## M2AM: Fluids and Dynamics

## Problem Sheet 5 - SOLUTIONS

1(a) The primitive equations for $h$ and $u$ are

$$
\begin{align*}
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+g \frac{\partial h}{\partial x}=0  \tag{1}\\
& \frac{\partial h}{\partial t}+u \frac{\partial h}{\partial x}+h \frac{\partial u}{\partial x}=0 \tag{2}
\end{align*}
$$

Now, by definition of $c$,

$$
\begin{equation*}
\frac{\partial c}{\partial x}=\frac{\sqrt{g}}{2 h} \frac{\partial h}{\partial x}=\frac{g}{2 c} \frac{\partial h}{\partial x}, \quad \text { hence } \quad g \frac{\partial h}{\partial x}=2 c \frac{\partial c}{\partial x} \quad \text { and } \quad g \frac{\partial h}{\partial t}=2 c \frac{\partial c}{\partial t} \tag{3}
\end{equation*}
$$

Equation (1) can be written as

$$
\begin{equation*}
\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+2 c \frac{\partial c}{\partial x}=0 \tag{4}
\end{equation*}
$$

Multiplication of (2) by $g$ gives

$$
\begin{equation*}
g \frac{\partial h}{\partial t}+u g \frac{\partial h}{\partial x}+g h \frac{\partial u}{\partial x}=0 \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
2 c \frac{\partial c}{\partial t}+2 c u \frac{\partial c}{\partial x}+c^{2} \frac{\partial u}{\partial x}=0 \tag{6}
\end{equation*}
$$

On cancellation of $c$, we find

$$
\begin{equation*}
2 \frac{\partial c}{\partial t}+2 u \frac{\partial c}{\partial x}+c \frac{\partial u}{\partial x}=0 \tag{7}
\end{equation*}
$$

Finally, addition and subtraction of (4) and (7) give, respectively,

$$
\begin{equation*}
\left[\frac{\partial}{\partial t}+(u+c) \frac{\partial}{\partial x}\right](u+2 c)=0, \quad\left[\frac{\partial}{\partial t}+(u-c) \frac{\partial}{\partial x}\right](u-2 c)=0 \tag{8}
\end{equation*}
$$

(b) Let perturbations to the base state $u=0, h=h_{0}$ be represented by

$$
\begin{equation*}
u=\hat{u}, \quad c=\sqrt{g h_{0}}+\hat{c} \tag{9}
\end{equation*}
$$

where hatted variables are assumed small. Defining $c_{0} \equiv \sqrt{g h_{0}}$, the linearized equations from part (a) are

$$
\begin{align*}
& {\left[\frac{\partial}{\partial t}+c_{0} \frac{\partial}{\partial x}\right](\hat{u}+2 \hat{c})=0}  \tag{10}\\
& {\left[\frac{\partial}{\partial t}-c_{0} \frac{\partial}{\partial x}\right](\hat{u}+2 \hat{c})=0} \tag{11}
\end{align*}
$$

Operating on equation (11) with $\partial / \partial t+c_{0} \partial / \partial x$ yields

$$
\begin{equation*}
\left[\frac{\partial^{2}}{\partial t^{2}}-c_{0}^{2} \frac{\partial^{2}}{\partial x^{2}}\right](\hat{u}+2 \hat{c})=0 \tag{12}
\end{equation*}
$$

and operating on equation (10) with $\partial / \partial t-c_{0} \partial / \partial x$ produces

$$
\begin{equation*}
\left[\frac{\partial^{2}}{\partial t^{2}}-c_{0}^{2} \frac{\partial^{2}}{\partial x^{2}}\right](\hat{u}-2 \hat{c})=0 . \tag{13}
\end{equation*}
$$

Addition and subtraction of (12) and (13) gives

$$
\begin{equation*}
\frac{\partial^{2} \hat{u}}{\partial t^{2}}-c_{0}^{2} \frac{\partial^{2} \hat{u}}{\partial x^{2}}=0, \quad \frac{\partial^{2} \hat{c}}{\partial t^{2}}-c_{0}^{2} \frac{\partial^{2} \hat{c}}{\partial x^{2}}=0 \tag{14}
\end{equation*}
$$

which we recognize as linearized wave equations for small disturbances with wavespeed $c_{0}$.

2(a). This is a "free boundary problem" because the nature of the right-most position of the water level needs to be determined as part of the solution; note that, owing to the presence of the dam, the region $x>0, t=0$ is not part of the solution domain. It is natural to define + characteristics to be given by

$$
\begin{equation*}
\frac{d x}{d t}=u+c \tag{15}
\end{equation*}
$$

and, on these, $u+2 c$ is constant. The - characteristics, on which $u-2 c$ is constant, are given by

$$
\begin{equation*}
\frac{d x}{d t}=u-c . \tag{16}
\end{equation*}
$$

Data is given for $t=0, x<0$ and both + characteristics and - characteristics pass through this half-line. Where both sets of characteristics cross, it must be true that

$$
\begin{equation*}
u=0, \quad c=c_{0} \tag{17}
\end{equation*}
$$

and this region is bounded on the right by the - characteristic through $(x, t)=(0,0)$ given by

$$
\begin{equation*}
x=-c_{0} t . \tag{18}
\end{equation*}
$$

To the right of this line, we still expect the + characteristics through $t=0, x<0$ to continue into the solution domain so we surmise that the condition

$$
\begin{equation*}
u+2 c=2 c_{0} \tag{19}
\end{equation*}
$$

holds everywhere in the solution domain. The remaining - characteristics are more interesting: condition (19) can be used to show that, on the - characteristics, both $u$ and $c$ are constant and, owing to this fact, all - characteristics are straight lines. The only possibility that produces a single-valued solution for $u$ and $c$ is to have all - characteristics to be straight lines, of differing gradients, through $(x, t)=(0,0)$ (this is an "expansion fan" as discussed in lectures).

(b) The equation of straight lines of differing gradient through the origin is

$$
\begin{equation*}
\frac{d x}{d t}=\frac{x}{t} . \tag{20}
\end{equation*}
$$

Hence, since these are the - characteristics, we conclude that

$$
\begin{equation*}
\frac{d x}{d t}=\frac{x}{t}=u-c \tag{21}
\end{equation*}
$$

On use of (19), which says that $u=-2 c+2 c_{0}$, we find

$$
\begin{equation*}
\frac{x}{t}=-3 c+2 c_{0} . \tag{22}
\end{equation*}
$$

Hence, in this "expansion fan" region, we have

$$
\begin{equation*}
3 c=2 c_{0}-\frac{x}{t}, \quad \text { or } h=\frac{1}{9 g}\left[2 c_{0}-\frac{x}{t}\right]^{2} . \tag{23}
\end{equation*}
$$

We must not allow $c<0$ so $x / t=2 c_{0}$ is the right-most boundary of this expansion fan, which gives the position of the rightmost "free boundary" beyond which there is no water. Hence the line $x=2 c_{0} t$ gives the position of the free boundary and the final solution is

$$
h(x, t)= \begin{cases}h_{0}, & x<-c_{0} t \\ \frac{1}{9 g}\left[2\left[g h_{0}\right]^{1 / 2}-\frac{x}{t}\right]^{2}, & -c_{0} t<x<2 c_{0} t \\ 0, & x>2 c_{0} t\end{cases}
$$

The final plot of characteristics is shown in the figure. Note that, in the expansion fan region, only the - characteristics are known to be straight lines.
3. As in the moving piston problem in lectures, the boundary of the solution domain is now determined by the position of the moving dam. In the region where both + and characteristics (through $x<0, t=0$ ) cross we have

$$
\begin{equation*}
u=0, \quad c=c_{0} . \tag{24}
\end{equation*}
$$

This region is bounded on the right by the line $x=-c_{0} t$. We will assume that the + characteristics cover the solution domain to the right of this line so that

$$
\begin{equation*}
u+2 c=2 c_{0}, \tag{25}
\end{equation*}
$$

as in Q2. Then, condition (25) can similarly be used to argue that $c$ is constant on these characteristics and they are straight lines through the origin $(x, t)=(0,0)$ and given by

$$
\begin{equation*}
\frac{d x}{d t}=\frac{x}{t} . \tag{26}
\end{equation*}
$$

On the particular - characteristic through the moving dam we must have $u=V$ and this line is given by

$$
\begin{equation*}
\frac{d x}{d t}=V \tag{27}
\end{equation*}
$$

But the - characteristics are defined by

$$
\begin{equation*}
\frac{d x}{d t}=u-c=-3 c+2 c_{0}=\frac{3 u}{2}-c_{0} \tag{28}
\end{equation*}
$$

where we have used (25). In the expansion fan region between the lines $x=-c_{0} t$ and $x=\left(3 V / 2-c_{0}\right) t$ we therefore have

$$
\begin{equation*}
\frac{d x}{d t}=\frac{x}{t}=-3 c+2 c_{0} \tag{29}
\end{equation*}
$$

implying that

$$
\begin{equation*}
c=\frac{1}{3}\left[2 c_{0}-\frac{x}{t}\right] . \tag{30}
\end{equation*}
$$

But since $V<2 c_{0}$ then there is an "expansion-fan" region between the lines $x=(3 V / 2-$ $\left.c_{0}\right) t$ and $x=V t$ where $c$ must have the same value it has on the moving dam, i.e., the value $c=c_{0}-(u / 2)=c_{0}-V / 2$. In summary, the final solution for $c(x, t)$ is

$$
c(x, t)= \begin{cases}c_{0}, & x<-c_{0} t, \\ \frac{1}{3}\left[2 c_{0}-\frac{x}{t}\right], & -c_{0} t<x<\left(3 V / 2-c_{0}\right) t \\ c_{0}-V / 2, & \left(3 V / 2-c_{0}\right) t<x<V t\end{cases}
$$

