M2AM: Fluids and Dynamics

Problem Sheet 6

1. Suppose the initial disturbance of a free surface at $t = 0$ can be written in the form of a Fourier integral:

$$\eta(x,0) = \int_{-\infty}^{\infty} a(k) e^{\mathrm{i}kx} dk,$$

where the real part is understood. Suppose now that the amplitude function $a(k)$ has the form

$$a(k) = e^{-\sigma(k-k_0)^2},$$

where $\sigma > 0$ is a real parameter. Sketch $a(k)$ as a function of wave number k. Show that

$$\eta(x,0) = \sqrt{\frac{\pi}{\sigma}} e^{-x^2/4\sigma} e^{ik_0 x}.$$

Use this result to show that if the wave packet described by $\eta(x,0)$ has a large number of crests then $a(k)$ is small except for k values very close to k_0.

[Hints: you may need to use ideas from M2P2, and the fact that

$$\int_{-\infty}^{\infty} e^{-\sigma s^2} ds = \sqrt{\frac{\pi}{\sigma}} \Bigg].$$

2. Consider deep water gravity waves but assume now that there are *two* fluids separated by the interface at $y = \eta(x,t)$ and suppose that the upper fluid extends to $y \to \infty$. Let the fluid in the lower layer have density ρ_1 and let the density of the upper fluid be $\rho_2 < \rho_1$. Show that the phase speed c of waves on the interface with wave number k will be given by

$$c^2 = \frac{g}{|k|} \left[\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right].$$

3. Let $z = x + \mathrm{i}y$ and let s denote arclength around a boundary so that

$$ds^2 = dx^2 + dy^2.$$

(a) As a complex number, the local tangent vector is proportional to

$$dz = dx + \mathrm{i}dy$$

so that the unit tangent is represented by the complex quantity dz/ds where arclength increases as a curve is traversed with the fluid region always on the left. Suppose the local tangent to a curve makes angle θ to the positive x-axis then

$$\frac{dz}{ds} = e^{\mathrm{i}\theta}.$$

Show that the curvature, defined by

$$\kappa = \frac{d\theta}{ds}$$

is given by the formula

$$\kappa = -\frac{\mathrm{i}d^2z/ds^2}{dz/ds}.$$

(b) Now consider the free surface $y = \eta(x)$ of a linearized water wave problem with $|\eta(x)|, |\eta'(x)| \ll 1$ so that the free surface is close to the flat state $y = 0$ and its slope is small compared to a typical wavelength in the x-direction. By setting

$$z = x + \mathrm{i}\eta(x),$$

show that the linearized approximation to the curvature is given by

$$\kappa = -\frac{d^2\eta}{dx^2}.$$

4. The dispersion relation for capillary-gravity waves is

$$\omega^2 = gk + \frac{Tk^3}{\rho},$$

where T be the surface tension on the free surface.

(a) Find the phase speed c and sketch a graph of c against wavelength $\lambda = 2\pi/k$.

(b) Show that there is a minimum phase speed c_{min} at

$$c_{min} = \left[\frac{4gT}{\rho}\right]^{1/4}.$$

(c) Show that the group velocity is

$$c_g = \frac{g + 3Tk^2/\rho}{2\sqrt{gk + Tk^3/\rho}}.$$

(d) A stone is dropped into a pond and generates waves that travel radially outwards. After a while, waves are only seen beyond a central circular region of undisturbed water whose radius increases with time. Assuming the waves generated are capillary-gravity waves, give an explanation for this phenomenon based on the results of this question?

5. Suppose now that the two fluids in Q.2 do *not* extend to $x \to \pm\infty$, but are confined to a container with straight impenetrable walls at $x = 0, a$ where $a > 0$. Suppose too that there is now a surface tension T on the free surface.

(a) What are the boundary conditions to be imposed on $x = 0$ and $x = a$?

(b) Show that admissible solutions for the free surface $\eta(x,t)$ are given by

$$\eta(x,t) = a_n \cos(n\pi x/a)\cos(\omega_n t), \qquad n = 1, 2, \ldots$$

where a_n is a constant and

$$\omega_n^2 = \frac{n\pi}{a(\rho_1+\rho_2)}\left\{(\rho_1-\rho_2)g + \frac{Tn^2\pi^2}{a^2}\right\}.$$

(c) Now suppose that the upper fluid is heavier than the lower fluid so that $\rho_2 > \rho_1$. Show that the system is unstable (so that small wave-like disturbances will generally grow in amplitude) if

$$T < \frac{a^2(\rho_2-\rho_1)g}{\pi^2}.$$

This instability of a heavier fluid over a lighter one is known as a *Rayleigh-Taylor instability*.