M2AM: Fluids and Dynamics

Problem Sheet 4

1. The solution to the kinematic wave equation with the "tent example" initial condition

$$u(x,0) = \left\{ \begin{array}{ll} 0 & x < -1, \\ u_0(1+x) & -1 \leq x \leq 0, \\ u_0(1-x) & 0 \leq x \leq 1, \\ 0 & x > 1, \end{array} \right\}$$

where $u_0 > 0$ is a constant was given in lectures. For $t < 1/u_0$ it can be written as

$$u(x,t) = \left\{ \begin{array}{lll} u_0(1-x+kt)/(1-u_0t) & \text{for} & (u_0+k)t < x < 1+kt, \\ u_0(1+x-kt)/(1+u_0t) & \text{for} & -1+kt < x < (u_0+k)t. \end{array} \right\}$$

At time $t = 1/u_0$ a shock forms. Let the subsequent position of the shock be $x = s(t)$. Show that the ordinary differential equation for $s(t)$ is

$$\frac{ds(t)}{dt} = k + \frac{u_0}{2}\left(\frac{1+s-kt}{1+u_0t}\right).$$

Solve this equation and show that the shock position for times $t > 1/u_0$ is

$$s(t) = kt - 1 + \sqrt{2}(1+u_ot)^{1/2}.$$

[Compare this with your solution for $S(t)$ in Q3 of Problem Sheet 3].

2. Solve the equation

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = 0$$

for $t > 0$ in the domain $-\infty < x < \infty$ given the initial conditions

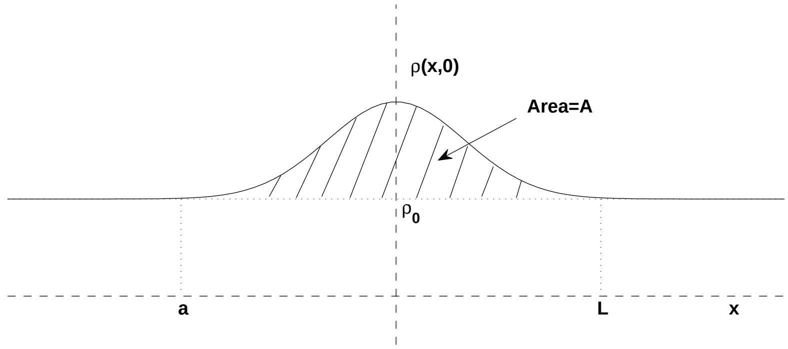
$$\rho(x,0) = \left\{ \begin{array}{ll} 0 & x \leq 0, \\ x & 0 \leq x \leq 1/2, \\ 1-x & 1/2 \leq x \leq 1, \\ 0 & x \geq 1. \end{array} \right\}$$

Find the critical time t_s of shock formation and the point x_s at which the wave breaks. Fit a shock to this solution and find the shock velocity.

3. [Large-t asymptotics] Consider solving the same equation as in Q2 but now with general initial condition

$$\rho(x,0) = \left\{ \begin{array}{ll} \rho_0 & x \leq a \\ g(x) & a \leq x \leq L \\ \rho_0 & x \geq L \end{array} \right\}$$

where $g(x)$ is some continuous function with $g(a) = g(L) = \rho_0$ having the general form illustrated in the figure. The area of the shaded region in the figure is known to be A. We expect a shock to form at some critical time t_s. This question examines what the solution looks like for large times $t \gg t_s$ (i.e., the large-t asymptotic solution).



(a) Show that a simple solution to the governing equation is given by $\rho(x,t) = \hat{\rho}(x,t)$ where

$$\hat{\rho}(x,t) = \left\{ \begin{array}{ll} \rho_0 & x \le \rho_0 t \\ x/t & \rho_0 t \le x \le s(t) \\ \rho_0 & x \ge s(t) \end{array} \right\}$$

and where $s(t)$ is some arbitrary function. Draw a sketch of this solution as a function of x at a fixed value of t.

(b) It is proposed that, at large times $t \gg t_s$, the solution to the original initial value problem looks like the solution $\hat{\rho}(x,t)$ with $s(t)$ now determined by the usual shock condition. Show that the ordinary differential equation for $s(t)$ is

$$\frac{ds}{dt} = \frac{1}{2}\left(\rho_0 + \frac{s}{t}\right).$$

(c) Find the general solution of this equation for $s(t)$. Hence, by applying the "equal area rule", show that

$$s(t) = \rho_0 t + \sqrt{2At}.$$

Draw a labelled sketch of this solution as a function of x at some large time t. [Note that this large-t solution does not depend on any details of the original initial condition except for A (and ρ_0)].

4. A simple solution to the equations governing an isentropic gas with $\gamma > 1$ is

$$u = 0, \quad \rho = \rho_0$$

where ρ_0 is a constant. The associated pressure is $p = p_0 = k^2 \rho_0^\gamma / \gamma$. Consider a small (linear) perturbation to the velocity and density fields of this basic state, i.e.,

$$u = \tilde{u}, \quad \rho = \rho_0 + \tilde{\rho}$$

where $|\tilde{u}| \ll 1$ and $\tilde{\rho} \ll \rho_0$. Show, from the equations governing the motion of an isentropic gas, that the linearized equation for $\tilde{\rho}$ is

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = a_0^2 \frac{\partial^2 \tilde{\rho}}{\partial x^2}$$

where $a_0^2 = k^2 \rho_0^{\gamma-1} = \gamma p_0 / \rho_0$. Hence deduce that density fluctuations propagate as waves with speed $\pm a_0$.

5. Use the Rankine-Hugoniot equations to show that the pressures and Mach numbers upstream and downstream of a stationary shock are related by

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1), \qquad \frac{1}{M_2^2} = \frac{1}{2}\left(\frac{(\gamma+1)^2 M_1^2}{(\gamma-1)M_1^2 + 2} - (\gamma - 1)\right).$$

[Here the subscript "1" refers to an upstream quantity while "2" refers to a downstream quantity]. Hence show that for *hypersonic flow* (corresponding to $M_1 \to \infty$) we have

$$\frac{p_2}{p_1} \sim \frac{2\gamma}{\gamma+1} M_1^2, \quad M_2^2 \to \frac{1}{2}\left(1 - \gamma^{-1}\right).$$

6. A steady shock moves with speed V into stationary air. The fluid velocity and sound speed upstream of the shock wave are u_1 and a_1 while the sound speed in the stationary air is a_0. Show that

$$a_1 = a_0 + \frac{\gamma - 1}{2} u_1.$$

[Hint: regard the shock front in the same way as in the moving piston problem from lectures and think about the quantity which remains constant in that problem].

Using the fact that the moving shock problem just described is equivalent to a stationary shock with upstream fluid velocity $u_1 - V$ and downstream velocity $-V$, use Bernoulli's equation to deduce that

$$\frac{u_1^2}{2} - u_1 V + \frac{a_1^2}{\gamma - 1} = \frac{a_0^2}{\gamma - 1}.$$

Hence show that the shock moves at speed

$$V = \frac{(\gamma+1)u_1}{4} + a_0.$$