M2AM: Fluids and Dynamics

Problem Sheet 2 - SOLUTIONS

1. We have

$$u = f(x - (k + u)t)$$

where f is an arbitrary function of a single variable. Differentiating partially with respect to t gives

$$\frac{\partial u}{\partial t} = f'(x - (k + u)t) \left(-t \frac{\partial u}{\partial t} - (u + k) \right)$$

which implies, on rearrangement, that

$$\frac{\partial u}{\partial t} = -\frac{(u + k) f'(x - (k + u)t)}{1 + t f'(x - (k + u)t)}.$$

Similarly, partial differentiation with respect to x yields

$$\frac{\partial u}{\partial x} = \frac{f'(x - (k + u)t)}{1 + t f'(x - (k + u)t)}.$$

It is therefore clear that

$$\frac{\partial u}{\partial t} + (u + k) \frac{\partial u}{\partial x} = 0.$$

2. In part (a), u is constant along characteristic curves given by $dx/dt = t/\cos(x)$. Integrating we obtain $\sin(x) - t^2/2 =$ constant. Thus $u = f(\sin(x) - t^2/2)$ is the general solution where f is an arbitrary function. Given $u = t^4$ when $x = 0$ implies $t^4 = f(-t^2/2)$. Solution is therefore $u(x, t) = 4(\sin(x) - t^2/2)^2$.

(b) u is constant along the characteristics $dx/dt = t^2/x$. Integrating: $x^2/2 - t^3/3 = \text{const} \Rightarrow u = f(x^2/2 - t^3/3)$. Given $u = x^2$ when $t = 0 \Rightarrow x^2 = f(x^2/2)$. Let $s = x^2/2 \Rightarrow 2s = f(s)$. Thus $u = 2(x^2/2 - t^3/3) = x^2 - 2t^3/3$.

(c) Characteristics: $dx/dt = xe^t \Rightarrow \ln x - e^t = \text{const} \Rightarrow u = f(\ln x - e^t)$. Apply $u = t$ when $x = 1 \Rightarrow t = f(-e^t)$. Let $s = -e^t$, then $t = \ln(-s) = f(s) \Rightarrow u = \ln(e^t - \ln x)$.

3. For an isentropic gas with $\gamma > 1$, we have

$$p = \frac{k^2}{\gamma} \rho^\gamma$$

so that the evolution equations become

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + k^2 \rho^{\gamma - 2} \frac{\partial \rho}{\partial x} = 0.$$

Defining $a^2 = \gamma p/\rho$ means that

$$a^2 = k^2 \rho^{\gamma - 1},$$

and hence that

$$2a \frac{\partial a}{\partial t} = k^2 (\gamma - 1) \rho^{\gamma - 2} \frac{\partial \rho}{\partial t}, \qquad 2a \frac{\partial a}{\partial x} = k^2 (\gamma - 1) \rho^{\gamma - 2} \frac{\partial \rho}{\partial x}$$

Substitution of these into the first of the evolution equations yields (after some simple algebra)

$$\frac{2}{\gamma-1}\left(\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}\right)+a\frac{\partial u}{\partial x}=0.$$

Substitution into the second of the evolution equations leads to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2}{\gamma-1}a\frac{\partial a}{\partial x}=0.$$

Addition and subtraction of the last two equations gives, respectively,

$$\left(\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x}\right)(u+\frac{2a}{\gamma-1})=0$$

and

$$\left(\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}\right)(u-\frac{2a}{\gamma-1})=0.$$

4. Suppose that

$$u-\frac{2a}{\gamma-1}=u_0-\frac{2a_0}{\gamma-1}$$

then

$$\frac{2a}{\gamma-1}=u-u_0+\frac{2a_0}{\gamma-1}.$$

Hence we get that

$$\left(\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x}\right)u=0.$$

But this can be rewritten

$$\left(\frac{\partial}{\partial t}+\left(u+\left(\frac{\gamma-1}{2}\right)\left(u-u_0+\frac{2a_0}{\gamma-1}\right)\right)\frac{\partial}{\partial x}\right)u=0.$$

This clearly simplifies to the required equation.