M2AM: Fluids and Dynamics

Problem Sheet 1 - SOLUTIONS

1. In each case, writing $\underline{u} = (u, v)$, we must solve the ODE's

$$\frac{dx}{dt} = u; \quad \frac{dy}{dt} = v$$

with initial conditions $x(0) = 0,\ y(0) = 1$. For (a), we get $x(t) = 0, y(t) = e^{-t}$ so fluid particle moves towards the origin along the y-axis and asymptotically approaches the origin. For (b), we get $x(t) = t,\ y(t) = 1$ so the fluid particle travels along $y = 1$ parallel to the x-axis and goes off to infinity. For (c), we get $x(t) = 0, y(t) = e^{t}$ so fluid particle moves away from the origin along the y-axis and asymptotically approaches infinity.

2. We have

$$x^L(\zeta, t) = \frac{\zeta}{1 + \zeta t}$$

which, incidentally, satisfies the condition $x^L(\zeta, 0) = \zeta$. Then

$$u^L(\zeta, t) = \frac{\partial x^L(\zeta, t)}{\partial t} = -\frac{\zeta^2}{(1 + \zeta t)^2} = -[x^L(\zeta, t)]^2$$

Since the velocity of every fluid particle is $-[x^L(\zeta, t)]^2$ then the Eulerian velocity field is just

$$u(x, t) = -x^2.$$

The material derivative is

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 - x^2(-2x) = 2x^3$$

which is easily seen to be equal to

$$\frac{\partial^2 x^L(\zeta, t)}{\partial t^2} = 2\left(\frac{\zeta}{1 + \zeta t}\right)^3.$$

Conversely, if the Eulerian flow field is given as $u(x, t) = -x^2$ then the equation governing the motion of fluid particles is

$$\frac{dx^L}{dt} = -[x^L]^2$$

which is a separable ODE so the solution follows from

$$\int^{x^L} \frac{dx}{x^2} = -\int^t dt'$$

that is,

$$-\frac{1}{x^L} = -t - c$$

where c is some constant. Imposing the initial condition $x^L(\zeta, 0) = \zeta$, we get the required $x^L(\zeta, t)$.

3. The equation for the evolution of a fluid particle is

$$\frac{\partial x^L}{\partial t} = u(x^L, t) = \frac{2t}{1+t^2}x^L + 1 + t^2.$$

This is a linear, first-order ODE and is solved by integrating factor method. The integrating factor is $(1+t^2)^{-1}$. The solution is

$$x^L = (c+t)(1+t^2)$$

where c is some constant. Imposing the initial condition $x^L(\zeta, 0) = \zeta$, we get the solution

$$x^L(\zeta, t) = (\zeta + t)(1+t^2).$$

We are now given that contaminated particles have $0 \le \zeta \le 1$. At $t = 2$, we have

$$x^L(\zeta, 2) = 5(\zeta + 2).$$

Thus, contaminated particles lie in the range $10 \le x^L \le 15$.

4. First note that

$$\nabla.(p\underline{C}) = \sum_{j=1}^{3} \frac{\partial [pC_i]}{\partial x_i} = \sum_{j=1}^{3} C_i \frac{\partial p}{\partial x_i} = \underline{C}.\nabla p$$

where, in the second equality, we have used the fact that $\underline{C}$ is constant. Now, applying the divergence theorem (as given in lectures) to the vector $p\underline{C}$ yields

$$\int_V \nabla.(p\underline{C})\, dV = \int_S p\underline{C}.\underline{n}\, dS$$

or

$$\int_V \underline{C}.\nabla p\, dV = \int_S p\underline{C}.\underline{n}\, dS$$

where we have used the first result just obtained in the LHS. But this equation can be written as

$$\underline{C}.\int_V \nabla p\, dV = \underline{C}.\int_S p\underline{n}\, dS$$

or

$$\underline{C}.\left[\int_V \nabla p\, dV - \int_S p\underline{n}\, dS\right] = 0$$

for arbitrary constant vectors $\underline{C}$. Thus the result

$$\int_V \nabla p\, dV = \int_S p\underline{n}\, dS$$

follows.

5. The momentum flux *into* the volume through left hand end is $[\rho(a,t)Au(a,t)]u(a,t)$; the in-flux of momentum through the right hand end it is $-[\rho(b,t)Au(b,t)]u(b,t)$. Thus, the net gain in momentum is

$$\left(\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right) A$$

as required. The pressure force on the left hand end of the fluid volume is $-p(a,t)A\underline{n} = p(a,t)A\underline{e_x}$ since $\underline{n} = -\underline{e_x}$ where $\underline{e_x}$ denotes the unit vector in the x-direction. The pressure force on the right hand end of the fluid volume is $-p(b,t)A\underline{n} = -p(b,t)A\underline{e_x}$ since $\underline{n} = \underline{e_x}$ on this face. The net force on the fluid volume in the $\underline{e_x}$ direction is therefore

$$\big(p(a,t) - p(b,t)\big)\, A.$$

By Newton's second law, the rate of change of momentum equals the net gain due to the momentum flux plus the applied forces, that is,

$$\frac{d}{dt}\int_a^b \rho u A dx = \left(\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right) A + \big(p(a,t) - p(b,t)\big)\, A.$$

Since A is constant, this cancels through the equation and we can write the last expression as

$$\int_a^b \left(\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x}\right) = 0.$$

Since the choice of interval $[a,b]$ is arbitrary, we conclude that the integrand is identically zero, as required. Expanding derivatives, we get

$$u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho u)}{\partial x} = -\frac{\partial p}{\partial x}.$$

But the equation of continuity takes the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} = 0$$

which can be used to simplify the above expression to

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}.$$

This is the 1-D form of the Euler equation derived (using alternative arguments) in lectures.

6. The rate of change of mass of an arbitrary section between $x = a$ and $x = b$ equals the mass flux entering at $x = a$ minus the mass flux leaving at $x = b$ **plus** the rate of change of any "extra" mass generated inside (in lectures it was assumed there were no such extra mass sources). Mathematically, this means that

$$\frac{d}{dt}\int_a^b \rho dx = [\rho u]_a^b + \int_a^b r(x,t)dx.$$

By the usual arguments we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = r.$$

The right hand side is non-zero due to the mass creation and is our *new* form of the continuity equation.

Now, in a similar spirit to Q5, by Newton's second law, the rate of change of momentum of the chosen section of fluid is equal to the net gain in momentum flux through the ends plus the pressure forces acting across the end-sections. Mathematically,

$$\frac{d}{dt}\int_a^b \rho u \, dx = \left[\rho u^2\right]_a^b + [-p]_a^b.$$

Following the working of Q5, we deduce

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial p}{\partial x}.$$

which, if we simplify using our *new* form of the continuity equation, reduces to

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}-ru$$

which is the new, modified form of the Euler equation.

7. (a) It is clear, from the figure, that

$$-\underline{n_1}.\underline{n_2}=\cos\theta$$

while it is clear that $d_1=\cos\theta dS_2$ so that the required relationship is

$$-\underline{n_1}.\underline{n_2}dS_2=dS_1.$$

(b) The force on the left hand end of the cylinder is

$$-\underline{n_1}p(\underline{x},\underline{n_1},t)dS_1$$

Here we are assuming the cylinder is so small that we can safely neglect any variation of the pressure over the face.

(c) Similarly, the force on the right hand end of the cylinder is

$$-\underline{n_2}p(\underline{x},\underline{n_2},t)dS_2$$

(d) The net force on the cylinder in the $\underline{n_1}$-direction is therefore

$$\begin{aligned}\left(-\underline{n_1}p(\underline{x},\underline{n_1},t)dS_1-\underline{n_2}p(\underline{x},\underline{n_2},t)dS_2\right).\underline{n_1}&=-p(\underline{x},\underline{n_1},t)dS_1-\underline{n_2}[p(\underline{x},\underline{n_2},t)dS_2].\underline{n_1}\\&=\left[p(\underline{x},\underline{n_2},t)-p(\underline{x},\underline{n_1},t)\right]dS_1\end{aligned}$$

where we have used the result from part (a).

(e) Any net force must be due to acceleration via "F=ma". Such a force is proportional to the volume, i.e., it scales like L^3. However, dS_1 scales like L^2. We can therefore deduce that

$$p(\underline{x},\underline{n_1},t)-p(\underline{x},\underline{n_2},t)\propto L$$

and hence that, as $L\rightarrow 0$,

$$p(\underline{x},\underline{n_1},t)=p(\underline{x},\underline{n_2},t)$$

as required.