# M2AM: Fluids and Dynamics

Problem Sheet 1

**1.** Find, and sketch, the trajectory of the fluid particle initially located (at $t = 0$) at position $(0, 1)$ in each of the following two-dimensional Eulerian flow fields:

(a) **u**(x,y,t) = (x, -y);

(b) **u**(x,y,t) = (y, 0);

(c) **u**(x,y,t) = (x,y).

**2.** In a 1-D continuum, a Lagrangian description of the flow is given by

$$x^L(\zeta, t) = \frac{\zeta}{1 + \zeta t}$$

for $\zeta > 0$ (the parameter $\zeta$ "labels" the fluid particles). Find the Lagrangian velocity $u^L(\zeta, t)$ and acceleration $a^L(\zeta, t)$. Show that the Eulerian description of the same velocity is given by

$$u(x, t) = -x^2.$$

Hence verify that

$$\frac{Du}{Dt} = a^L,$$

that is, verify that the material derivative of the Eulerian velocity is the same as the acceleration of the fluid particles.

Conversely, assuming the Eulerian flow field is given by $u(x, t) = -x^2$, rederive the above expression for the position, at time $t$, of the particle that is initially at $x = \zeta$.

**3.** The Eulerian velocity for an unsteady 1-D fluid flow is given by

$$u(x, t) = \frac{2tx}{1 + t^2} + 1 + t^2.$$

By integration, find the Lagrangian description of the fluid flow. Suppose that, at $t = 0$, all fluid particles in the interval $[0, 1]$ are contaminated with dye. Over what range will these contaminated particles lie when $t = 2$?

**4.** Let $\underline{C}$ be a constant vector. Show that

$$\nabla.(p\underline{C}) = \underline{C}.\nabla p$$

where $p$ is any scalar field. Hence, use the divergence theorem to prove the relation

$$\int_S p\underline{n}\, dS = \int_V \nabla p\, dV$$

where $S$ is an arbitrary closed surface (with unit outward normal $\underline{n}$) enclosing a volume $V$. This relation was used in lectures in the derivation of the Euler equation.

**5.** This question shows a different derivation of the Euler equation for a 1-D flow to the one presented in the lectures. Consider a one-dimensional flow and a fixed slice of fluid between $x = a$ and $x = b$ (where $b > a$). The fluid velocity is $u(x,t)$ and its density is $\rho(x,t)$. Show that the net gain in momentum flux of the slice is

$$\left(\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right) A.$$

Assuming that the only forces acting on the slice are pressure forces at the ends, deduce that

$$\frac{d}{dt}\int_a^b \rho u dx = \left(\rho(a,t)u(a,t)^2 - \rho(b,t)u(b,t)^2\right) + \left(p(a,t) - p(b,t)\right)$$

where $p = p(x,t)$ is the fluid pressure. Hence show that

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = -\frac{\partial p}{\partial x}.$$

Finally, use the continuity equation to show that the above equation simplifies to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x}.$$

This is the same as the equation derived in lectures using a different approach.

**6.** The equations for the motion of a 1-D fluid through a region with constant cross-sectional area $A$ are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x}.$$

Now suppose that mass is being created within the fluid such that, in the absence of fluid motion, the change in mass of a slice $\delta x$ is

$$\delta m = r(x,t)A\delta x\delta t$$

in a time interval $\delta t$. Explain how the above equations can be modified to account for this mass creation.

**7.** In this problem we demonstrate that the pressure at a point in the fluid does not depend on the orientation of the fluid particle. Consider the small "lipstick"-shaped cylinder of fluid shown in the figure. The typical dimension of the fluid cylinder is $L \ll 1$ and it is centred at some position $\mathbf{x}$. Two "ends" of the particle are shown with their respective unit normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$. Let $dS_1$ and $dS_2$ be the (respective) surface areas of these two ends. The angle $ABC$ is $\theta$ and is assumed to be arbitrary. For now, assume that the pressure in the fluid is $p(\mathbf{x},\mathbf{n},t)$, i.e., it is a function of position and time as well as a function of the normal vector $\mathbf{n}$. The aim is to show that it does not, in fact, depend on $\mathbf{n}$.

(a) Find a relationship between $dS_1$ and $dS_2$.

(b) Find an expression for the force $\mathbf{F_1}$ (assumed to be in the $\mathbf{n_1}$ direction) on the left hand end of the cylinder in terms of the fluid pressure.

(c) Similarly, find an expression for the force $\mathbf{F_2}$ on the right hand end.

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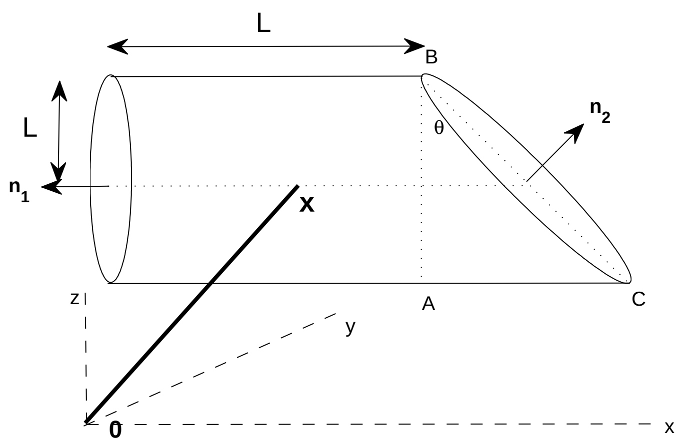


(d) Hence show that the net force on the cylinder in the direction $\mathbf{n_1}$ is

$$\left(p(\mathbf{x}, \mathbf{n_2}, t) - p(\mathbf{x}, \mathbf{n_1}, t)\right) dS_1.$$

(e) Now use Newton's Second Law to argue that the force in part (d) must be proportional to $L^3$.

(f) Finally, making use of the fact that $dS_1$ scales like $L^2$, argue that

$$p(\mathbf{x}, \mathbf{n_2}, t) - p(\mathbf{x}, \mathbf{n_1}, t) \propto L$$

and, hence, as $L \to 0$, that

$$p(\mathbf{x}, \mathbf{n_2}, t) = p(\mathbf{x}, \mathbf{n_1}, t).$$

In this way, we have shown that the fluid pressure does not depend on the normal vector $\mathbf{n}$.