Mechanical Engineering Mathematics ME3.6
Problem Sheet 2: Nonlinear Autonomous Systems

1. Find the critical points of the following nonlinear ODEs and classify them according to their linear approximations

(i) $dx/dt = -6x + 2xy - 8,\ dy/dt = y^2 - x^2$;

(ii) $dx/dt = -2x - y + 2,\ dy/dt = xy$;

(iii) $dx/dt = 4 - 4x^2 - y^2,\ dy/dt = 3xy$;

(iv) $dx/dt = \sin y,\ dy/dt = x + x^3$;

(v) $dx/dt = y;\ dy/dt = (\omega^2 - g - y^2)x/(1 + x^2)$, (for $\omega^2 < g$ and $\omega^2 > g$).

2. The nonlinear harmonic oscillator equation is given by

$$\frac{d^2x}{dt^2} + x - x^3 = 0.$$

Write this as a system of first order equations and show that there are two saddle points and one centre in the linear approximation about the critical points. By multiplying the original equation by dx/dt and integrating, show that the equation may be rewritten as

$$\frac{1}{2}y^2 + V(x) = \text{constant},$$

where $y = dx/dt$, and identify the potential function $V(x)$. Use this information to sketch the trajectories that pass through the saddle points.

3. Two chemicals with concentrations $x(t)$ and $y(t)$ interact according to the equations

$$\frac{dx}{dt} = x - x^2 - xy,\ \frac{dy}{dt} = 3y - xy - 2y^2.$$

Find the critical points that lie in the first quadrant ($x \geq 0, y \geq 0$) and determine their stability in the linear approximation. Find also the eigenvectors in each case. Give an approximate sketch of the first quadrant of the phase plane. If, at $t = 0$, there are non-zero concentrations of both chemicals, what happens to the chemicals as $t \to \infty$?

4. A model for the change in population of a group of islands is given by

$$\frac{dF}{dt} = -\alpha F + \beta\mu(M)F,\ \frac{dM}{dt} = -\alpha M + \gamma\mu(M)F,\ \text{with } \alpha, \beta > 0,$$

where $\mu(M) = 1 - \exp(-kM)$. Here, $F(t)$and $M(t)$ represent the female and male parts of the population at time $t > 0$. Show that when $\beta > \alpha$ there are two critical points, and that one is an inflected node (the real eigenvalues are a double root). For the second critical point, show that one eigenvalues is $\lambda = -\alpha$, and hence deduce the nature of the point.

5. The interaction between a host species whose population is $H(t)$ for $t > 0$ and a parasite with population $P(t)$ is given by

$$\frac{dH}{dt} = a_1H - b_1H^2 - c_1HP,\ \frac{dP}{dt} = -a_2P + c_2HP,$$

with all constants assuming positive values. What does the term proportional to $-H^2$ represent? Find the three critical points of the system and classify them in the following ranges of D (where $D = a_1c_2 - b_1a_2$) :

(i) $D < 0$, (ii) $0 < D < a_2b_1^2/4c_2$, (iii) $D > a_2b_1^2/4c_2$.

6. A rumour spreads through the Mechical Engineering Department. At time t the population of the Department can be classified into three categories:

(a) x persons who are ignorant of the rumour;

(b) y persons who actively spread the rumour to each person they meet;

(c) z persons who have heard the rumour but have stopped spreading it on the grounds that it is old news.

If a person from group (b) meets a person either from his/her own group or from group (c) then they stop spreading the rumour.

(**i**) Show that the equations which govern the system are

$$\frac{dx}{dt} = -\mu xy, \quad \frac{dy}{dt} = \mu xy - \mu y(y-1) - \mu yz, \quad \frac{dz}{dt} = \mu y(y-1) + \mu yz,$$

where the constant μ represents the contact rate within the Department.

(**ii**) Deduce that the population of the Department must be of constant size (= N, say). Show that x and y are related via

$$y = (N-1)\ln x - 2x + C.$$

(**iii**) If the rumour starts from just one person, show that the number of engineers x_f who always remain ignorant of the rumour is the solution of

$$(N-1)\ln(x_f) - 2x_f = (N-1)\ln(N-1) - 2N + 1.$$

7. A magnet of mass m is suspended from a spring of stiffness k. The magnet is at rest at a distance L above a small piece of iron. Suppose that the attractive force between the magnet and the iron has magnitude A/x^2 when they are separated by a distance x.

Using $z(t)$ to represent downward displacement of the magnet from its equilibrium position, show that

$$m\frac{d^2z}{dt^2} = -kz + \frac{A}{(L-z)^2} - \frac{A}{L^2}.$$

Write this as a system of first-order equations. Show that the origin is a critical point and determine its nature. Show that the magnet will undergo oscillatory motion about this point if $A < \frac{1}{2}kL^3$.

8. The Van der Pol equation

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0, \quad \varepsilon > 0,$$

has been used as a model for a variety of physical phenomena including the pulsations of variable stars and the feedback in electrical circuits.

Show that the origin is an unstable spiral for this system if $0 < \varepsilon < 2$, and an unstable node if $\varepsilon > 2$. Using software such as Mathematica, solve this equation numerically using the initial condition $x = dx/dt = 0.1$ when $t = 0$, and plot the phase plane for $\varepsilon = 0.1,\ 1.0$ and 5.0. What happens to the solutions in each case as t becomes large?