M2AM: Fluids and Dynamics

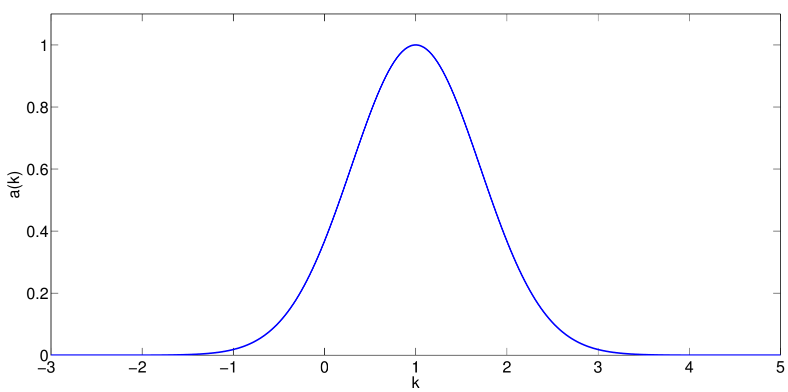
Problem Sheet 6 - SOLUTIONS

1. A graph of $a(k)$ against k is a simple Gaussian curve centred at k_0, as shown below for the case $k_0 = 1$. Note that we can write

$$\begin{aligned} a(k)e^{\mathrm{i}kx} = e^{-\sigma(k-k_0)^2+\mathrm{i}kx} &= e^{-\sigma k^2+2kk_0\sigma-\sigma k_0^2+\mathrm{i}kx} \\ &= e^{-\sigma(k^2-2kk_0-\mathrm{i}kx/\sigma+k_0^2)} \\ &= e^{-\sigma(k-(k_0+\mathrm{i}x/2\sigma))^2-\sigma k_0^2+\sigma(k_0+\mathrm{i}x/2\sigma)^2} \end{aligned} \tag{1}$$

where we have "completed the square". Hence

$$\begin{aligned} \eta(x,0) &= e^{-\sigma k_0^2+\sigma(k_0+\mathrm{i}x/2\sigma)^2}\int_{-\infty}^{\infty} e^{-\sigma(k-(k_0+\mathrm{i}x/2\sigma))^2}dk \\ &= e^{\mathrm{i}k_0x-x^2/4\sigma}\int_{-\infty}^{\infty} e^{-\sigma(k-(k_0+\mathrm{i}x/2\sigma))^2}dk. \end{aligned} \tag{2}$$



To compute the integral

$$\int_{-\infty}^{\infty} e^{-\sigma(k-(k_0+\mathrm{i}x/2\sigma))^2}dk, \tag{3}$$

first introduce the change of variable $m = k - (k_0 + \mathrm{i}x/2\sigma)$ so that it becomes the line integral in the complex m-plane

$$\int_{-\infty+\mathrm{i}x/2\sigma}^{\infty+\mathrm{i}x/2\sigma} e^{-\sigma m^2}dm. \tag{4}$$

To compute this, consider the closed contour integral

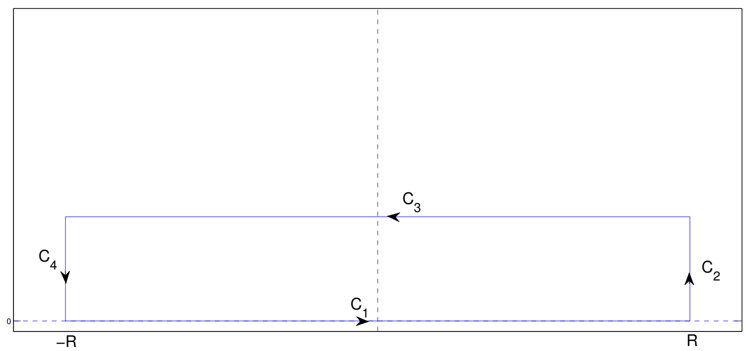
$$\oint_{C_R} e^{-\sigma m^2}dm = 0 \tag{5}$$

where C_R is the boundary of the rectangular region in the complex m-plane shown in the figure below and where the integral is zero by Cauchy's theorem (the integrand is analytic everywhere inside the curve C_R). Split C_R up into $C_1 \cup C_2 \cup C_3 \cup C_4$ as shown in the figure where C_1 runs along the real m-axis and C_3 is parallel to it along the line $\mathrm{Im}[m] = x/2\sigma$.

In the limit $R \to \infty$, the integral along C_3 is the integral we need; simple estimates can be used to show that, as $R \to \infty$, the contributions from C_2 and C_4 tend to zero and the contribution from C_1 is the integral given in the hint. It follows that the integrals along C_1 and C_3 are equal and, hence,

$$\eta(x, 0) = \sqrt{\frac{\pi}{\sigma}} e^{-x^2/4\sigma} e^{\mathrm{i}k_0 x}, \tag{6}$$

as required.



If $\eta(x, 0)$ has a large number of crests in a wave packet, then σ must be large in order that the exponential factor $e^{-x^2/4\sigma}$ is not exponentially small over the large x-interval. On the other hand, this means that $a(k)$ is exponentially small except for values of k very close to k_0.

2. Following the single-fluid example in lectures, the linearized kinematic boundary conditions are that, on $y = 0$,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y}, \qquad \frac{\partial \eta}{\partial t} = \frac{\partial \phi_2}{\partial y}. \tag{7}$$

Similarly, with no surface tension, the boundary condition that the pressures are continuous, i.e. $p_1 = p_2$ reduces, in the linearized form, to the following condition evaluated at $y = 0$:

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \eta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g \eta. \tag{8}$$

Now assume $k > 0$ and let

$$\eta = A\cos(kx - \omega t), \qquad \phi_1 = Be^{ky}\sin(kx - \omega t), \qquad \phi_2 = Ce^{-ky}\sin(kx - \omega t) \tag{9}$$

where A, B and C are constants. On substitution of these forms into (7) and (8) we find

$$A\omega = Bk, \qquad A\omega = -Ck, \qquad -\rho_1\omega B + \rho_1 g A = -\rho_2 \omega C + \rho_2 g A \tag{10}$$

On eliminating B and C from these three equations we arrive at the dispersion relation

$$\omega^2 \left[\frac{\rho_2}{k} + \frac{\rho_1}{k}\right] = (\rho_1 - \rho_2)g, \tag{11}$$

or, on use of the fact that $c \equiv \omega/k$,

$$c^2 = \frac{(\rho_1 - \rho_2)g}{k(\rho_1 + \rho_2)}. \tag{12}$$

Performing a similar analysis for $k < 0$ produces the required result

$$c^2 = \frac{g}{|k|}\frac{(\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}. \tag{13}$$

3. We know

$$\frac{dz}{ds} = e^{\mathrm{i}\theta}. \tag{14}$$

Hence

$$\frac{d^2 z}{ds^2} = \mathrm{i}\frac{d\theta}{ds}e^{\mathrm{i}\theta} = \mathrm{i}\frac{d\theta}{ds}\frac{dz}{ds} \tag{15}$$

so that

$$\kappa \equiv \frac{d\theta}{ds} = -\frac{\mathrm{i}d^2 z/ds^2}{dz/ds}, \tag{16}$$

as required. Now let $y = \eta(x)$ so that

$$z = x + \mathrm{i}y = x + \mathrm{i}\eta(x), \tag{17}$$

then

$$ds^2 = dx^2 + \eta'(x)^2 dx^2, \qquad \frac{ds}{dx} = -(1 + \eta'(x)^2)^{1/2} \tag{18}$$

where the minus sign is chosen so that the fluid is on the left as s increases. By the chain rule,

$$\frac{dz}{ds} = \frac{dz}{dx}\Big/\frac{ds}{dx} = -\frac{(1 + \mathrm{i}\eta'(x))}{(1 + \eta'(x)^2)^{1/2}} \approx -(1 + \mathrm{i}\eta'(x)), \tag{19}$$

where we have linearized for small $\eta'(x)$. Similarly, the chain rule implies

$$\frac{d^2 z}{ds^2} = \frac{d(dz/ds)/dx}{ds/dx} \approx \mathrm{i}\eta''(x) \tag{20}$$

in a linearized approximation. It follows from (16) that the linearized curvature is

$$\kappa = -\eta''(x). \tag{21}$$

4(a). Clearly

$$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} + \frac{Tk}{\rho} \tag{22}$$

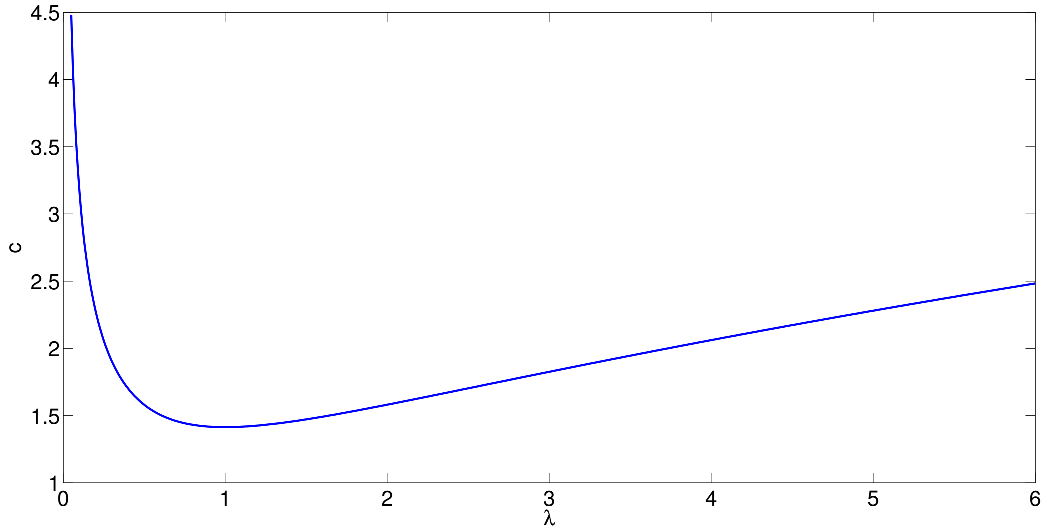
so that

$$c = \left[\frac{g}{k} + \frac{Tk}{\rho}\right]^{1/2} \tag{23}$$

Now $\lambda = 2\pi/k$ so

$$c = \left[\frac{g\lambda}{2\pi} + \frac{2T\pi}{\rho\lambda}\right]^{1/2}. \tag{24}$$

A sketch of a typical graph of c against λ is shown in the figure.



(b) The quantity c^2 will have a local minimum when c has a local minimum and this occurs when

$$\frac{dc^2}{dk} = -\frac{g}{k^2} + \frac{T}{\rho} = 0, \tag{25}$$

that is, when

$$k^2 = k_{min}^2 = \frac{g\rho}{T}. \tag{26}$$

The corresponding minimum value of c is

$$c_{min} = c(k_{min}) = \left[\frac{4gT}{\rho}\right]^{1/4}. \tag{27}$$

(c) Now the group velocity c_g is obtained by direct differentiation:

$$c_g = \frac{d\omega}{dk} = \frac{g + 3Tk^2/\rho}{2(gk + Tk^3/\rho)^{1/2}} \tag{28}$$

(d) From part (c), we can write

$$c_g = \frac{g + 3Tk^2/\rho}{2\sqrt{k}(g + Tk^2/\rho)^{1/2}}, \tag{29}$$

which is positive and, like c, clearly becomes infinite as $k \to 0$ and as $k \to \infty$ so it also has a single local minimum, c_g^{min} say. Assuming that qualitatively similar properties hold for 2D capillary-gravity waves on a pond, since the energy of a wave packet travels with velocity c_g, after a time T, all wave packets generated by the dropped stone will have travelled at least a distance $c_g^{min}T$ from the point at which the stone was dropped and this explains why no remaining ripples would be observed within this distance.

5 (a). Now, since the walls are impenetrable, the x-velocity of the fluid must vanish at $x = 0, a$ so that

$$\frac{\partial \phi_j}{\partial x} = 0, \quad x = 0, a, \quad j = 1, 2, \tag{30}$$

where $\phi_j,\ j = 1, 2$ are the harmonic velocity potentials in each fluid.

(b) Following Q2, the linearized boundary conditions on $y = 0$ are

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\partial \phi_j}{\partial y}, \quad j = 1, 2, \\ \rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \eta - \rho_2 \frac{\partial \phi_2}{\partial t} - \rho_2 g \eta &= -T \eta''(x), \end{aligned} \tag{31}$$

where we have now included surface tension T. Possible solutions, satisfying (30), are

$$\phi_1 = b_n e^{n\pi y/a} \cos(n\pi x/a) \sin \omega_n t, \qquad \phi_2 = c_n e^{-n\pi y/a} \cos(n\pi x/a) \sin \omega_n t \tag{32}$$

where b_n, c_n and ω_n are constants and $n \geq 1$ is any positive integer. The boundary conditions (31) then force the functional form

$$\eta = a_n \cos(n\pi x/a) \cos \omega_n t \tag{33}$$

for some constant a_n. Substitution of all these forms into (31) produces

$$-\omega_n a_n = (n\pi/a) b_n \qquad \omega_n a_n = (n\pi/a) c_n, \qquad \rho_1 \omega_n b_n - \rho_1 g a_n - \rho_2 \omega_n c_n - \rho_2 g a_n = T(n^2\pi^2/a^2) a_n \tag{34}$$

Elimination of b_n and c_n produces the dispersion relation

$$\omega_n^2 = \frac{n\pi}{a(\rho_1 + \rho_2)} \left[(\rho_1 - \rho_2) g + \frac{T n^2 \pi^2}{a^2} \right]. \tag{35}$$

(c) If $\rho_2 > \rho_1$ then ω_n can be pure imaginary – and hence the system unstable – if

$$\frac{T n^2 \pi^2}{a^2} < (\rho_2 - \rho_1) g, \qquad \text{or } T < \frac{(\rho_2 - \rho_1) a^2 g}{n^2 \pi^2}. \tag{36}$$

Thus the $n = 1$ mode is unstable if

$$T < \frac{a^2 (\rho_2 - \rho_1) g}{\pi^2}. \tag{37}$$

Higher order modes become successively unstable as T gets smaller.