M4A32 Vortex dynamics: solutions (2006)

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1. (a) Since

$$u - iv = 2i\frac{\partial\psi}{\partial z} = Uy = U\left(\frac{z-\bar{z}}{2i}\right)$$

then, on integration, we get the following contribution to the streamfunction

$$\psi(z,\bar{z}) = \frac{Uz\bar{z}}{4} - \frac{Uz^2}{8} - \frac{U\bar{z}^2}{8}.$$

Adding in the contribution from the two point vortices at $\pm ic$ we get

$$\psi(z,\bar{z}) = \frac{Uz\bar{z}}{4} - \frac{Uz^2}{8} - \frac{U\bar{z}^2}{8} - \frac{\Gamma}{2\pi}\log|z-ic| - \frac{\Gamma}{2\pi}\log|z+ic|$$

This is the required streamfunction.

5 marks

(b) Now

$$u - iv = 2i\frac{\partial\psi}{\partial z} = 2i\left(\frac{U\bar{z}}{4} - \frac{Uz}{4} - \frac{\Gamma}{4\pi(z-ic)} - \frac{\Gamma}{4\pi(z+ic)}\right).$$

The non-self-induced velocity component at $z = ic$ is therefore

$$2i\left(-\frac{iUc}{4} - \frac{iUc}{4} - \frac{\Gamma}{8\pi ic}\right)$$

Setting this to zero gives

$$\Gamma = 4\pi Uc^2.$$

3 marks

(c) Expanding the velocity field in part (b) about $z = ic$ yields

$$\begin{aligned} u - iv &= 2i\left(\frac{U\bar{z}}{4} - \frac{Uz}{4} - \frac{\Gamma}{4\pi(z-ic)} - \frac{\Gamma}{4\pi(z+ic)}\right) \\ &= -\frac{i\Gamma}{2\pi(z-ic)} + \frac{iU}{2}(\bar{z}+ic-ic) - \frac{iU}{2}(z-ic+ic) - \frac{i\Gamma}{2\pi}\frac{1}{z-ic+2ic} \\ &= -\frac{i\Gamma}{2\pi(z-ic)} + Uc - \frac{\Gamma}{4\pi c} + \epsilon(z-ic) + \delta(\bar{z}+ic) + ... \end{aligned}$$

where

$$\epsilon = -\frac{i\Gamma}{8\pi c^2} - \frac{iU}{2}, \quad \delta = \frac{iU}{2}$$

4 marks

(d) Consider the velocity field around the elliptical vortex patch. It will have the form

$$u - iv = \begin{cases} -\frac{\omega_0}{4}\bar{z} - \frac{\omega_0}{4}C_i(z) & z \in D \\ \epsilon z + \delta\bar{z} - \frac{\omega_0}{4}C_o(z) & z \notin D \end{cases}$$

where $C_o(z)$ will decay for large z. By continuity of velocity, on ∂D we have

$$C_i(z) - C_o(z) = \hat{\epsilon} z + \hat{\delta}\bar{z}.$$

where

$$\hat{\epsilon} = -\frac{4\epsilon}{\omega_0}, \quad \hat{\delta} = -1 - \frac{4\delta}{\omega_0}.$$

The conformal mapping from the interior of a unit ζ-disc to the exterior of this elliptical patch has the form

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta$$

where α is a real parameter and β is complex. Now, on ∂D,

$$\bar{z} = \overline{z(\zeta)} = \alpha\zeta + \frac{\overline{\beta}}{\zeta}$$

But

$$\frac{1}{\zeta} = \frac{z}{\alpha} - \frac{\beta\zeta}{\alpha}$$

so that, on ∂D,

$$C_i(z) - C_o(z) = \hat{\delta}\left(\alpha - \frac{|\beta|^2}{\alpha}\right)\zeta + \left(\hat{\epsilon} + \frac{\bar{\beta}\hat{\delta}}{\alpha}\right) z$$

It follows that

$$C_i(z) = \left(\hat{\epsilon} + \frac{\bar{\beta}\hat{\delta}}{\alpha}\right) z$$

Hence, the velocity field inside the patch is

$$u - iv = -\frac{\omega_0}{4}\bar{z} + \left(\epsilon + \frac{\bar{\beta}}{\alpha}\left(\frac{\omega_0}{4} + \delta\right)\right) z.$$

8 marks

2. **(a)** Consider a (complex) ζ-plane through the equator of the sphere. Pick a point P on the spherical surface. Draw a straight line between P and the north pole N of the sphere. Extrapolate this line if necessary so that it hits the plane through the equator. This construction gives a one-to-one mapping (the stereographic projection) of points P on the spherical surface and points ζ in the plane. The north pole N corresponds to the point at infinity in the plane. The south pole S corresponds to $\zeta = 0$. If (θ, ϕ) are the usual angles in spherical polar coordinates, simple geometrical considerations lead to the relation

$$\zeta = \cot(\theta/2)\, e^{i\phi}$$

4 marks

(b) The south pole corresponds to $\zeta = 0$. The streamfunction associated with a point vortex, of unit circulation, at this point is

$$\psi(\zeta, \bar{\zeta}) = -\frac{1}{4\pi} \log\left(\frac{\zeta\bar{\zeta}}{(1+\zeta\bar{\zeta})}\right).$$

2 marks

(c) Note that the streamfunction of part (b) can be written

$$\psi = -\frac{1}{4\pi}\Big(\log(\zeta\bar{\zeta}) - \log(1+\zeta\bar{\zeta})\Big)$$

The first logarithmic term correponds to the point vortex, the second logarithmic term to the background of uniform vorticity covering the entire spherical surface. If only **half** the surface (the southern hemisphere) is covered in uniform vorticity then, to satisfy the Gauss constraint that the total vorticity on the surface is zero, then it will be necessary to **double** the magnitude of this second term. Thus, the global streamfunction will now be

$$\psi(\zeta, \bar{\zeta}) = \begin{cases} -\frac{1}{4\pi}\Big(\log(\zeta\bar{\zeta}) - 2\log(1+\zeta\bar{\zeta})\Big) & |\zeta| \le 1 \\ 0 & |\zeta| > 1 \end{cases}$$

since the projected region $|\zeta| \le 1$ corresponds to the southern hemisphere.

6 marks

(d) Since the equatorial circle is now a vortex jump then, for an equilibrium, we require that the velocity field is continuous there and that the equatorial circle is a streamline. To verify these conditions, note that

$$\frac{\partial\psi}{\partial\zeta} = -\frac{1}{4\pi}\left(\frac{1}{\zeta} - \frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}}\right) = -\frac{1}{4\pi}\left(\frac{1-\zeta\bar{\zeta}}{\zeta(1+\zeta\bar{\zeta})}\right)$$

but this quantity (which is proportional to the complex fluid velocity on the equatorial circle) is identically zero on the equator $|\zeta| = 1$. Since the flow in the northern hemisphere vanishes, the velocities are continuous.

4 marks

Note also that

$$d\psi = \frac{\partial \psi}{\partial \zeta} d\zeta + \frac{\partial \psi}{\partial \bar{\zeta}} d\bar{\zeta}$$

which is zero on $|\zeta| = 1$ since $\partial\psi/\partial\zeta = \partial\psi/\partial\bar{\zeta} = 0$ there.

4 marks

3.(a) The Green's function $G(\zeta;\alpha,\bar{\alpha})$ is the function which is harmonic everywhere in $|\zeta|<1$ except that, near $\zeta=\alpha$,

$$G(\zeta;\alpha,\bar{\alpha}) = -\frac{1}{2\pi}\log|\zeta-\alpha| + \textit{regular}$$

and is such that $G=0$ everywhere on the boundary $|\zeta|=1$.

3 marks

(b) First, it is clear that $G(\zeta;\alpha,\bar{\alpha})$ is the imaginary part of the analytic function of ζ given by

$$-\frac{i}{2\pi}\log R_0(\zeta;\alpha,\bar{\alpha})$$

where

$$R_0(\zeta;\alpha,\bar{\alpha}) \equiv \frac{(\zeta-\alpha)}{|\alpha|(\zeta-\bar{\alpha}^{-1})}$$

so it is harmonic everywhere in $|\zeta|<1$ except at $\zeta=\alpha$ where, clearly,

$$\begin{aligned} G(\zeta;\alpha,\bar{\alpha}) &= -\frac{1}{2\pi}\log|\zeta-\alpha| + \frac{1}{2\pi}\log|\alpha(\zeta-\bar{\alpha}^{-1})| \\ &= -\frac{1}{2\pi}\log|\zeta-\alpha| + \textit{regular}. \qquad [1] \end{aligned}$$

2 marks

Note also that, for ζ on the unit circle where $\bar{\zeta}=\zeta^{-1}$,

$$\overline{R_0(\zeta;\alpha,\bar{\alpha})} = \frac{\zeta^{-1}-\bar{\alpha}}{|\alpha|(\zeta^{-1}-\alpha^{-1})} = \frac{1}{R_0(\zeta;\alpha,\bar{\alpha})}$$

so that $G=0$ on the unit circle, as required.

3 marks

(c) From [1], it is clear that

$$g(\zeta;\alpha,\bar{\alpha}) = \frac{1}{2\pi}\log|\alpha(\zeta-\bar{\alpha}^{-1})|$$

so that

$$H(\alpha,\bar{\alpha}) = \frac{1}{4\pi}\log|\alpha\bar{\alpha}-1|.$$

4 marks

(d) The Joukowski slit mapping

$$\eta(\zeta) = \frac{1}{\zeta}+\zeta$$

takes the interior of the unit ζ-disc to the η-plane exterior to a slit between -2 and +2 on the real η-axis. Composing this with the Möbius mapping

$$z(\eta) = \frac{1}{\eta}$$

gives the required result since this map takes the slit $[-2,2]$ on the real η-axis to $(-\infty,-1/2]\cup[1/2,\infty)$ on the real z-axis. The final mapping is

$$z(\zeta)=\frac{\zeta}{1+\zeta^2}.$$

Note that $z(0)=0$.

3 marks

(e) Taking the derivative of the above map,

$$\frac{dz}{d\zeta}=\frac{1-\zeta^2}{(1+\zeta^2)^2}.$$

On use of this in the transformation formula for the Hamiltonians we get

$$H^{(z)}(z_\alpha,\bar{z}_\alpha)=H^{(\zeta)}(\alpha,\bar{\alpha})+\frac{1}{4\pi}\log|z_\zeta(\alpha)|=\frac{1}{4\pi}\log|\alpha\bar{\alpha}-1|+\frac{1}{4\pi}\log\left|\frac{1-\alpha^2}{(1+\alpha^2)^2}\right|$$

The vortex trajectories, in this case of a single vortex, are the contours of $H^{(z)}(z_\alpha,\bar{z}_\alpha)$. They are given by

$$(1-\alpha\bar{\alpha})\left|\frac{(1-\alpha^2)}{(1+\alpha^2)^2}\right|=c$$

where c is some constant. By the symmetry of the configuration, the critical trajectory must pass through $z_\alpha=0$ which corresponds to $\alpha=0$. Therefore, the critical trajectory corresponds to $c=1$. This trajectory is therefore given by

$$(1-\alpha(z)\overline{\alpha(z)})|1-\alpha(z)^2|=|1+(\alpha(z)^2)|^2$$

where

$$z=\frac{\alpha}{1+\alpha^2}$$

or

$$\alpha^2-\frac{\alpha}{z}+1=0.$$

Solving this quadratic equation for α as a function of z

$$\alpha(z)=\frac{1}{2z}\left(1-\sqrt{1-4z^2}\right)$$

where we have taken the $-$ve sign so that $\alpha(0)=0$.

5 marks

4. (a) The flow is steady, two-dimensional and incompressible which means there exists a streamfunction $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

But, in two dimensions, the vorticity field $\omega(x, y)$ is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

which implies

$$\omega = -\nabla^2 \psi$$

where ∇^2 denotes the planar Laplacian operator.

2 marks

(b) The steady vorticity equation in two dimensions reduces to

$$\mathbf{u}.\nabla \omega = 0$$

which can be rewritten as

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0.$$

Alternatively

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = 0 \qquad [*]$$

But, suppose that

$$\omega = h(\psi)$$

for some differentiable function h then

$$\frac{\partial \omega}{\partial x} = h'(\psi)\frac{\partial \psi}{\partial x}, \quad \frac{\partial \omega}{\partial y} = h'(\psi)\frac{\partial \psi}{\partial y}$$

and then [*] is satisfied identically.

2 marks

(c) Let

$$\psi = -\frac{1}{2}\log\left(\frac{f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2}\right)$$

which implies

$$-2\psi_z = \frac{f''(z)}{f'(z)} - \frac{2\overline{f(z)}f'(z)}{(1 + f(z)\overline{f(z)})}$$

which, in turn, implies

$$-2\psi_{z\bar{z}} = -\frac{2f'(z)\overline{f'(z)}}{(1 + f(z)\overline{f(z)})^2}$$

so that

$$\psi_{z\bar{z}} = \frac{f'(z)\overline{f'(z)}}{(1+f(z)\overline{f(z)})^2} = e^{-2\psi}$$

Therefore

$$\omega = -\nabla^2\psi = -4\psi_{z\bar{z}} = -4e^{-2\psi}$$

so ω is purely a function of ψ and therefore ψ is a possible steady solution from part (b).

6 marks

(d) If, near $z = a$,

$$f(z) = \frac{b}{z-a} + \text{analytic}$$

then

$$f'(z) = -\frac{b}{(z-a)^2} + \text{analytic}$$

so that

$$\frac{f'(z)\overline{f'(z)}}{(1+f(z)\overline{f(z)})^2} \sim \frac{\frac{|b|^2}{(z-a)^2(\bar{z}-\bar{a})^2}}{(1+\frac{|b|^2}{(z-a)(\bar{z}-\bar{a})})^2} \sim \frac{1}{|b|^2} + \text{regular}$$

as $z \to a$. Thus, ψ is not singular as $z \to a$ if $f(z)$ has a simple pole at a.

4 marks

(e) Let

$$f(z) = \frac{a}{z^N} + b$$

where a and b are complex constants. Then

$$f'(z) = -\frac{Na}{z^{N+1}}$$

and

$$\psi = -\frac{1}{2}\log\left[\frac{f'(z)\overline{f'(z)}}{(1+f(z)\overline{f(z)})^2}\right] = -\frac{1}{2}\log\left[\frac{N^2|a|^2 z^{N-1}\bar{z}^{N-1}}{(z^N\bar{z}^N + (a+bz^N)(\bar{a}+\bar{b}\bar{z}^N))^2}\right]$$
$$\sim -\frac{(N-1)}{2}\log(z\bar{z}) \quad \text{as } z \to 0$$

Comparing with

$$\psi = \text{Im}\left[-\frac{i\Gamma}{2\pi}\log z\right] = -\frac{\Gamma}{4\pi}\log(z\bar{z})$$

for a point vortex of circulation Γ at $z = 0$, it is clear that the circulation of the point vortex is

$$\Gamma = 2(N-1)\pi.$$

6 marks

5. (a) It is known that

$$\begin{aligned} u - iv &= 2i\frac{\partial \psi}{\partial z} \\ &= 2i\left(-\frac{\omega_0}{4}\left(\bar{z} - F'(z)\right)\right) \\ &= 2i\left(-\frac{\omega_0}{4}\left(\bar{z} - S(z)\right)\right) \\ &= -\frac{i\omega_0}{2}\left(\bar{z} - S(z)\right). \end{aligned}$$

Since the vorticity ω is given by

$$\omega = -4\frac{\partial^2 \psi}{\partial z \partial \bar{z}}$$

then, on use of the expression for ψ, this equals ω_0 except possibly at any singularities of $S(z)$.

5 marks

(b) At a vortex jump, the fluid velocities must be continuous and the jump must be a streamline. However, since $S(z) = \bar{z}$ on the vortex jump ∂D, then from part (a) it is clear that the velocities are continuous at ∂D since $u - iv = 0$ there. Moreover, since

$$d\psi = \frac{\partial \psi}{\partial z}dz + \frac{\partial \psi}{\partial \bar{z}}d\bar{z} = 0 \quad \text{on } \partial D$$

then ∂D is also a streamline.

5 marks

(c) Since

$$z(\zeta) = \zeta + \frac{b\zeta}{\zeta^2 - a^2}$$

and since $|\zeta| = 1$ corresponds to ∂D then, on ∂D,

$$\bar{z} = S(z) = \overline{z(\zeta)} = \bar{\zeta} + \frac{\overline{b\zeta}}{\bar{\zeta}^2 - a^2} = \frac{1}{\zeta} + \frac{b\zeta}{1 - \zeta^2 a^2}$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ on ∂D and the fact that a and b are real.

3 marks

(d) It is clear, from inspection, that $S(z)$ has simple poles at $\zeta = 0, \pm 1/a$. Since $|a| > 1$ then $\pm 1/a$ are points inside the unit ζ-disc and, hence, inside the vortex patch D. These are at $z(0)$ and $z(\pm 1/a)$ or explicitly,

$$0, \quad \frac{1}{a} + \frac{ba}{1 - a^4}, \quad -\frac{1}{a} - \frac{ba}{1 - a^4}$$

4 marks

(e) A simple pole of $S(z)$ means that $u - iv$ has a simple pole. It also has pure imaginary residue and therefore corresponds, physically, to a point vortex singularity. There are therefore point vortex singularities at the three points found in (d). In equilibrium, the point vortices must be steady. Thus, the parameters a and b must be such that the non-self-induced velocity at each of the three point vortices must be **zero**. These are the extra conditions on a and b.

3 marks