ME3.6 Sheet 1 Answers

1(i) We have $dx/dt = 2x + 5y, dy/dt = x - 2y$.
Critical point is when the RHS's vanish simultaneously, i.e. when $\underline{x = y = 0}$.
So the critical point is at the origin.

For this problem $A = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$, and so the eigenvalues satisfy

$$(2-\lambda)(-2-\lambda) - 5 = 0 \Rightarrow \lambda^2 - 9 = 0 \Rightarrow \underline{\lambda = \pm 3},$$

i.e. we have a **<u>saddle</u>** at $(0,0)$.

The eigenvectors satisfy $\begin{pmatrix} 2-\lambda & 5 \\ 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$.

For $\lambda = 3$ this gives $-x_1 + 5y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_1\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

For $\lambda = -3$ we have $x_1 + y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 5 \\ 1 \end{pmatrix}e^{3t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t}.$$

Since $dy/dx = (x-2y)/(2x+5y)$ we see that $dy/dx = 0$ on $y = x/2$ and $dy/dx = \infty$ on $y = -2x/5$. For sketch see separate sheet.

1(ii) We have $dx/dt = -3x + y, dy/dt = -x - 3y$.
Critical point is when the RHS's vanish simultaneously, i.e. when $\underline{x = y = 0}$.
So the critical point is at the origin.

For this problem $A = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}$, and so the eigenvalues satisfy

$$(-3-\lambda)(-3-\lambda) - (-1) = 0 \Rightarrow \lambda^2 + 6\lambda + 10 = 0 \Rightarrow \underline{\lambda = -3 \pm i},$$

i.e. we have a **<u>stable spiral</u>** at $(0,0)$.

The eigenvectors satisfy $\begin{pmatrix} -3-\lambda & 1 \\ -1 & -3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$.

For $\lambda = -3 + i$ this gives $-ix_1 + y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ i \end{pmatrix}$.

For $\lambda = -3 - i$ we have $ix_1 + y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_2\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-3t}e^{it} + C_2\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{-3t}e^{-it}.$$

Since $dy/dx = (-x - 3y)/(-3x + y)$ we see that $dy/dx = 0$ on $y = -x/3$
and $dy/dx = \infty$ on $y = 3x$.
To get direction of spiral note that if $y > 3x$ then $dx/dt > 0$ i.e. x increases as t increases.
Therefore the trajectories spiral in clockwise. For sketch see separate sheet.

1(iii) We have $dx/dt = -4x + 3y, dy/dt = -2x + y$.
Critical point is when the RHS's vanish simultaneously, i.e. when $\underline{x = y = 0}$.
So the critical point is at the origin.
For this problem $A = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix}$, and so the eigenvalues satisfy

$$(-4 - \lambda)(1 - \lambda) - (-6) = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \underline{\lambda = -2 \text{ and } -1},$$

i.e. we have a **stable node** at $(0,0)$.
The eigenvectors satisfy $\begin{pmatrix} -4-\lambda & 3 \\ -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$.
For $\lambda = -2$ this gives $-2x_1 + 3y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
For $\lambda = -1$ we have $-3x_1 + 3y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}.$$

Since $dy/dx = (-2x + y)/(-4x + 3y)$ we see that $dy/dx = 0$ on $y = 2x$
and $dy/dx = \infty$ on $y = 4x/3$.
Also note that as $t \to \infty$ the e^{-2t} term is negligible so that $x \sim y$ as $t \to \infty$.
This means that all solutions (except those that start on $y = 2x/3$) are drawn in along $y = x$.
For sketch see separate sheet.

1(iv) We have $dx/dt = -2x - 2y, dy/dt = 3x + 2y$.
Critical point is when the RHS's vanish simultaneously, i.e. when $\underline{x = y = 0}$.
So the critical point is at the origin.
For this problem $A = \begin{pmatrix} -2 & -2 \\ 3 & 2 \end{pmatrix}$, and so the eigenvalues satisfy

$$(-2 - \lambda)(2 - \lambda) - (-6) = 0 \Rightarrow \lambda^2 + 2 = 0 \Rightarrow \underline{\lambda = \pm i\sqrt{2}},$$

i.e. we have a **center** at $(0,0)$.
The eigenvectors satisfy $\begin{pmatrix} -2-\lambda & -2 \\ 3 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$.
For $\lambda = i\sqrt{2}$ this gives $-(2 + i\sqrt{2})x_1 - 2y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_1\begin{pmatrix} 2 \\ -(2 + i\sqrt{2}) \end{pmatrix}$.

For $\lambda = -i\sqrt{2}$ we have $-(2 - i\sqrt{2})x_1 - 2y_1 = 0$ and so $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = C_2 \begin{pmatrix} 2 \\ -(2 - i\sqrt{2}) \end{pmatrix}$.

Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ -(2 + i\sqrt{2}) \end{pmatrix} e^{i\sqrt{2}t} + C_2 \begin{pmatrix} 2 \\ -(2 - i\sqrt{2}) \end{pmatrix} e^{-i\sqrt{2}t}.$$

Since $dy/dx = (3x + 2y)/(-2x - 2y)$ we see that $dy/dx = 0$ on $y = -3x/2$
and $dy/dx = \infty$ on $y = -x$.
To get direction of orbit, look, for example at 1st quadrant.
Here we have $dx/dt < 0$ and $dy/dt > 0$ so that as t increases x decreases and y increases.
Therefore the trajectories are clockwise. For sketch see separate sheet.

1(**v**) We have $dx/dt = -2x - 2y + 3, dy/dt = 3x + 2y - 4$.
Critical point is when the RHS's vanish simultaneously,
i.e. when $-2x - 2y + 3 = 0, 3x + 2y - 4 = 0$.
Adding these equations together: $x - 1 = 0 \Rightarrow x = 1$.
Substitute back to get $2y = 4 - 3x = 1$. So the critical point is at $\underline{(1, 1/2)}$.
Now write $x = 1 + X, y = 1/2 + Y$. Then X, Y satisfy

$$dX/dt = -2X - 2Y, \; dY/dt = 3X + 2Y,$$

and so the equations are now the same as in part (iv).
Sketch is same except that the orbits are now centered around $(1, 1/2)$ instead of $(0, 0)$.

2. Using Newton's law we have

$$md^2x/dt^2 = -kx - \gamma dx/dt.$$

If we write $y = dx/dt$ then we obtain the two equations given in the question.
The critical point occurs when the 2 RHS's are zero simultaneously, i.e. when $x = y = 0$.
Thus the critical point is at $(0,0)$.
We have $dy/dx = -(\gamma/m) - (k/m)(x/y)$
$\Rightarrow dy/dx = 0$ when $y = -(k/\gamma)x$. Also, when $x = 0$ we have $dy/dx = -\gamma/m$ provided $y \neq 0$.
i.e. the slope is negative along the y –axis.

For this problem $A = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix}$, and so the eigenvalues satisfy

$$\lambda^2 + (\gamma/m)\lambda + (k/m) = 0 \Rightarrow \underline{2\lambda = -(\gamma/m) \pm \sqrt{((\gamma/m)^2 - 4(k/m)}}.$$

<u>Case 1</u>: If $(\gamma/m)^2 > 4(k/m)$, then the values of λ are both real and negative.
$\Rightarrow$ $(0,0)$ is a <u>**stable node**</u>.

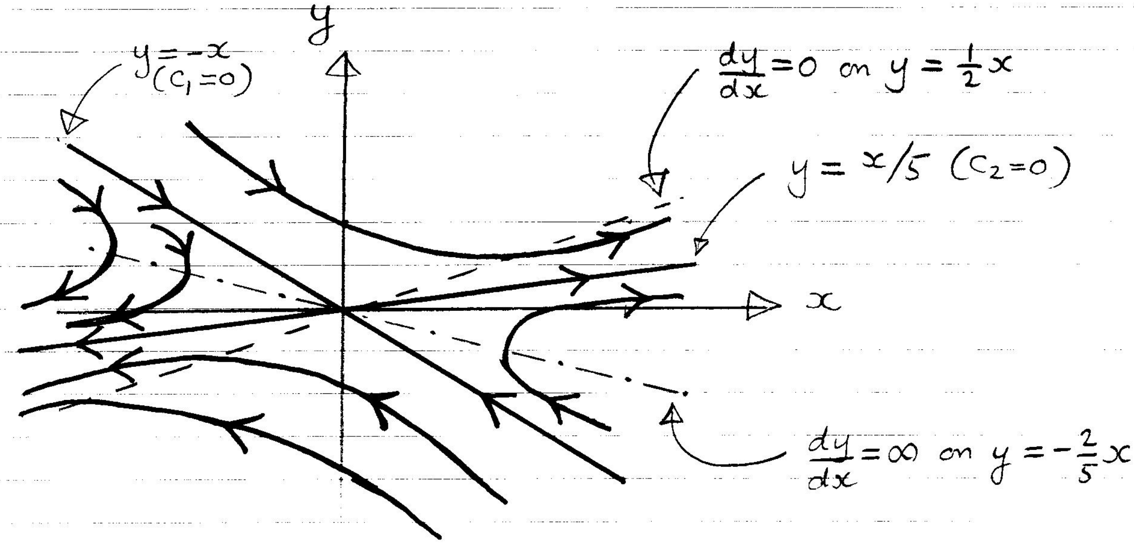
The eigenvectors satisfy $\begin{pmatrix} -\lambda & 1 \\ -k/m & -\gamma/m - \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$

This means that for both eigenvalues we have $y_1 = \lambda x_1$.
When t is large the less negative eigenvalue dominates.
The trajectories are therefore drawn in along $y = \lambda_1 x$
where λ_1 is the eigenvalue with $+\sqrt{}$ above.
<u>Case 2</u>: If $(\gamma/m)^2 < 4(k/m)$, then the values of λ are complex with negative real part.
$\Rightarrow$ $(0,0)$ is a <u>**stable spiral**</u>.

The two phase planes are shown on a separate sheet.
The essential difference is that in case 1 if the spring has a positive displacement initially, it subsequently comes to rest at $x = 0$ without passing into a region where $x < 0$.
i.e. it does not oscillate about the origin because it is heavily damped.

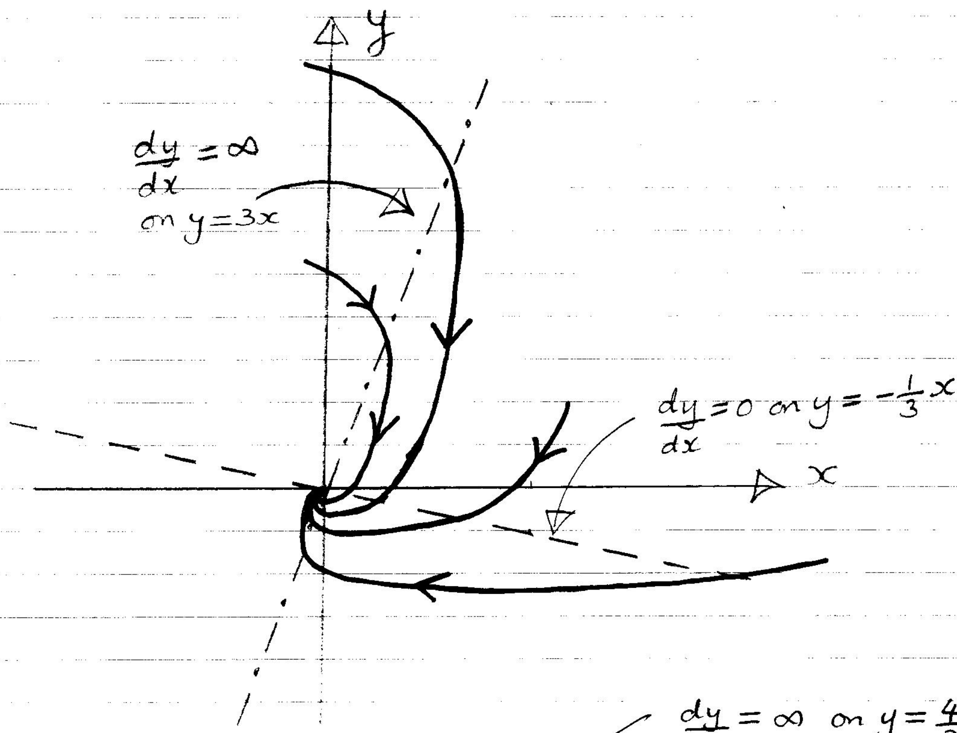
Q1

(i)



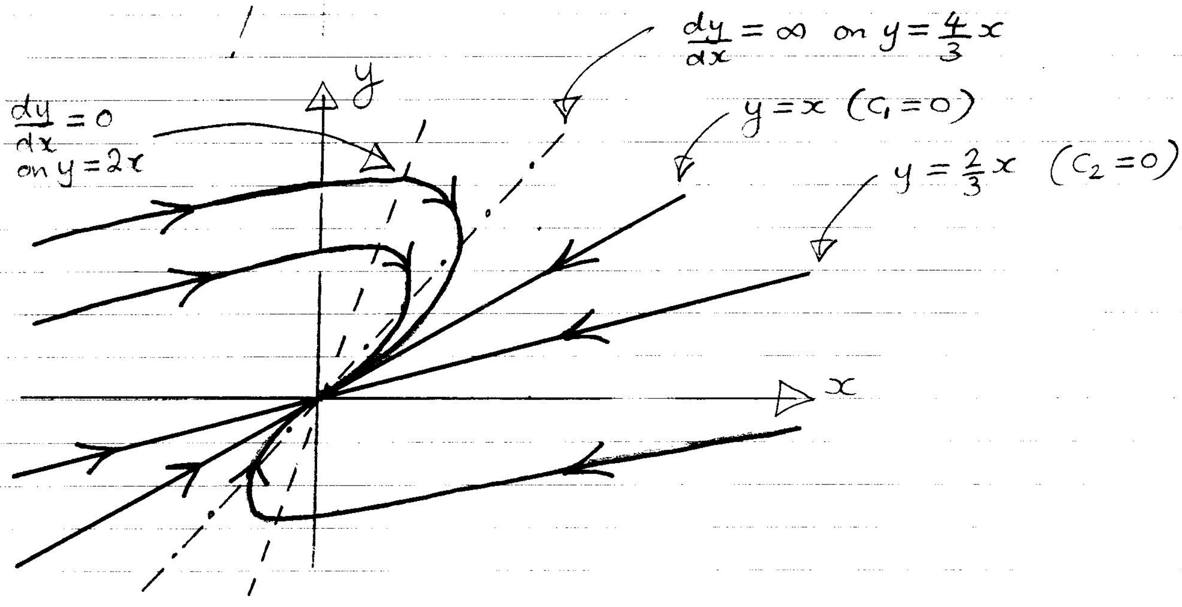
y
x
y = -x (c₁=0)
dy/dx = 0 on y = ½x
y = x/5 (c₂=0)
dy/dx = ∞ on y = -2/5 x

(ii)



y
x
dy/dx = ∞ on y = 3x
dy/dx = 0 on y = -1/3 x

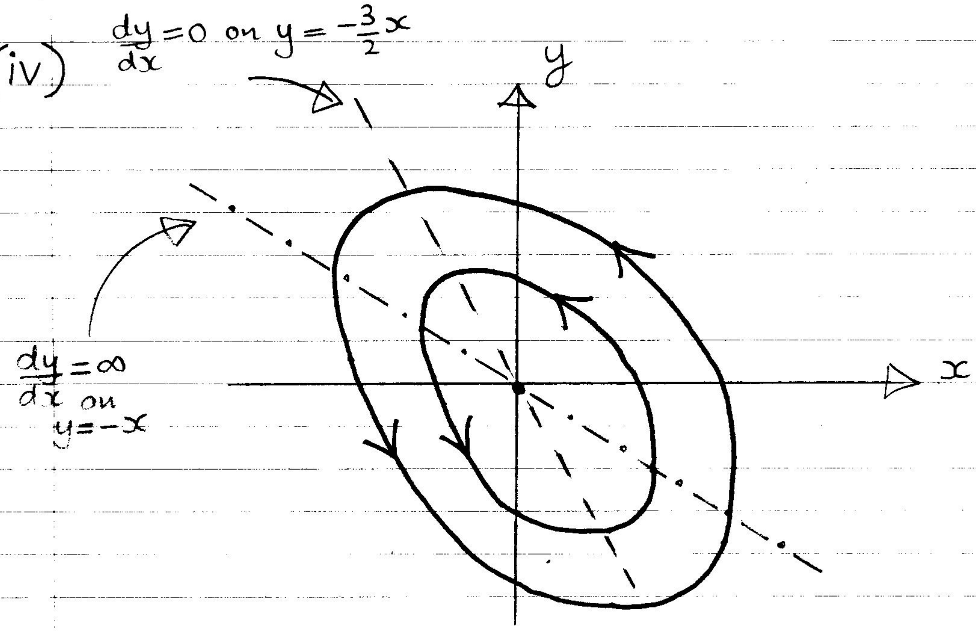
(iii)



y
x
dy/dx = ∞ on y = 4/3 x
dy/dx = 0 on y = 2x
y = x (c₁=0)
y = 2/3 x (c₂=0)

Q1

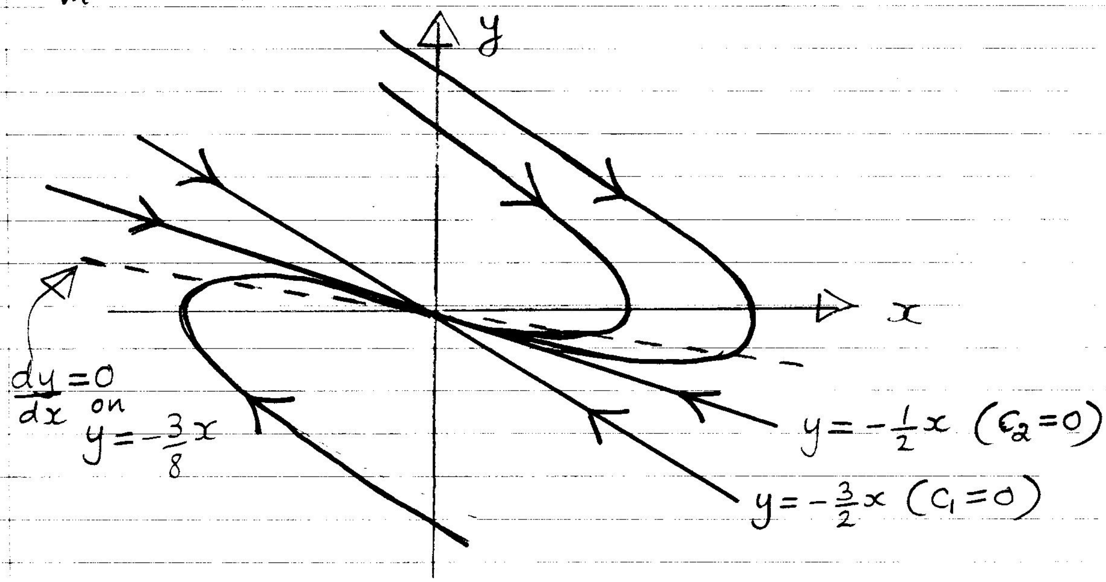
(iv)



(v) Sketch is same as above except that orbits are now centred around $(1, \frac{1}{2})$ instead of the origin.

Q2

(i) $\frac{\gamma}{m} = 2, \frac{k}{m} = \frac{3}{4}$ (heavy damping)



(ii) $\frac{\gamma}{m} = 1, \frac{k}{m} = \frac{3}{4}$ (light damping)