M2AM: Fluids and Dynamics

Problem Sheet 2

1. The kinematic wave equation governing the field $u(x,t)$ is

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0$$

where k is a constant. The general solution is given implicitly by the formula

$$u = f(x-(k+u)t)$$

where f is an arbitrary function (this solution was derived in lectures using the method of characteristics). Verify *directly* that $u(x,t)$ as defined by this implicit formula is a solution of the kinematic wave equation.

2. Solve the following partial differential equations by the method of characteristics:

(a)

$$\cos x\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0, \quad u = t^4 \text{ when } x = 0.$$

(b)

$$x\frac{\partial u}{\partial t} + t^2\frac{\partial u}{\partial x} = 0, \quad u = x^2 \text{ when } t = 0.$$

(c)

$$\frac{1}{x}\frac{\partial u}{\partial t} + e^t\frac{\partial u}{\partial x} = 0, \quad u = t \text{ when } x = 1.$$

3. The equations governing the 1-D motion of an ideal gas, in which the pressure-density relation is given by $p = k^2\rho$, were found to be

$$\left(\frac{\partial}{\partial t} + (u+k)\frac{\partial}{\partial x}\right)\left(\frac{u}{k} + \chi\right) = 0,$$
$$\left(\frac{\partial}{\partial t} + (u-k)\frac{\partial}{\partial x}\right)\left(\frac{u}{k} - \chi\right) = 0.$$

where $\chi \equiv \log\rho$.

Consider now an *isentropic gas* which has a pressure-density relation of the more general form

$$p = \frac{k^2}{\gamma}\rho^\gamma.$$

Introduce the quantity

$$a^2 = \frac{\gamma p}{\rho}$$

and show that the equations governing the 1-D motion of an isentropic gas are

$$\left(\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right)\left(u + \frac{2a}{\gamma-1}\right) = 0,$$
$$\left(\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right)\left(u - \frac{2a}{\gamma-1}\right) = 0.$$

4. By using the equations governing a 1-D isentropic gas derived in Q3, show that for the restricted class of solutions in which

$$u - \frac{2a}{\gamma - 1} = u_0 - \frac{2a_0}{\gamma - 1} = \text{constant}$$

then the equation for $u(x, t)$ reduces to

$$\frac{\partial u}{\partial t} + \left(\frac{\gamma + 1}{2} u - \frac{\gamma - 1}{2} u_0 + a_0 \right) \frac{\partial u}{\partial x} = 0.$$

This is the generalization of the kinematic wave equation for an isentropic gas.