M2AM: Fluids and Dynamics

Problem Sheet 5

1. The shallow water equations for constant density fluid with speed $u(x,t)$ and free surface height $h(x,t)$ are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0,$$
$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0.$$

(a) Introduce the change of variable defined by

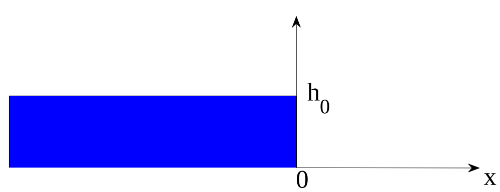
$$c(x,t) \equiv [gh(x,t)]^{1/2}$$

and show that these governing equations can be rewritten as

$$\left[\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right](u+2c) = 0,$$
$$\left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right](u-2c) = 0.$$

(b) Show that the speed of small linearized disturbances to a shallow water layer of stationary fluid at height h_0 is $c_0 = \sqrt{gh_0}$.

2. [The dam-break problem] Suppose that uniform fluid of depth h_0 is contained at rest by a dam at $x = 0$ as shown in the figure. At time $t = 0$, the dam instantaneously breaks.



(a) Use the shallow water equations for u and c in question 1 and find the characteristics for the problem. Sketch them in an (x,t)-plane.

(b) By use of the method of characteristics, show that the explicit solution for $h(x,t)$ is given by

$$h(x,t) = \begin{cases} h_0, & x < -c_0 t, \\ \dfrac{1}{9g}\left[2[gh_0]^{1/2} - \dfrac{x}{t}\right]^2, & -c_0 t < x < 2c_0 t, \\ 0, & x > 2c_0 t. \end{cases}$$

3. Carrying on from question 2, suppose now that, instead of simply removing the dam at $t = 0$, the dam is now moved steadily into the region with $x > 0$ at some constant speed $V < 2c_0$. The initial conditions are taken to be the same as in Q2. By assuming, as in the moving piston problem in lectures, that certain characteristics cover the solution domain, show that the solution for $c(x,t)$ in this case is given by

$$c(x,t) = \begin{cases} c_0, & x < -c_0 t, \\ \dfrac{1}{3}\left[2c_0 - \dfrac{x}{t}\right], & -c_0 t < x < (3V/2 - c_0)t, \\ c_0 - V/2, & (3V/2 - c_0)t < x < Vt. \end{cases}$$