M2AM: Fluids and Dynamics

Problem Sheet 4 - SOLUTIONS

1. The ODE for $s(t)$ comes from the shock condition as derived in lectures. The ODE can be written

$$\frac{ds}{dt} - \frac{u_0 s}{2(1+u_0 t)} = k + \frac{u_0(1-kt)}{2(1+u_0 t)}.$$

This is a first-order linear ODE and the integrating factor is easily found to be $(1+u_0 t)^{-1/2}$. Then,

$$\begin{aligned}\frac{d}{dt}\left(\frac{s(t)}{(1+u_0 t)^{1/2}}\right) &= \frac{k}{(1+u_0 t)^{1/2}} + \frac{u_0 - k(u_0 t + 1) + k)}{2(1+u_0 t)^{3/2}}, \\ &= \frac{k}{2(1+u_0 t)^{1/2}} + \frac{(u_0 + k)}{2(1+u_0 t)^{3/2}}.\end{aligned}$$

Integration yields

$$\frac{s(t)}{(1+u_0 t)^{1/2}} = \frac{k}{u_0}(1+u_0 t)^{1/2} - \frac{1}{u_0}\frac{(u_0+k)}{(1+u_0 t)^{1/2}} + C$$

where C is a constant. Hence

$$s(t) = kt - 1 + C(1+u_0 t)^{1/2}.$$

When we use the initial condition that $s = 1 + k/u_0$ when $t = 1/u_0$ we get $C = \sqrt{2}$.

2. The characteristics for the equation are

$$\frac{dx}{dt} = \rho$$

and, making use of the fact that ρ is constant on these, we get the set of straight lines

$$x = \left\{\begin{array}{ll} \zeta & \zeta < 0, \\ \zeta t + \zeta & 0 \le \zeta \le 1/2, \\ (1-\zeta)t + \zeta & 1/2 \le \zeta \le 1, \\ \zeta & \zeta > 1. \end{array}\right\}$$

One method of finding the shock formation time is to compute $(\partial x/\partial \zeta)|_t$ and examine where it vanishes. We find

$$(\partial x/\partial \zeta)|_t = \left\{\begin{array}{ll} 1 & \zeta < 0, \\ t+1 & 0 \le \zeta \le 1/2, \\ -t+1 & 1/2 \le \zeta \le 1, \\ 1 & \zeta > 1. \end{array}\right\}$$

Hence, this equals zero when $t = t_s = 1$. This is the shock formation time.

An implicit form of the solution is easily found to be (e.g. using the method of characteristics)

$$\rho = f(x - \rho t)$$

where f is an arbitrary function. This function is determined by initial conditions. At $t = 0$,

$$\rho(x,0) = f(x) = \left\{ \begin{array}{ll} 0 & x < 0, \\ x & 0 \le x \le 1/2, \\ (1-x) & 1/2 \le x \le 1, \\ 0 & x > 1. \end{array} \right\}$$

Hence, on rearrangement (as in Q1), we get the explicit solution

$$\rho(x,t) = \left\{ \begin{array}{ll} 0 & x < 0, \\ x/(1+t) & 0 \le x \le (t+1)/2, \\ (1-x)/(1-t) & (t+1)/2 \le x \le 1, \\ 0 & x > 1. \end{array} \right\}$$

Now, rewriting the equation in conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

with $Q = \rho^2/2$, the shock condition gives

$$\frac{ds}{dt} = \frac{[Q]_-^+}{[\rho]_-^+}$$

where $s(t)$ denotes the shock position. Using the explicit solution just found, this gives

$$\frac{ds}{dt} = \frac{s}{2(1+t)}$$

which is a separable ODE easily solved to give

$$s(t) = \sqrt{\frac{1+t}{2}}$$

where we have used the initial conditions $s(1) = 1$.

3. (a) It is clear that $\rho = \rho_0 =$ constant is a solution of the governing equation. Since

$$\frac{\partial}{\partial t}\left(\frac{x}{t}\right) = -\frac{x}{t^2}, \quad \frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}$$

it is easily verified that $\rho = (x/t)$ is also a solution. This is just a triangular shaped wave with a jump discontinuity at the point $x = s(t)$ (which, for now, is arbitrary). See Figure below.

(b) The shock condition is that the shock speed ds/dt is given by

$$\frac{ds}{dt} = \frac{[Q]_-^+}{[\rho]_-^+}$$

where $Q(\rho) = \rho^2/2$. For the solution in part (a), this gives

$$\frac{ds}{dt} = \frac{\rho_0^2/2 - (s/t)^2/2}{\rho_0 - (s/t)} = \frac{1}{2}\left(\rho_0 + \frac{s}{t}\right),$$

which is the required equation.

(c) This ODE can be rewritten as

$$\frac{ds}{dt} - \frac{s}{2t} = \frac{\rho_0}{2}.$$

It is a first-order linear ODE and the integrating factor is $1/t^{1/2}$ so

$$\frac{d}{dt}\left(\frac{s}{t^{1/2}}\right) = \frac{\rho_0}{2t^{1/2}},$$

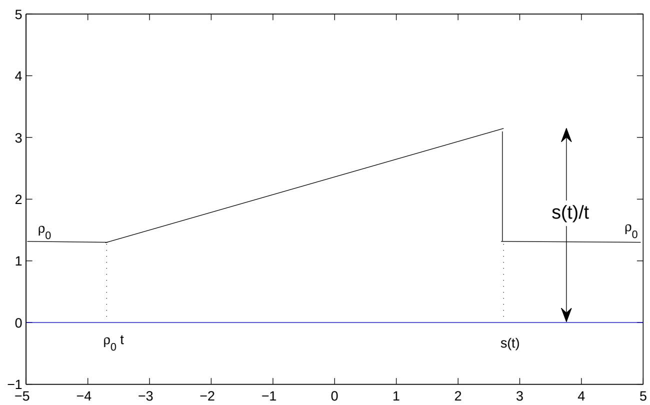
hence integration produces

$$\left(\frac{s}{t^{1/2}}\right) = \rho_0 t^{1/2} + C.$$

Alternatively,

$$s = \rho_0 t + Ct^{1/2}.$$

The value of C must come by enforcing that the total area under the graph is the same as for the initial condition. The total area of the triangle (above the level ρ_0) is clearly $(1/2)[Ct^{1/2}][Ct^{-1/2}] = C^2/2$ and, since this must equal A (the total area of the initial condition above the level ρ_0), we must have $C = \sqrt{2A}$, hence the result.



4. Consider the perturbation

$$u = \tilde{u}, \quad \rho = \rho_0 + \tilde{\rho}$$

where $|\tilde{u}| \ll 1$ and $|\tilde{\rho}| \ll \rho_0$. Then, substitution into the conservation of mass equation gives, on linearization,

$$\frac{\partial\tilde{\rho}}{\partial t} + \rho_0\frac{\partial\tilde{u}}{\partial x} = 0.$$

Differentiation with respect to t gives

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0\frac{\partial^2\tilde{u}}{\partial t\partial x} = 0.$$

We need to find an expression for $\frac{\partial^2\tilde{u}}{\partial t\partial x}$. This comes from the Euler equation. Substitution of the forms for u and ρ into the Euler equation gives

$$(\rho_0 + \tilde{\rho})\left(\frac{\partial\tilde{u}}{\partial t} + \tilde{u}\frac{\partial\tilde{u}}{\partial x}\right) = -k^2\rho_0^{\gamma-1}\left(1 + \tilde{\rho}/\rho_0\right)^{\gamma-1}\frac{\partial\tilde{\rho}}{\partial x}.$$

But, on use of a Taylor series (binomial theorem),

$$(1+\tilde{\rho}/\rho_0)^{\gamma-1} = 1+(\gamma-1)\frac{\tilde{\rho}}{\rho_0}+...$$

hence, the linearized Euler equation is

$$\rho_0\frac{\partial\tilde{u}}{\partial t} = -k^2\rho_0^{\gamma-1}\frac{\partial\tilde{\rho}}{\partial x}.$$

Differentiation of this with respect to x gives

$$\rho_0\frac{\partial^2\tilde{u}}{\partial x\partial t} = -k^2\rho_0^{\gamma-1}\frac{\partial^2\tilde{\rho}}{\partial x^2}.$$

This is another expression for $\frac{\partial^2\tilde{u}}{\partial t\partial x}$ which, when substituted into the earlier equation derived above, yields the required wave equation for $\tilde{\rho}$. The general solution is

$$\tilde{\rho}(x,t) = f(x-a_0t)+g(x+a_0t)$$

where $a_0^2 = k^2\rho_0^{\gamma-1}$ is the speed of wave propagation.

5. From the third of the Rankine-Hugoniot equations: $p_1-p_2 = \rho_2u_2^2-\rho_1u_1^2 = \rho_1u_1(u_2-u_1)$, since $\rho_1u_1 = \rho_2u_2$. Dividing this by p_1 :

$$1-(p_2/p_1) = (\rho_1u_1/p_1)(u_2-u_1) = (\gamma/a_1^2)(u_1u_2-u_1^2),$$

using the definition $a_1^2 = \gamma p_1/\rho_1$. In the lectures we established that

$$u_1u_2 = 2\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{1}{2}u_1^2+\frac{a_1^2}{\gamma-1}\right).$$

Using this expression to substitute for u_1u_2 we obtain

$$\begin{aligned} 1-(p_2/p_1) &= (\gamma/a_1^2)\left(2\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{1}{2}u_1^2+\frac{a_1^2}{\gamma-1}\right)-u_1^2\right) \\ &= \frac{2\gamma}{\gamma+1}\left(1-M_1^2\right). \end{aligned}$$

Hence the result for the pressure ratio.

To find the Mach number relation we start with the second of the Rankine-Hugoniot equations in the form $a_2^2 = a_1^2+\frac{1}{2}(\gamma-1)(u_1^2-u_2^2)$. Divide by u_2^2 :

$$\frac{1}{M_2^2} = \frac{a_1^2}{u_2^2}+\frac{1}{2}(\gamma-1)\left(\left(\frac{u_1}{u_2}\right)^2-1\right).$$

The first term on the RHS can be written as $(1/M_1^2)(u_1/u_2)^2$ and hence

$$\frac{1}{M_2^2} = \left(\frac{u_1}{u_2}\right)^2\left(\frac{1}{M_1^2}+\frac{1}{2}(\gamma-1)\right)-\frac{1}{2}(\gamma-1).$$

From the lecture notes we know u_2/u_1 in terms of M_1. Substitute this in to get:

$$\frac{1}{M_2^2} = \frac{1}{4}\left(\frac{\gamma+1}{\gamma-1}\right)^2\left(\frac{2M_1^2(\gamma-1)}{M_1^2(\gamma-1)+2}\right)^2\left(\frac{1}{M_1^2}+\frac{1}{2}(\gamma-1)\right)-\frac{1}{2}(\gamma-1),$$

which simplifies to the desired result.

Hypersonic flow implies that $M_1 \gg 1$. Take limit as $M_1 \to \infty$ to obtain asymptotic results.

6. Examining the piston problem from the notes we see that $s = u - 2a/(\gamma - 1)$ is constant throughout the flow region. Hence $u_1 - 2a_1/(\gamma - 1) = -2a_0/(\gamma - 1)$. Hence we get the result upon rearrangement. Now add $-V$ to the moving shock problem to bring the shock to rest. From Bernoulli we know that $\frac{1}{2}u^2 + a^2/(\gamma - 1)$ is conserved. Hence

$$\frac{1}{2}(u_1 - V)^2 + \frac{a_1^2}{\gamma - 1} = \frac{1}{2}(-V)^2 + \frac{a_0^2}{\gamma - 1}.$$

Upon cancellation, this gives the expression in the question. Substituting for a_1 we get

$$\frac{1}{2}u_1^2 - u_1 V + \frac{(a_0 + \frac{1}{2}(\gamma - 1)u_1)^2}{\gamma - 1} = \frac{a_0^2}{\gamma - 1}.$$

Cancelling and simplifying gives $V = \frac{1}{4}(\gamma + 1)u_1 + a_0$, as required.