ME3.6 Sheet 2 Answers

1(i) We have $dx/dt = -6x + 2xy - 8 = f(x,y), dy/dt = y^2 - x^2 = g(x,y)$.
The critical points satisfy $g = 0 \Rightarrow y = \pm x$ and $f = 0 \Rightarrow -6x + 2xy - 8 = 0$.
Substitute $y = \pm x$ to get $-6x + 2x^2 - 8 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or -1.
If instead we substitute $y = -x$ we get $-6x + 2x^2 - 8 = 0$ which has no real solutions.
Therefore the critical points are $\underline{(4,4) \text{ and } (-1,-1)}$.

The Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -6+2y & 2x \\ -2x & 2y \end{pmatrix}$

$\Rightarrow J(4,4) = \begin{pmatrix} 2 & 8 \\ -8 & 8 \end{pmatrix}$ The eigenvalues of $J(4,4)$ satisfy

$(2-\lambda)(8-\lambda) + 64 = 0 \Rightarrow \lambda^2 - 10\lambda + 80 = 0 \Rightarrow \lambda = 5 \pm i\sqrt{55}$

$\Rightarrow$ $\underline{(4,4) \text{ is an unstable spiral.}}$

$J(-1,-1) = \begin{pmatrix} -8 & -2 \\ 2 & -2 \end{pmatrix}$. The eigenvalues satisfy

$(-8-\lambda)(-2-\lambda) + 4 = 0 \Rightarrow \lambda^2 + 10\lambda + 20 = 0 \Rightarrow \lambda = -5 \pm \sqrt{5}$.
Both roots are therefore real and negative so that we have a $\underline{\text{stable node at } (-1,-1)}$.

1(ii) We have $dx/dt = -2x - y + 2 = f(x,y), dy/dt = xy = g(x,y)$.
The critical points satisfy $g = 0 \Rightarrow x = 0$ and/or $y = 0$ and $f = 0 \Rightarrow -2x - y + 2 = 0$.
Substitute $x = 0$ to get $y = 2$.
If instead we substitute $y = 0$ we get $-2x + 2 = 0 \Rightarrow x = 1$.
Therefore the critical points are $\underline{(0,2) \text{ and } (1,0)}$.

The Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ y & x \end{pmatrix}$

$\Rightarrow J(0,2) = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}$ The eigenvalues of $J(0,2)$ satisfy

$(-2-\lambda)(-\lambda) + 2 = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$

$\Rightarrow$ $\underline{(0,2) \text{ is a stable spiral.}}$

$J(1,0) = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}$. The eigenvalues satisfy

$(-2-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = -2$ and $\lambda = 1$,
Both roots are therefore real and of opposite sign so that we have a $\underline{\text{saddle at } (1,0)}$.

1(iii) We have $dx/dt = 4 - 4x^2 - y^2 = f(x,y), dy/dt = 3xy = g(x,y)$.
The critical points satisfy $g = 0 \Rightarrow x = 0$ and/or $y = 0$ and $f = 0 \Rightarrow 4 - 4x^2 - y^2 = 0$.
Substitute $x = 0$ to get $4 - y^2 = 0 \Rightarrow y = \pm 2$.
If instead we substitute $y = 0$ we get $4 - 4x^2 = 0 \Rightarrow x = \pm 1$.
Therefore the critical points are $\underline{(0,2),(0,-2),(1,0),(-1,0)}$.

The Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -8x & -2y \\ 3y & 3x \end{pmatrix}$

$\Rightarrow J(0,2) = \begin{pmatrix} 0 & -4 \\ 6 & 0 \end{pmatrix}$ which has eigenvalues $\lambda = \pm i\sqrt{24} \Rightarrow$ $\underline{\text{center at } (0,2)}$.

$J(0,-2) = \begin{pmatrix} 0 & 4 \\ -6 & 0 \end{pmatrix}$. Eigenvalues $\lambda = \pm i\sqrt{24} \Rightarrow$ center at $(0,-2)$.

$J(1,0) = \begin{pmatrix} -8 & 0 \\ 0 & 3 \end{pmatrix}$. Eigenvalues $\lambda = -8$ and $3 \Rightarrow$ saddle at $(1,0)$.

$J(-1,0) = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$. Eigenvalues $\lambda = 8$ and $-3 \Rightarrow$ saddle at $(-1,0)$.

1(iv) We have $dx/dt = \sin y = f(x,y), dy/dt = x + x^3 = g(x,y)$.
The critical points satisfy $f = 0 \Rightarrow y = n\pi$, $(n = 0,\pm1,\pm2,\ldots)$
and $g = 0 \Rightarrow x = 0$ (since $1 + x^2 = 0$ has no real roots).
$\Rightarrow$ C.P.'s are at $(0,n\pi)$.

The Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & \cos y \\ 1+3x^2 & 0 \end{pmatrix}$.

$\Rightarrow J(0,n\pi) = \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}$ which has eigenvalues $\lambda^2 = (-1)^n$.

n even $\Rightarrow \lambda = \pm1 \Rightarrow$ C.P.'s are saddles at $((0,0),(0,2\pi),(0,4\pi),\ldots)$
n odd $\Rightarrow \lambda = \pm i \Rightarrow$ C.P.'s are centers at $((0,\pi),(0,3\pi),(0,5\pi),\ldots)$.

1(v) We have $dx/dt = y = f, dy/dt = (\omega^2 - g - y^2)x/(1+x^2) = G$, say.
C.P.'s occur when $y = 0$ and $(\omega^2 - g - y^2)x = 0 \Rightarrow x = 0 \Rightarrow$ C.P. is at $(0,0)$.

$J(x,y) = \begin{pmatrix} f_x & f_y \\ G_x & G_y \end{pmatrix} = \cdots = \begin{pmatrix} 0 & 1 \\ (1-x^2)(\omega^2-g-y^2)/(1+x^2)^2 & -2yx/(1+x^2) \end{pmatrix}$.

$J(0,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 - g & 0 \end{pmatrix} \Rightarrow \lambda = \pm(\omega^2-g)^{1/2}$.

If $\omega^2 - g > 0$ we have a saddle at $(0,0)$;
If $\omega^2 - g < 0$ we have a center at $(0,0)$.

2. Write as 1st order system: $dx/dt = y = f$ and $dy/dt = -x + x^3 = g$.
Critical points occur when $f = 0 \Rightarrow y = 0$ and $g = 0 \Rightarrow x(x^2-1) = 0 \Rightarrow x = 0,\pm1$.
Therefore C.P's are $(0,0),(1,0),(-1,0)$.

Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{pmatrix}$.

$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = \pm i \Rightarrow$ center at $(0,0)$.

$J(1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow \lambda = \pm\sqrt{2} \Rightarrow$ saddle at $(1,0)$.

$J(-1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Eigenvalues as above. Saddle at $(-1,0)$.

Original equation multiplied by dx/dt is

$$\ddot{x}\dot{x} + (x - x^3)\dot{x} = 0.$$

Integrate with respect to t :

$$\frac{1}{2}(\dot{x})^2 + (x^2/2 - x^4/4) = C \ (*).$$

Therefore $\underline{V(x) = x^2/2 - x^4/4}$. We want the solution that passes through $(\pm 1, 0)$.
Substituting $y = dx/dt = 0, x = \pm 1$ into $(*) \Rightarrow C = 1/4$.
Thus the trajectories are $y^2 + x^2 - x^4/2 = 1/2 \Rightarrow \underline{y = \pm(x^2 - 1)/\sqrt{2}}$.
For sketch see separate sheet.

3. We have $dx/dt = x(1 - x - y) = f$, $dy/dt = y(3 - x - 2y) = g$.
C.P.'s occur when $f = g = 0$.
Both equations satisfied if $x = 0$ & $y = 0$ or $x = 0$ & $3 - x - 2y = 0$ ($\Rightarrow y = 3/2$),
or $1 - x - y = 0$ & $y = 0$ ($\Rightarrow x = 1$) or $1 - x - y = 0$ & $3 - x - 2y = 0$ ($\Rightarrow y = 2, x = -1$).
This last solution is not in the 1st quadrant.
Thus the $\underline{\text{C.P.'s are } (0,0), (0,3/2), (1,0)}$.

$$\text{Jacobian } J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 - 2x - y & -x \\ -y & 3 - x - 4y \end{pmatrix}.$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda = 1,3 \Rightarrow \underline{\text{unstable node at } (0,0)}.$$

$$\text{Eigenvectors:} \begin{pmatrix} 1 - \lambda & 0 \\ 0 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

$$\underline{\lambda = 1 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \ \underline{\lambda = 3 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

$$J(0,3/2) = \begin{pmatrix} -1/2 & 0 \\ -3/2 & -3 \end{pmatrix} \Rightarrow \lambda = -1/2, -3 \Rightarrow \underline{\text{stable node at } (0,3/2)}.$$

$$\text{Eigenvectors:} \begin{pmatrix} -1/2 - \lambda & 0 \\ -3/2 & -3 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

$$\underline{\lambda = -1/2 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/5 \end{pmatrix}}. \ \underline{\lambda = -3 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda = -1,2 \Rightarrow \underline{\text{saddle at } (1,0)}.$$

$$\text{Eigenvectors:} \begin{pmatrix} -1 - \lambda & -1 \\ 0 & 2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

$$\underline{\lambda = -1 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \ \underline{\lambda = 2 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}.$$

For sketch see separate sheet.
As $t \to \infty$ all solutions which have $y \neq 0$ end up at the stable node at $(0,3/2)$.
i.e. chemical x is used up and chemical $y \to 3/2$.

4. $dF/dt = -\alpha F + \beta\mu(M)F = f$, $dM/dt = -\alpha M + \gamma\mu(M)F = g$.
C.P.'s occur when $f = g = 0$. Clearly $(F,M) = (0,0)$ is a solution.
Now $f = 0 \Rightarrow F(-\alpha + \beta\mu) = 0$, so if $F \neq 0$ we have $\mu = \alpha/\beta$.
Substitute into $g = 0 \Rightarrow F = (\beta/\gamma)M$. $(*)$
Now, $\mu = \alpha/\beta \Rightarrow 1 - \exp(-kM) = \alpha/\beta$.

Solving for M : $M = \underline{-(1/k)\ln(1-\alpha/\beta) = M_0}$, say.
Then from $(*)$: $F = \underline{-(\beta/\gamma k)\ln(1-\alpha/\beta) = F_0}$, say.
The solutions for F_0 and M_0 are real provided $1-\alpha/\beta > 0$, i.e. $\beta > \alpha$.
C.P.'s are therefore $\underline{(0,0),(F_0,M_0)}$.

Jacobian $J(F,M) = \begin{pmatrix} f_F & f_M \\ g_F & g_M \end{pmatrix} = \begin{pmatrix} -\alpha+\beta\mu & \beta F\mu' \\ \gamma\mu & -\alpha+\gamma F\mu' \end{pmatrix}$.

Note that $\mu(0) = 0, \mu'(0) = k$.

$J(0,0) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \Rightarrow \lambda = -\alpha$ (repeated) $\Rightarrow$ $\underline{(0,0)\text{ is an inflected stable node}}$.

For the stability of (F_0,M_0) recall that $\mu = \alpha/\beta$ & $\mu'(M_0) = k(1-\alpha/\beta)$.

$J(F_0,M_0) = \begin{pmatrix} 0 & -(\beta^2/\gamma)(1-\alpha/\beta)\ln(1-\alpha/\beta) \\ \gamma\alpha/\beta & -\alpha-\beta(1-\alpha/\beta)\ln(1-\alpha/\beta) \end{pmatrix}$.

The eigenvalues satisfy

$$\lambda^2 + (\alpha+\beta(1-\alpha/\beta)\ln(1-\alpha/\beta))\lambda + \alpha\beta(1-\alpha/\beta)\ln(1-\alpha/\beta) = 0.$$

We can spot that $\lambda = -\alpha$ satisfies this equation.
The product of the roots equals $\alpha\beta(1-\alpha/\beta)\ln(1-\alpha/\beta) < 0$.
Therefore the second eigenvalue must be positive. It follows that $\underline{(F_0,M_0)\text{ is a saddle}}$.

5. $dH/dt = a_1H - b_1H^2 - c_1HP = f$, $dP/dt = -a_2P + c_2HP = g$.
The term proportional to H^2 has coefficient $-b_1$ and so reduces the host population.
H^2 indicates a self-interaction and represents population reduction due to overcrowding.
C.P.'s occur when $f = g = 0$. $g = 0 \Rightarrow P = 0$ or $H = a_2/c_2$.
Substitute $P = 0$ into $f = 0 \Rightarrow H(a_1 - b_1H) = 0 \Rightarrow H = 0, a_1/b_1$.
Substitute $H = a_2/c_2$ into $f = 0 \Rightarrow P = (a_1c_2 - b_1a_2)/(c_1c_2) = D/(c_1c_2)$.
Therefore the C.P.'s are $\underline{(H,P) = (0,0),(a_1/b_1,0),(a_2/c_2,D/(c_1c_2))}$.

Jacobian $J(H,P) = \begin{pmatrix} f_H & f_P \\ g_H & g_P \end{pmatrix} = \begin{pmatrix} a_1-2b_1H-c_1P & -c_1H \\ c_2P & -a_2+c_2H \end{pmatrix}$.

$J(0,0) = \begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix} \Rightarrow \lambda = a_1, -a_2 \Rightarrow \underline{(0,0)\text{ is a saddle}}$.

$J(a_1/b_1,0) = \begin{pmatrix} -a_1 & -a_1c_1/b_1 \\ 0 & D/b_1 \end{pmatrix}$

$\Rightarrow \lambda = -a_1, D/b_1 \Rightarrow$ $\underline{\text{stable node if } D < 0\text{, saddle if } D > 0}$.
So, $\underline{\text{for (i) we have a stable node. For (ii) and (iii) it is a saddle}}$.

$J(a_2/c_2,D/(c_1c_2)) = \cdots = \begin{pmatrix} -b_1a_2/c_2 & -a_2c_1/c_2 \\ D/c_1 & 0 \end{pmatrix} \Rightarrow \lambda^2 + (b_1a_2/c_2)\lambda + a_2D/c_2 = 0$.

$\Rightarrow 2\lambda = -(b_1a_2/c_2) \pm \sqrt{\Delta}$ where $\Delta = (a_2/c_2)(b_1^2a_2/c_2 - 4D)$.
Case (i) $\Rightarrow$ real roots of opposite sign $\Rightarrow$ $\underline{\text{saddle}}$
Case (ii) $\Rightarrow$ real roots both negative $\Rightarrow$ $\underline{\text{stable node}}$
Case (iii) $\Rightarrow$ complex roots with negative real part $\Rightarrow$ $\underline{\text{stable spiral}}$.

6. (**i**) Interaction between (a) and (b) leads to a decrease in (a), i.e.

$$\underline{dx/dt = -\mu xy} \quad (1)$$

y increases due to interaction with group (a) - so on the RHS of the equation for dy/dt we get a term μxy

y decreases due to interaction with group (c), which contributes a term $-\mu yz$

y also decreases due to interaction with all the other members of (b) so this gives $-\mu y(y-1)$.

Putting all this together

$$\underline{dy/dt = \mu xy - \mu yz - \mu y(y-1)}. \ (2)$$

z increases due to interaction with group (b) $\Rightarrow$ on the RHS of the equation for dz/dt we have a term μyz.

z also increases due to group (b) members interacting, contributing a term $\mu y(y-1)$.

$$\Rightarrow \underline{dz/dt = \mu yz + \mu y(y-1)} \ (3)$$

(**ii**) Adding together equations (1)-(3) we have $dx/dt + dy/dt + dz/dt = 0$

$\Rightarrow \underline{x + y + z = \text{constant} = N}$. Thus $z = N - x - y$.

Substituting for z in (2):

$$dy/dt = 2\mu xy - \mu(N-1)y \ (4)$$

Equation (4) divided by equation (1) gives $dy/dx = -2 + (N-1)/x$. Integrating $\Rightarrow$

$$\underline{y = -2x + (N-1)\ln x + C} \ (5)$$

(**iii**) At time $t = 0$ (say) we have $x(0) = N-1, y(0) = 1, z(0) = 0$.

Substituting into (5) $\Rightarrow C = 2N - 1 - (N-1)\ln(N-1)$.

Suppose that $x \to x_f$ as $t \to \infty$. Also, $y \to 0$ as $t \to \infty$ (eventually everyone meets).

Substitute into (5): $0 = -2x_f + (N-1)\ln x_f + 2N - 1 - (N-1)\ln(N-1)$.

Rearrange to obtain desired result.

7. For sketch of set-up see separate sheet.

In equilibrium the force upwards due to the spring balances the forces downwards due to the magnetic attraction and gravity.

$$\Rightarrow kx = mg + A/L^2 \ (1).$$

When the magnet is in motion we have that the net force down equals mass multiplied by downwards acceleration.

$$\Rightarrow m\frac{d^2z}{dt^2} = mg + \frac{A}{(L-z)^2} - k(x+z) \ (2).$$

Substituting for kx from (1) we obtain the equation given in the question.

As a first order system this is

$$\frac{dz}{dt} = w = f, \ \frac{dw}{dt} = \frac{A/m}{(L-z)^2} - (k/m)z - \frac{A/m}{L^2} = g.$$

C.P. occurs when $f = g = 0 \Rightarrow (z,w) = \underline{(0,0) \text{ is a critical point}}$.

$$\text{Jacobian } J(z,w) = \begin{pmatrix} f_z & f_w \\ g_z & g_w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2A/m}{(L-z)^3} - k/m & 0 \end{pmatrix}.$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{2A}{mL^3} - k/m & 0 \end{pmatrix} \Rightarrow \lambda = \pm\left(\frac{2A}{mL^3} - \frac{k}{m}\right)^{1/2}.$$

Thus, (0,0) is a saddle if $2A/mL^3 > k/m$, and a center if $2A/mL^3 < k/m$.

Oscillations will occur in the latter case, i.e. when $A < kL^3/2$.

8. Writing as a first-order system: $dx/dt = y = f, dy/dt = \varepsilon(1 - x^2)y - x = g$.
Easy to see that only C.P. is at (0,0).

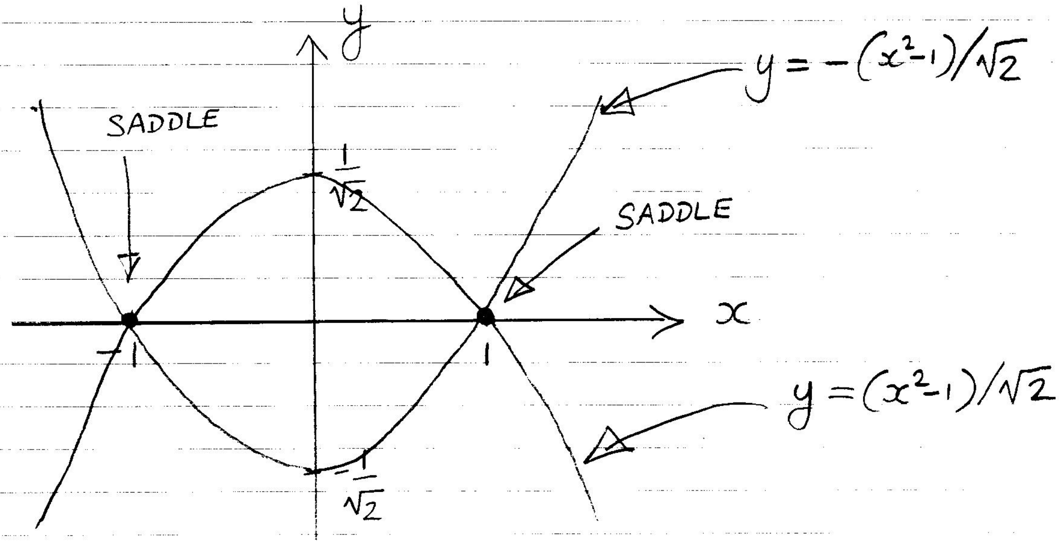
$$\text{Jacobian } J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\varepsilon xy - 1 & \varepsilon(1 - x^2) \end{pmatrix}.$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} \Rightarrow \lambda^2 - \varepsilon\lambda + 1 = 0 \Rightarrow 2\lambda = \varepsilon \pm \sqrt{\varepsilon^2 - 4}.$$

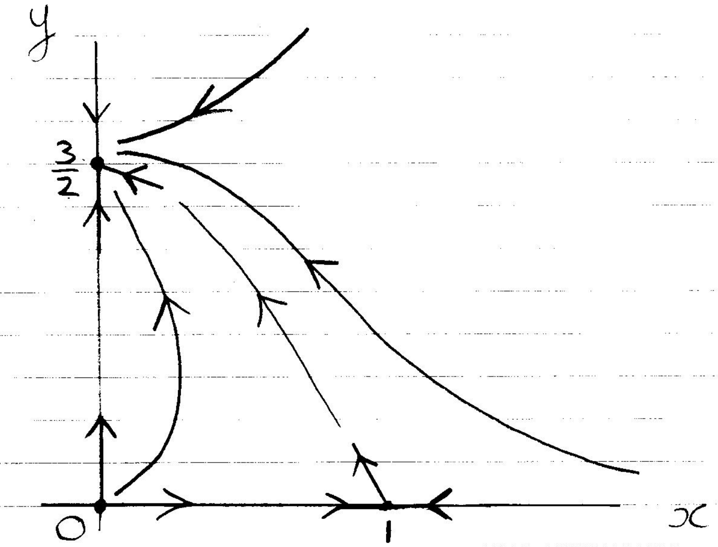
Thus, if $0 < \varepsilon < 2$ we have complex roots with positive real part
$\Rightarrow$(0,0) is an unstable spiral.
If $\varepsilon > 2$ both roots are real and positive $\Rightarrow$(0,0) is an unstable node.
For the numerical part of this question see the Mathematica Notebook `vanderpol.nb` on the website `http://www.ma.ic.ac.uk/~agw/me.html`.

Problem Sheet 2

Q2.



Q3.



Q7.