M2AM: Fluids and Dynamics

Problem Sheet 3

1. Consider the kinematic wave equation for $u(x,t)$ given as

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0.$$

It is required to solve this equation, for $t > 0$ in $-\infty < x < \infty$, given the initial condition that

$$u(x,0) = \left\{ \begin{array}{ll} 1 & x < -1, \\ (1-x)/2 & |x| \leq 1, \\ 0 & x > 1. \end{array} \right\}$$

(a) Plot the characteristics for this equation in the upper-half (x,t)-plane.

(b) Show that a shock forms at some critical time t_s and find the value of t_s.

(c) An implicit form of the general solution to the kinematic wave equation is known to be

$$u = f(x - (u+k)t)$$

where f is an arbitrary function. Use this fact to deduce an explicit analytical expression for the solution $u(x,t)$ and hence verify your answer to part (b).

2. Consider the same kinematic wave equation (as in Q1) but now with initial condition given by

$$u(x,0) = 1 - \tanh x.$$

Show that a shock forms in this case at a critical time t_s and find t_s.

3. The solution to the kinematic wave equation (as in Q1) with the "tent example" initial condition

$$u(x,0) = \left\{ \begin{array}{ll} 0 & x < -1, \\ u_0(1+x) & -1 \leq x \leq 0, \\ u_0(1-x) & 0 \leq x \leq 1, \\ 0 & x > 1. \end{array} \right\}$$

where $u_0 > 0$ is a constant was given in lectures. For $t < 1/u_0$ it can be written as

$$u(x,t) = \left\{ \begin{array}{ll} u_0(1-x+kt)/(1-u_0t), & \text{for } \ (u_0+k)t < x < 1+kt \\ u_0(1+x-kt)/(1+u_0t), & \text{for } \ -1+kt < x < (u_0+k)t \end{array} \right\}$$

At time $t = 1/u_0$ a shock forms. Thereafter, the solution for $u(x,t)$ becomes triple-valued for values of x in the interval $1+kt < x < (u_0+k)t$ and, for $t > 1/u_0$, we now have

$$u(x,t) = \left\{ \begin{array}{ll} u_0(1-x+kt)/(1-u_0t), & \text{for } \ 1+kt < x < (u_0+k)t \\ u_0(1+x-kt)/(1+u_0t), & \text{for } \ -1+kt < x < (u_0+k)t \end{array} \right\}$$

This mathematical solution for $t > 1/u_0$ is illustrated in the figure. Suppose that a point $S(t)$ in the interval

$$1 + kt < S(t) < (u_0+k)t,$$

is now chosen in such a way that the areas of the two shaded regions (labelled A and B) shown in the figure are exactly *equal*. Find an expression for $S(t)$.