Mechanical Engineering Mathematics ME3.6
Sheet 3 Fourier Transforms I

1. Find the Fourier transforms of the following functions: (with $a > 0$)

(i) $f(t) = \exp(-a|t|)$;
(ii) $f(t) = \mathrm{sgn}(t)\exp(-a|t|)$; [$\mathrm{sgn}(t) = 1$ if $t > 0$ and -1 if $t < 0$].
(iii) $f(t) = 2a/(a^2+t^2)$; (Hint: use the result of (i) and the symmetry formula from lectures)
(iv) $f(t) = 1 - t^2$ for $|t| \leq 1$ and zero otherwise;
(v) $f(t) = \sin(at)/(\pi t)$; (Hint: use the transform of a rectangular pulse from the lectures and the symmetry formula).

2. If a function has Fourier transform $\hat{f}(\omega)$, find the Fourier transform of $f(t)\sin(at)$ in terms of $\hat{f}$.

3. By applying the inversion formula to the transforms obtained in 1(i) and 1(iv), establish the following results:

$$\text{(i)} \int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a} \text{ if } a > 0; \quad \text{(ii)} \int_{-\infty}^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{2}.$$

4. Sketch the function given by

$$f(t) = \begin{cases} 2d - |t| & \text{for } |t| \leq 2d, \\ 0 & \text{otherwise.} \end{cases},$$

and show that $\hat{f}(\omega) = (4/\omega)^2\sin^2(\omega d)$.

Use the energy theorem (i.e. $2\pi\rho_f(0) = \int_{-\infty}^\infty E(\omega)\,d\omega$), to demonstrate that

$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{2\pi}{3}.$$

5. Show that the Fourier transform of $\exp(-ct)H(t)$, where H is the Heaviside function and c is a positive constant, is given by $1/(c+i\omega)$. Hence use the convolution theorem to find the inverse Fourier transform of

$$\frac{1}{(a+i\omega)(b+i\omega)},$$

where $a > b > 0$.

6. Use the symmetry rule to show that

$$\mathrm{FT}[f(t)g(t)] = \frac{1}{2\pi}(\hat{f}(\omega) * \hat{g}(\omega)).$$

7. Suppose that $f(t)$ is a function such that $\hat{f}(\omega) = 0$ for all ω with $|\omega| > M$, where M is a positive constant. Let $g(t) = \sin(at)/(\pi t)$. Show that if the constant $a > M$:

$$f(t) * g(t) = f(t).$$

Hint: Use the transform of $g(t)$ from Q1(v).

8. By considering suitable integration formulae, establish the following results involving the Dirac delta function:

(i) $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$; (ii) $t\delta'(t) = -\delta(t)$; (iii) $\delta(-t) = \delta(t)$.

[In each case multiply by an arbitrary test function $\phi(t)$and integrate from $-\infty$ to ∞].