{FT}\{\hat{f}(t)\} = 2\pi f(-\omega) = \underline{2\pi\exp(-a|\omega|)}$.

**(iv)** $f(t) = 1-t^2$ for $|t| \le 1 \Rightarrow \hat{f}(\omega) = \int_{-1}^{1}(1-t^2)e^{-i\omega t}dt$
$= \int_{-1}^{1}(1-t^2)\cos\omega t\,dt - i\int_{-1}^{1}(1-t^2)\sin\omega t\,dt$
The second integral is zero since we are integrating an odd function.
The first integral is an even function so can be written as twice the integral over $[0,1]$.
Thus $\hat{f}(\omega) = 2\int_{0}^{1}(1-t^2)\cos\omega t\,dt = \cdots$(by parts twice)$\cdots = \underline{-(4/\omega^2)\cos\omega + (4/\omega^3)\sin\omega}$.

**(v)** From lectures, if $h(t) = 1$ for $|t| \le a$ and zero otherwise, then $\hat{h}(\omega) = (2/\omega)\sin(a\omega)$.
Then by symmetry formula, $\text{FT}\{(2/t)\sin(at)\} = 2\pi h(-\omega) = 2\pi h(\omega)$ since $h$ is even.
Thus, $\text{FT}\{\sin(at)/\pi t\} = \underline{h(\omega)}$.

**2.** $\text{FT}\{f(t)\sin at\} = \int_{-\infty}^{\infty} f(t)\sin at\, e^{-i\omega t}\,dt = \frac{1}{2i}\int_{-\infty}^{\infty} f(t)(e^{iat}-e^{-iat})e^{-i\omega t}\,dt$
$= \frac{1}{2i}\int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)t}\,dt - \frac{1}{2i}\int_{-\infty}^{\infty} f(t)\,e^{-i(\omega+a)t}\,dt = \underline{\frac{1}{2i}\hat{f}(\omega-a) - \frac{1}{2i}\hat{f}(\omega+a)}$.

**3 (i)** From 1(i) $\text{FT}\{\exp(-a|t|\} = 2a/(a^2+\omega^2)$.
Therefore using inversion formula:
$\exp(-a|t|) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(2a/(a^2+\omega^2))\,e^{i\omega t}\,d\omega = (a/\pi)\left(\int_{-\infty}^{\infty}\frac{\cos(\omega t)}{a^2+\omega^2}\,d\omega + i\int_{-\infty}^{\infty}\frac{\sin(\omega t)}{a^2+\omega^2}\,d\omega\right)$
Second integral is zero since integrand is odd in $\omega$,
first integral has even integrand so doubles up over $[0,\infty]$.
Thus $\exp(-a|t|) = (2a/\pi)\int_{0}^{\infty}\cos(t\omega)/(a^2+\omega^2)\,d\omega$.
This expression is true for any $t$. Setting $t = 1$ :

$$\underline{\frac{\pi e^{-a}}{2a} = \int_{0}^{\infty}\frac{\cos\omega}{a^2+\omega^2}\,d\omega}$$

as required.

**(ii)** From 1(iv) if we define $g(t) = 1-t^2$ for $|t| \le 1$ and zero otherwise, then by inversion:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega\right)e^{i\omega t}\,d\omega.$$

Set $t = 0$ and rearrange to obtain desired result.

**4.** Energy theorem states that $2\pi(f(0) * f(0)) = \int_{-\infty}^{\infty} E(\omega)\,d\omega$. First need to find $\hat{f}(\omega)$.
$\hat{f}(\omega) = \int_{-2d}^{2d}(2d-|t|)e^{-i\omega t}dt = \int_{-2d}^{2d}(2d-|t|)\cos\omega t\,dt - i\int_{-2d}^{2d}(2d-|t|)\sin\omega t\,dt$
The second integral is zero since the integrand is odd in $t$.
The first integral has an even integrand and so doubles up over $[0,2d]$.
Thus $\hat{f}(\omega) = 2\int_{0}^{2d}(2d-t)\cos\omega t\,dt = \cdots$(by parts)
$\cdots = (2/\omega^2)(1-\cos(2\omega d)) = (4/\omega^2)\sin^2(\omega d)$
$\Rightarrow E(\omega) = \left|\hat{f}(\omega)\right|^2 = \underline{(16/\omega^4)\sin^4(\omega d)}$.
Now $\rho_f(0) = f(0) * f(0) = \int_{-\infty}^{\infty}(f(u))^2\,du = \int_{-2d}^{2d}(2d-|u|)^2du = 2\int_{0}^{2d}(2d-u)^2du$

$= \cdots = \underline{(16/3)d^3}$.

Energy theorem $\Rightarrow 2\pi\rho_f(0) = \int_{-\infty}^{\infty} E(\omega)\,d\omega \Rightarrow 32\pi d^3/3 = 16\int_{-\infty}^{\infty} \sin^4(\omega d)/\omega^4 d\omega$.

Setting $d = 1$ we get

$$\underline{\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{2\pi}{3}},$$

as required.

**5**. If $f(t) = \exp(-ct)H(t)$ then $\hat{f}(\omega) = \int_{-\infty}^{\infty}(e^{-ct}H(t))\,e^{-i\omega t}\,dt = \int_0^{\infty} e^{-(c+i\omega)t}dt = \underline{1/(c+i\omega)}$.

Convolution $\Rightarrow$ $(\text{FT})^{-1}(\hat{g}(\omega)\hat{h}(\omega)) = g(t) * f(t)$.

Let $\hat{g}(\omega) = 1/(a+i\omega) \Rightarrow g(t) = \exp(-at)H(t)$.

Let $\hat{h}(\omega) = 1/(b+i\omega) \Rightarrow h(t) = \exp(-bt)H(t)$.

$\Rightarrow (\text{FT})^{-1}((a+i\omega)^{-1}(b+i\omega)^{-1}) = (\exp(-at)H(t)) * (\exp(-bt)H(t))$.

RHS = $\int_{-\infty}^{\infty} \exp(-a(t-u))H(t-u)\exp(-bu)H(u)\,du$

$= \int_0^{\infty} \exp(-a(t-u))H(t-u)\exp(-bu)\,du$

The function $H(t-u)$ is non-zero (and equal to 1) only if $0 < u < t$.

Therefore RHS = $\int_0^t \exp(-at)\exp((a-b)u)\,du = \cdots = \underline{(\exp(-bt) - \exp(-at))/(a-b)\ \ (t > 0)}$.

$\underline{\text{RHS} = 0 \text{ if } t < 0.}$

**6**. Convolution $\Rightarrow$ $\text{FT}\{\hat{f}(t) * \hat{g}(t)\} = \text{FT}\{\hat{f}(t)\}\ \text{FT}\{\hat{g}(t)\}$

= (symmetry formula) = $4\pi^2 f(-\omega)g(-\omega)$.

Take RHS, change $\omega$ to $t$ and take transform again:

$\text{FT}\{4\pi^2 f(-t)g(-t)\} = 2\pi(\hat{f}(-\omega) * \hat{g}(-\omega))$ using symmetry rule again.

Thus: $\underline{\text{FT}\{f(t)g(t)\} = (\hat{f}(\omega) * \hat{g}(\omega))/(2\pi)}$, as required.

**7**. From 1(v) we have that $\hat{g}(\omega) = 1$ if $|\omega| \le a$ and zero otherwise.

By convolution: $f(t) * g(t) = (\text{FT})^{-1}(\hat{f}(\omega)\hat{g}(\omega))$.

Inversion formula $\Rightarrow$ RHS = $(1/2\pi)\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)e^{i\omega t}d\omega = (1/2\pi)\int_{-a}^{a}\hat{f}(\omega)e^{i\omega t}\,d\omega$

$= (1/2\pi)\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}d\omega$ (since $a > M$).

Thus: $f(t) * g(t) = (\text{FT})^{-1}(\hat{f}(\omega)) \underline{= f(t)}$, as required.

**8 (i)** $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\phi(t)\,dt = f(t_0)\phi(t_0) = f(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)\phi(t)\,dt = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)\phi(t)\,dt$

$\Rightarrow \int_{-\infty}^{\infty}[f(t)\delta(t-t_0) - f(t_0)\delta(t-t_0)]\phi(t)\,dt = 0$ for arbitrary $\phi$.

$\Rightarrow \underline{f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)}$.

**(ii)** $\int_{-\infty}^{\infty} t\phi(t)\delta'(t)\,dt$ = (by parts) = $[\delta(t)t\phi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(t)(\phi(t) + t\phi'(t))\,dt$.

Term in square brackets is zero.

Integral reduces to $-\phi(0)$ which can also be written as $-\int_{-\infty}^{\infty}\delta(t)\phi(t)\,dt$.

Thus $\underline{t\delta'(t) = -\delta(t)}$.

**(iii)** $\int_{-\infty}^{\infty}\delta(-t)\phi(t)\,dt$ = (subst $t = -s$) $= \int_{-\infty}^{\infty}\delta(s)\phi(-s)\,ds = \phi(0) = \int_{-\infty}^{\infty}\delta(t)\phi(t)\,dt$

$\Rightarrow \underline{\delta(-t) = \delta(t)}$.