## M2AM: Fluids and Dynamics

## Problem Sheet 3 - SOLUTIONS

1. (a) The characteristics satisfy

$$
\frac{d x}{d t}=u+k
$$

and, since $u$ is constant on them, they are all straight lines (whose slope depends on where they intersect the $x$-axis). Suppose the characteristics intersect the $x$-axis at $x=\zeta$ where $\zeta<-1$ then characteristics are

$$
x=\zeta+(k+1) t
$$

If $|\zeta| \leq 1$ then characteristics are

$$
x=\zeta+((1-\zeta) / 2+) t
$$

Finally, if $\zeta>1$ the characteristics are

$$
x=\zeta+k t
$$

A plot of these characteristics is shown in the Figure:


Graph of characteristics, in ( $x, t$ )-plane, for Q1(a).
(b) It is clear from the Figure that all characteristics hitting the $x$-axis in the interval $[-1,1]$ cross at $t=2$ and this is the earliest time of crossing characteristics. Therefore $t_{s}=2$ is the shock formation time. The shock forms at $x=x_{s}=1+2 k$.
(c) Since

$$
u(x, t)=f(x-(u+k) t)
$$

then, at $t=0$, we have

$$
u(x, 0)=f(x)=\left\{\begin{array}{ll}
1 & x<-1 \\
(1-x) / 2 & |x| \leq 1 \\
0 & x>1
\end{array}\right\}
$$

hence, we must have

$$
u(x, t)=\left\{\begin{array}{ll}
1 & x-(u+k) t<-1, \\
(1-[x-(u+k) t]) / 2 & |x-(u+k) t| \leq 1, \\
0 & x-(u+k) t>1
\end{array}\right\}
$$

This can be rewritten explicitly. For $|x-(u+k) t| \leq 1$ we rearrange to find

$$
u=\frac{1-x+k t}{2-t}
$$

Substitution of this expression into the implicit formula above yields, after some algebra, the explicit formula

$$
u(x, t)=\left\{\begin{array}{ll}
1 & x<-1+t+k t \\
\frac{1-x+k t}{2-t} & -1+t+k t \leq x \leq 1+k t \\
0 & x>1+k t .
\end{array}\right\}
$$

It is clear that this solution, as a graph of $u$ against $x$, has infinite slope when $t=2$, i.e., at the expected shock formation time.
2. First note that, qualitatively, this initial profile is similar to that of Q1 so we anticipate shock formation. In this case, however, it is not possible to easily obtain an explicit form for the solution. Nevertheless, the equation for the characteristics in this case is

$$
\frac{d x}{d t}=1-\tanh \zeta+k
$$

so the characteristics are

$$
x=\zeta+(1+k-\tanh \zeta) t .
$$

To find where characteristics cross, we will seek points at which

$$
\left.\frac{\partial x}{\partial \zeta}\right|_{t}=0 .
$$

This happens when

$$
1-t \operatorname{sech}^{2} \zeta=0
$$

that is, when $t=\cosh ^{2} \zeta$. The earliest crossing of characteristics is therefore when $\zeta=0$ corresponding to $t=t_{s}=1$. This is the shock formation time.
3. First, let $u_{2}$ be the value of $u$ at the top of triangular area $B$ and $u_{3}$ the value of $u$ at the upper-left vertex of triangular area $A$. Then the area of $A$ is

$$
\frac{1}{2}(S(t)-(1+k t)) u_{2}
$$

while the area of $B$ is

$$
\frac{1}{2}\left(u_{0} t+k t-S(t)\right)\left(u_{3}-u_{2}\right) .
$$

Setting these equal implies the equation

$$
\frac{1}{2}(S(t)-(1+k t)) u_{2}=\frac{1}{2}\left(u_{0} t+k t-S(t)\right)\left(u_{3}-u_{2}\right) .
$$

Now substituting the values

$$
u_{2}=\frac{u_{0}}{1-u_{0} t}(1-S+k t), \quad u_{3}=\frac{u_{0}}{1+u_{0} t}(1+S-k t)
$$

(these follow from the analytical form of the solution given in the question) and rearranging leads (after some algebra) to the following quadratic equation for $S(t)$ :

$$
S^{2}+2 S(1-k t)+k^{2} t^{2}-2 k t-1-2 t u_{0}=0
$$

The solution of this quadratic, lying between $1+k t$ and $u_{0} t+k t$, is found (using the usual formula for the roots of a quadratic polynomial) to be

$$
S(t)=-1+k t+\sqrt{2}\left(1+u_{0} t\right)^{1 / 2} .
$$

