Quadrature domains and fluid dynamics

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Abstract. Few physical scientists interested in the mathematical description of fluid flows will know what a quadrature domain is; just as few mathematicians interested in quadrature domain theory would profess to know much about fluid dynamics. And yet, recent research has shown that a surprisingly large number of the by-now classic exact solutions of two-dimensional fluid dynamics can be understood within the context of quadrature domain theory.

This article surveys a number of different physical applications of quadrature domain theory arising in the general field of fluid dynamics.

1. Introduction

The simplest example of a quadrature domain is a circular disc. Let $z = x + iy$ and suppose the disc is centred at the origin $z = 0$ with radius r. The well-known "mean value theorem" says that, if $h(z)$ is any function analytic in the disc D, then

$$\int\int_D h(z)dxdy = \pi r^2 h(0). \tag{1}$$

(1) is a simple example of a *quadrature identity.* The idea of quadrature domain theory is to consider more complicated domains satisfying more complicated quadrature identities. Shapiro [1] gives an illuminating introduction to quadrature domain theory. See also Sakai [2].

Perhaps the first connection between quadrature domain theory and applications was made by Richardson [3] who was interested in understanding the motion of the free boundaries of blobs of fluid trapped between two plates in a Hele-Shaw cell. When the flow is driven by a distribution of sources and/or sinks and surface tension effects on the free boundaries are ignored, this free boundary problem admits wide classes of "exact solution", i.e., initial fluid domains can be found whose evolution under the dynamics of the physical problem can be computed by tracking a finite set of time-evolving parameters. This constitutes a remarkable simplification of the problem.

1991 *Mathematics Subject Classification.* Primary 99Z99; Secondary 00A00.
Key words and phrases. quadrature domains, complex analysis, fluid dynamics.

It is important for our general message to point out that the exact solutions found by Richardson had, in fact, been found many years before by Polubarinova-Kochina [4] and Kufarev [5] who were interested in the motion of the interface between oil and water in porous media (mathematically, this problem is identical to the Hele-Shaw problem). But even if the solutions were already known, Richardson's 1972 paper introduced a crucial new theoretical ingredient: an understanding of the problem within the framework of quadrature domain theory.

Varchenko and Etingof [6] provide a comprehensive review of the impact of this new perspective in the context of flows in porous media and Hele-Shaw flows. The purpose of this article is different. The goal is to describe a broad spectrum of distinct physical problems, all emanating from the field of fluid dynamics, which can usefully be interpreted within the context of quadrature domain theory. The history of the Hele-Shaw problem illustrates the power of rephrasing a well-known problem in a new mathematical language. Here we describe the relevance of quadrature domain theory to a variety of different physical applications and describe its associated impact.

2. Quadrature domains

First, some background on quadrature domains. Consider a planar domain D. Let $h(z)$ be any function that is analytic in D and integrable over it. Suppose that

$$\int\int_D h(z)dxdy = \sum_{k=1}^{N}\sum_{j=0}^{n_k-1} c_{jk}h^{(j)}(z_k) \tag{2}$$

where $\{z_k \in \mathbb{C}\}$ is a set of points strictly inside D, $\{c_{jk} \in \mathbb{C}\}$ and $h^{(j)}(z)$ denotes the j-th derivative of h. Here, N and $\{n_k \geq 1\}$ are integers. Then D is very special and is known as a *quadrature domain* because of the remarkable fact, embodied in (2), that the two-dimensional integral on the left hand side of (2) in fact requires only the sum of a finite number of terms given on the right hand side for its evaluation. The quadrature identity (2) generalizes (1).

An alternative way to understand quadrature domains [6] is to consider their *Cauchy transforms* $C(z)$ defined as

$$C(z) = \frac{1}{\pi}\int\int_D \frac{dx'dy'}{z'-z}, \quad z \notin D. \tag{3}$$

This function is well-defined if D is bounded and is analytic for $z \notin D$. By Green's theorem, we also have

$$C(z) = \frac{1}{2i\pi}\oint_{\partial D} \frac{d\bar{z}'}{z'-z}, \quad z \notin D \tag{4}$$

and this form can be used to define Cauchy transforms for unbounded domains. Choosing $h(z) = z^n$ for $n \geq 0$ in the left hand side of (2) defines the geometrical *moments* M_n of a domain D, i.e.,

$$M_n = \frac{1}{\pi}\int\int_D z^n dxdy. \tag{5}$$

$C(z)$ is a generating function for these moments because its Laurent expansion coefficients valid as $|z| \to \infty$ are the moments (5). If the moments of a domain encode information concerning its shape, then so does $C(z)$. In physical problems, it is usually the evolution of the Cauchy transform which can be established most directly from the problem statement. The physical problem then reduces to that of reconstructing the domain from knowledge of its Cauchy transform. Mathematically, this is identical to the inverse problem of two-dimensional potential theory and such a viewpoint offers a helpful perspective. There are deep theoretical connections [1] between quadrature domain theory, potential theory and the concept of "balayage", and the theory of the Schwarz function [7].

Quadrature domains come in a variety of flavours. Basically, they are domains where the continuation of $C(z)$ into the domain has a special set of singularities. The most common quadrature domains satisfy quadrature identities of the type (2) and have Cauchy transforms that have a finite set of poles so that $C(z)$ is a rational function. Varchenko and Etingof [6] call these *algebraic domains.* They also introduced an *abelian domain* to be one where $C'(z)$, rather than $C(z)$, is rational. An ellipse is a quadrature domain, but it is neither algebraic nor abelian. Instead, it has a Cauchy transform with two square-root branch points at the foci. For example, an ellipse D with major and minor axes a and b respectively and with foci at ± 1 satisfies the quadrature identity [1]

$$\int\int_D h(z)dxdy = 2ab\int_{-1}^{1} h(x)(1-x^2)^{1/2}dx. \tag{6}$$

A different class of domains, called *quadrature domains for arclength* [9], satisfy identities of the form

$$\oint_{\partial D} h(z)|dz| = \sum_{k=1}^{N}\sum_{j=0}^{n_k-1} c_{jk}h^{(j)}(z_k). \tag{7}$$

Remarkably, all these classes of domain have been found to arise in applications, each having very different physics.

3. Constructing quadrature domains

Suppose $C(z)$ is known, then it remains to reconstruct the associated quadrature domain. There are various ways to do this. In a number of applications, Crowdy has made pragmatic use of the fact that the boundaries of quadrature domains are algebraic curves [10] [11]. Conformal maps from a pre-image ζ-plane (say) to the domain can also be used [12] [14]. Here we briefly describe the approach to reconstructing quadrature domains presented by Crowdy and Marshall [12] (who also survey the various other known methods of construction). It is known [6] [8] that the conformal mappings to a bounded g-connected quadrature domain is a meromorphic function on a Riemann surface of genus g. One model of such functions uses a mapping from a pre-image region in a ζ-plane consisting of the interior unit ζ-disc with g smaller interior discs excised. This is the Schottky model. The associated mappings must be invariant with respect to a group of Mobius transformations associated with this pre-image region. This group (Θ, say) is a classical *Schottky group*. Mumford, Series and Wright [15] give a very accessible and modern discussion of Schottky groups and their applications.

Given a Schottky group Θ the *Schottky-Klein prime function* is defined as [16]

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\prod_{\Theta'}\{\zeta,\gamma/\gamma_i,\zeta_i\} \tag{8}$$

where Θ' denotes all transformations in Θ excluding the identity and all inverses while ζ_i and γ_i denote images of ζ and γ respectively under the i-th map ($\theta_i(\zeta)$, say) in this set. $\{\zeta,\gamma/\gamma_i,\zeta_i\}$ denotes a cross-ratio. Then one representation for a function invariant with respect to transformations in Θ is a ratio of products of Schottky-Klein prime functions, i.e.,

$$z(\zeta,t) = R(t)\frac{\prod_{j=1}^{N}\omega(\zeta,\beta_j(t))}{\prod_{j=1}^{N}\omega(\zeta,\alpha_j(t))} \tag{9}$$

where the N poles $\{\alpha_j|j=1,..,N\}$ and N zeros $\{\beta_j|j=1,..,N\}$ satisfy the g conditions

$$\prod_{j=1}^{N}\prod_{\theta_i\in\Theta_k}\frac{(\beta_j-\theta_i(B_k))}{(\beta_j-\theta_i(A_k))}\Big/\frac{(\alpha_j-\theta_i(B_k))}{(\alpha_j-\theta_i(A_k))} = 1,\ k=1,...,g. \tag{10}$$

A_k and B_k are the two fixed points of the k-th Mobius map generating the group and Θ_k is another subset of Θ (see [12] for a precise definition). Figure 1 illustrates various multiply-connected square packings of near-circular discs all constructed using mappings of the general form (9).

When the Schottky group is trivial, the associated prime function is $\omega(\zeta,\gamma) = (\zeta-\gamma)$ and the functions (9) are just the rational functions. When Θ is generated by the single Mobius map $\theta_1(\zeta)=\rho^2\zeta$ then $\omega(\zeta,\gamma)\propto P(\zeta/\gamma,\rho)$ where

$$P(\zeta,\rho)\equiv(1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}/\zeta) \tag{11}$$

which is closely related to the classical Jacobi theta functions. With the choice (11), (9) then yields the class of *loxodromic functions* [13]. Such functions have been used to construct explicit solutions to the rotating Hele-Shaw problem [18], the viscous sintering problem [19] [20], the problem of finding vortical equilibria of the Euler equation [21] and the problem of free surface Euler flows with surface tension [24].

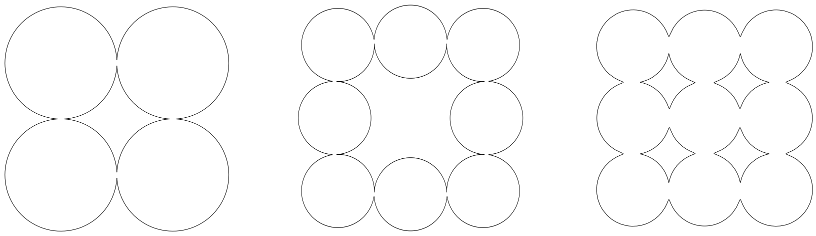

FIGURE 1. Three distinct square packings of near-circular particles constructed using conformal mappings based on the Schottky-Klein prime function (9). All are quadrature domains. The first two are doubly-connected, the third is quintuply-connected.

4. Applications in fluid dynamics

This section describes a series of physical problems where quadrature domains arise.

4.1. Hele-Shaw flows and flows in porous media

This class of problems is where use of quadrature domains and Cauchy transforms is best known and so we review it only briefly. Consider the two-dimensional flow-field $\mathbf{u}$ of a blob of fluid $D(t)$ of viscosity μ sandwiched between two plates of glass separated by b. Under appropriate assumptions, the flow is modelled by $\mathbf{u} = \nabla\phi$ where

$$\nabla^2\phi = 0, \text{ in } D(t). \tag{12}$$

If there is no surface tension then $\phi =$ constant on each free surface, while it must also be true that

$$V_n = \nabla\phi.\mathbf{n} \tag{13}$$

where V_n denotes the normal velocity of the free surface and $\mathbf{n}$ is the normal to the boundary. Richardson [3] studied the case of flows driven by sources of strength $Q_j(t)$ at positions $z = z_j$.

The above is a canonical example of a "laplacian growth problem". An extensive array of analytic results are known (see a website by Howison [25] for a comprehensive list of related references). These equations describe a number of different physical situations including, for example, flow in porous media, electrodeposition and the slow solidification in a supercooled liquid. Entov, Etingof and Kleinbock [26] discuss a number of variants of the Hele-Shaw problem for which there exist exact solutions. These include flow in a rotating Hele-Shaw cell where the flow is driven by centrifugal effects, Hele-Shaw flows with gravity and "squeeze flow" in a Hele-Shaw cell where the plates making up the cell are moved together (or apart).

Extensions to flows in non-planar cells, multiply-connected fluid regions and to three dimensions have all been made (see [25] for references).

As an example of how useful it can be to understand such problems in terms of quadrature domains, we examine the case of Hele-Shaw flows in rotating cells. Recent experiments have investigated the various interfacial instabilities that can occur in an initial concentric annulus of fluid placed in a rotating cell [17]. The time-evolving concentric annulus is a trivial exact solution to this problem, but it fails to exhibit any of the nonlinear phenomena observed in the experiments. Knowledgeable of the fact (see [26]) that simply-connected quadrature domains are preserved in a rotating cell, Crowdy [18] generalized this result to doubly-connected domains relevant to the experiments involving an annulus. Having derived a general class of solutions, to mimic the experiments of [17] it was necessary to construct an initial quadrature domain that is close to a concentric annulus. Gustafsson [8] showed that multiply-connected quadrature domains are dense (in an appropriate sense) in the general class of multiply-connected domains so the existence of such a quadrature domain was guaranteed. Indeed, in [18], it is shown that conformal maps from the annulus $\rho < |\zeta| < 1$ of the form

$$z(\zeta) = \zeta \frac{P_N(\zeta \rho^{2/N} a^{-1}, \rho)}{P_N(\zeta a^{-1}, \rho)} \tag{14}$$

where $1 < a < \rho^{-1}$ (chosen so that the map in univalent) give images that are quadrature domains getting arbitrarily close to the annulus $\rho < |z| < 1$ as $N \to \infty$. Here, $P_N(\zeta, \rho)$ is equivalent to a product of the functions $P(\zeta, \rho)$ and is defined by

$$P_N(\zeta, \rho) = (1 - \zeta^N) \prod_{k=1}^{\infty} (1 - \rho^{2kN} \zeta^N)(1 - \rho^{2kN} \zeta^{-N}). \tag{15}$$

Under evolution, the map takes the form

$$z(\zeta, t) = R(t) \zeta \frac{P_N(\zeta \rho(t)^{2/N} a(t)^{-1}, \rho(t))}{P_N(\zeta a(t)^{-1}, \rho(t))} \tag{16}$$

the parameters R, a and ρ evolving in time according to a coupled system of ordinary differential equations. In [18], this class of exact solutions is studied and compared to the qualitative results of the experiments in [17].

4.2. Rotating vortex arrays

A famous exact solution in vortex dynamics is the celebrated Kirchhoff elliptical vortex patch [27] which rotates, under the dynamics of the Euler equation, at constant angular velocity without changing its shape. The interior of an ellipse is a generalized quadrature domain satisfying the identity (1), while the exterior can be viewed as an unbounded quadrature domain with a Cauchy transform $C(z)$ which is a linear polynomial. By considering generalizations of these facts and using ideas involving the Schwarz function, broad new classes of exact solution have been found [28] for rotating vortex arrays with finite area cores having distributed vorticity. These solutions generalize the classic 19th century investigations of polygonal vortex arrays by Thomson [27]. Here, the flow $\mathbf{u}$ is given by $\mathbf{u} = (\psi_y, -\psi_x)$ where ψ is a *streamfunction* governed by the steady nonlinear Euler equation for a two-dimensional incompressible fluid of constant density which takes the form

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0. \tag{17}$$

The key observation is that, in a frame of reference co-rotating with the configuration, carefully-chosen streamfunctions ψ having the form of *modified Schwarz potentials* [1], i.e.,

$$\psi(x, y) = -\frac{\omega}{4}\left(z\bar{z} - \int^{z} S(z')dz' - \int^{\bar{z}} \overline{S}(z')dz'\right) \tag{18}$$

in the fluid region D, where $S(z)$ is the Schwarz function of the boundary ∂D (and $\bar{S}(z)$ its conjugate function), can represent dynamically-consistent equilibrium solutions of the Euler equation (17). (Such potentials also play an important role in the general theory of quadrature domains [1]). The constant ω is the magnitude of the uniform vorticity in the fluid. This new perspective has led to the discovery of a wide range of new exact solutions of the steady Euler equations, including those for rotating vortex configurations involving multiple interacting vortex patches. Some typical configurations are shown in Figure 2. The fluid regions exterior to the five co-rotating vortex patches in Figure 2 are unbounded, quintuply-connected quadrature domains constructed using conformal mappings based on the Schottky-Klein prime function. The pre-image region consists of the unit ζ-circle with four smaller discs of radius q excised. Figure 2 shows six different vortical configurations for various values of q.

4.3. Multipolar vortices in the plane and on the sphere

Motivated the observation, experiments and numerical investigations of a class of coherent structures known collectively as *multipolar vortices* (e.g. [30]), Quadrature domain theory has been used [29] to construct a class of exact stationary equilibrium solutions of the Euler equations displaying all the qualitative properties of the multipolar vortices observed in practice. The essence of the approach in [29] is to reappraise the classical circular vortex patch known as the Rankine vortex [27]. If one thinks of it instead as the simplest form of quadrature domain (i.e. a circular disc) then the multipolar vortex solutions correspond to generalized quadrature domains (subject, of course, to the physical constraints of the Helmholtz laws of vortex motion [27]). Again, streamfunctions of the form (18) turn out to be significant. These ideas have proven to be generalizable in a number of directions including finding multipolar vortices in annular arrays [21], vortices with more complicated topology [10] as well as finding equilbrium regions

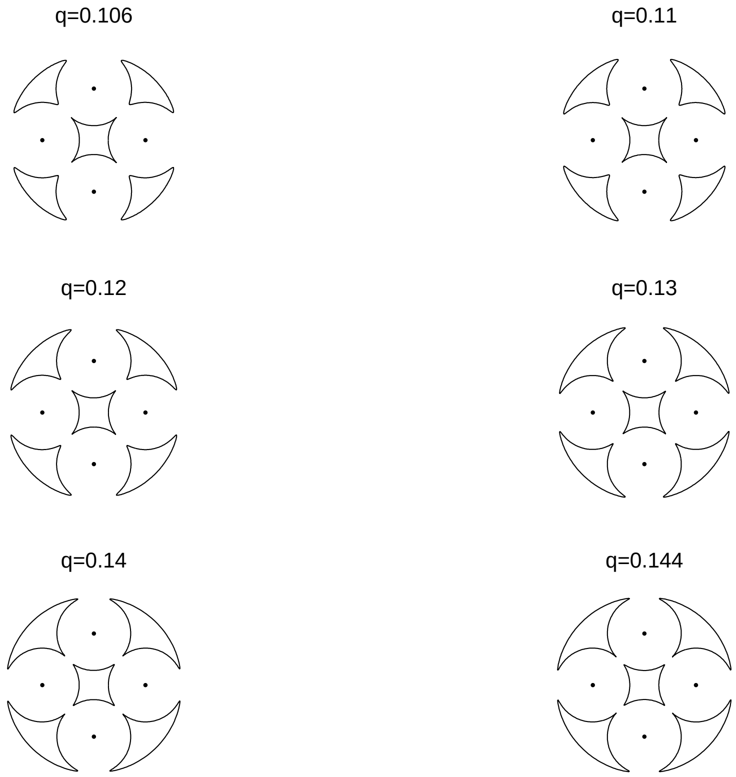

FIGURE 2. Steadily rotating vortex configuration consisting of 5 vortex patches (one central and four satellite patches) and 4 point vortices. The point vortices correspond to the singularities of the global Schwarz function.

of distributed vorticity on surfaces with non-zero curvature [31]. Examples of a triangular (or quadrupolar) vortices in equilibrium on the plane and on a sphere are shown in Figure 3. The right-hand diagram in Figure 3 has an interpretation as a quadrature domain on the surface of a sphere.

Shapiro [32] introduced the notion of a *special point* of a quadrature domain. The boundary of a bounded quadrature domain is known to be given by all the continuous, non-isolated solutions of

$$P(z, \bar{z}) = 0 \tag{19}$$

where $P(z, w) = 0$ is an algebraic curve whose order equals that of the quadrature identity. There are often a number of isolated solutions of (19) occurring inside the domain and these have been dubbed *special points* of the quadrature domain. It is interesting to remark that, in the context of steady vortical flows of the Euler

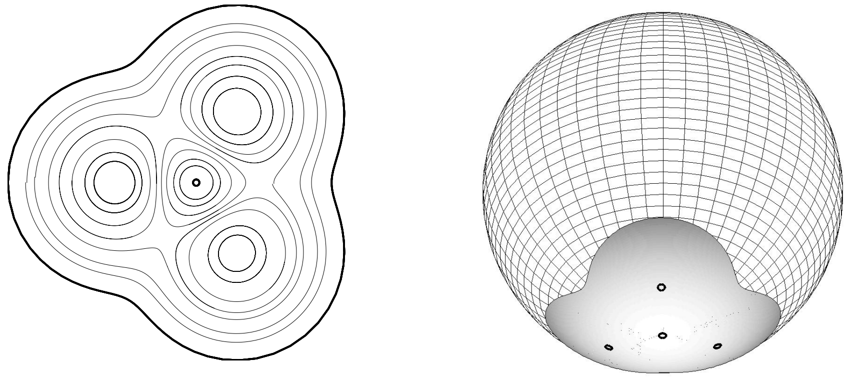

FIGURE 3. Streamlines and shape of a triangular vortex in equilibrium on the plane [29] (left). A triangular vortex in equilibrium on the surface of a sphere [31] (right). These are quadrature domains on the plane and on the sphere.

equation, these special points have a physical interpretation; they are precisely the stagnation points of the flow [10].

4.4. Free surface Euler flows with capillarity

Another famous exact solution known to fluid dynamicists is that for steady deep-water capillary waves found by Crapper in 1957. By reappraising this solution [33], broad new classes of exact solution for equilibrium configurations of free surface irrotational Euler flows with interfacial tension on the free boundaries have been identified. In the fluid region, the incompressible velocity field is given by $\mathbf{u} = \nabla\phi$ where

$$\nabla^2\phi = 0. \tag{20}$$

In equilibrium, any free boundary must be a streamline. If there is uniform surface tension T on the free boundary, the fluid pressure must balance the capillary forces. Using a well-known theorem due to Bernoulli, we can write

$$T\kappa + \Gamma = |\nabla\phi|^2 \tag{21}$$

where κ is the surface curvature and Γ is the Bernoulli constant.

Crowdy [34] has found new exact solutions for the shape deformations of both a bubble placed in an ambient circulatory flow of circulation γ and of a blob of fluid with internal circulation modelled by a contained line vortex singularity of strength γ. Non-trivial equilibrium shapes of a bubble in a circulatory flow

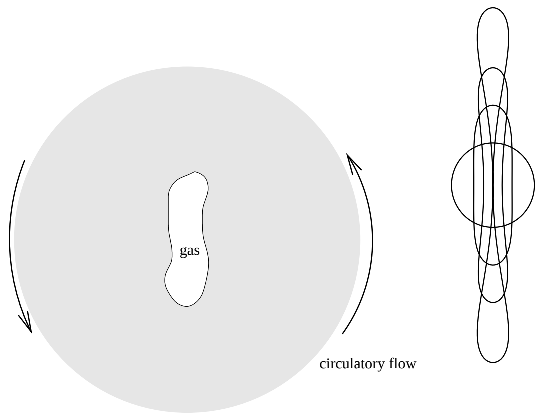

FIGURE 4. Schematic illustrating the problem of a bubble with capillarity in an ambient circulatory flow (left). Equilibrium bubble shapes, computed using exact solutions, for different values of the circulation [34] (right). At a critical circulation, the bubble is found to pinch.

are shown in Figure 4. In the case of both a bubble and a blob, the conformal mappings $z(\zeta)$ from a unit ζ-circle to the equilibrium shapes are such that $z(\zeta)$ and $\sqrt{z_\zeta(\zeta)}$ are rational functions. This means that the equilibrium shapes are simultaneously quadrature domains in the sense of satisfying identities of the form (2) *and* quadrature domains for arclength in the sense of satisfying identities of the form (7). Indeed, using the formulae in [34], the fluid domains D exterior to the equilibrium bubble configurations shown in Figure 4 satisfy both

$$\oint_{\partial D} h(z)|dz| = -\frac{2\pi}{\Gamma}h(z_1) - \frac{2\pi}{\Gamma}h(-z_1) + \frac{\pi}{\Gamma}(1+\sqrt{1+2\Gamma\gamma^2})h(\infty). \tag{22}$$

where $h(z)$ is some function analytic in the fluid domain D, and

$$\int\int_D h(z)dxdy = \Gamma^2 h(z_1) + \Gamma^2 h(-z_1) \tag{23}$$

where $h(z)$ is some function analytic in the fluid domain D and decaying sufficiently fast at infinity. In (22) and (23), $z_1 = z_1(\Gamma, \gamma)$ is some algebraic function of the physical parameters Γ and γ.

This new understanding has led to a range of new mathematical results for this class of flows (e.g. [24]) including a simplified representation of the classic exact solutions of Kinnersley [23] for waves on fluid sheets. Moreover, all these new results are automatically applicable to a quite separate physical problem in electrophysics involving the shaping of conducting metal jets using electric fields [35] which has identical governing equations.

4.5. Steady Hele-Shaw flows with surface tension

Exact solutions for the equilibrium shapes of simply-connected blobs and bubbles in a Hele-Shaw flow where there is non-zero interfacial tension and the flow is driven by quadrupoles (or higher order poles) have been found by Entov *et al.* This problem also admits equilibrium shapes which are images of the unit ζ-circle under conformal mappings where $\sqrt{z_\zeta(\zeta)}$ is a rational function so that the equilibria can be interpreted as quadrature domains for arclength (but are *not*, in general, also quadrature domains in the usual sense). The solutions of Entov *et al* have been generalized in various directions [36] (for example, to doubly-connected fluid configurations). The mathematical similarities and differences between the two physically-distinct problems of §4.4 and 4.5 have also been discussed [36].

4.6. Viscous sintering

Hopper [37] found a remarkable exact solution for the surface tension-driven coalescence of two near-circular viscous fluid blobs. This is the planar analogue of the two-sphere coalescence "unit problem" which is an important microscale model of an industrially-important manufacturing process known as *viscous sintering.* Howison [25] has also compiled a comprehensive list of related references.

Hopper's mathematical model is as follows. In a time-evolving region $D(t)$, of incompressible fluid of viscosity μ, a streamfunction ψ satisfies

$$\nabla^4\psi = 0 \text{ in } D(t), \tag{24}$$

so that $\psi = \mathrm{Im}[\bar{z}f(z,t) + g(z,t)]$ for some $f(z,t)$ and $g(z,t)$ analytic in $D(t)$. On the boundary of the fluid,

$$-pn_i + 2\mu e_{ij}n_j = \kappa n_i \text{ and } V_n = \mathbf{n}.\nabla^\perp\psi \text{ on } \partial D(t) \tag{25}$$

where κ is the boundary curvature, p is the fluid pressure and e_{ij} is the fluid rate-of-strain tensor. V_n denotes the normal velocity of the boundary. Crowdy [38] [44] has shown explicitly that the dynamics of these equations can, in certain circumstances, preserve quadrature domains. Reappraising Hopper's work within this framework has led to generalizations of his solutions.

As an initial sinter compact of touching particles is heated, the particles coalesce and the compact densifies as the interparticulate pores close up under the effects of surface tension. In certain circumstances, the above mathematical problem admits exact solutions in the form of time-evolving quadrature domains. Figures 5 and 6 show time sequences of the sintering of two doubly-connected packings of near-circular viscous blobs computed using the methodology presented in [20]. The sequences are shown up to the time at which the central pore has closed up. Figures 5 and 6 are calculated using conformal mappings, dependent on just a finite set of time-evolving parameters, based on the Schottky-Klein prime function representation (9) [20]. Indeed, Figure 5 was computed using conformal mappings

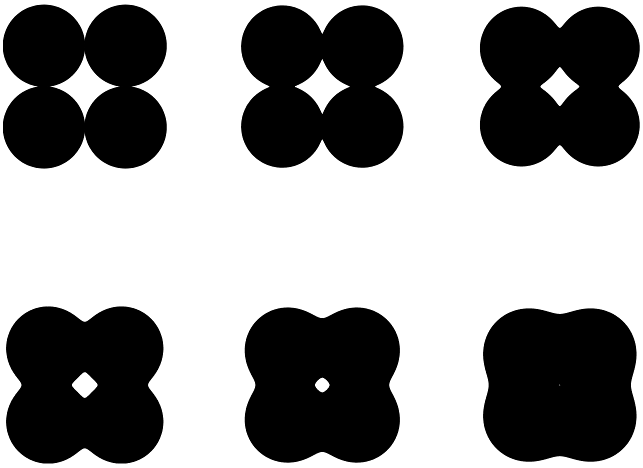

FIGURE 5. Viscous sintering of four near-cylindrical viscous blobs in an initially doubly-connected configuration up to the time of pore closure. Times shown are $t = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.14$.

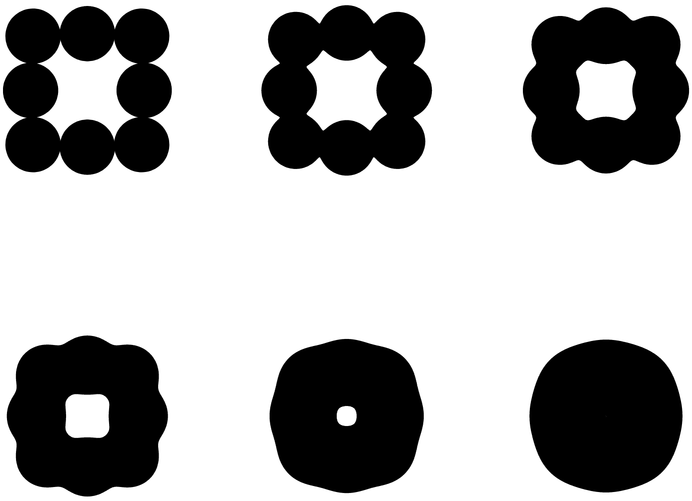

FIGURE 6. Viscous sintering of a looser square packing of eight near-circular viscous blobs up to the time of pore closure. Times shown are $t = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1.36.

of precisely the form (16) with $N = 4$, thus by virtue of the association with quadrature domains, the full dynamics of the problem is reduced to the solution of just three ordinary differential equations for $a(t), R(t)$ and $\rho(t)$. The evolution in Figure 5 has also been computed using elliptic function theory by Richardson [40] and using purely numerical methods by Van de Vorst [39].

4.7. Bubbles in Stokes flows

Quadrature domain theory can also be used to understand a range of exact solutions for bubbles in ambient Stokes flows. An example is the work of Tanveer and Vasconcelos [41] who consider time-evolving bubbles in ambient straining and shear flows (among others). The latter exact solutions can be generalized to the case of compressible bubbles [42] and to steady two-bubble configurations [43]. Figure 4.7 shows various steady two-bubble configurations placed in an ambient flow where the Goursat functions have the far-field form

$$\begin{aligned} f(z) &\sim f_3 z^3 + f_1 z + O(z^{-1}), \\ g'(z) &\sim g_4 z^4 + g_2 z^2 + O(z^{-1}) \end{aligned} \tag{26}$$

where f_3, f_1, g_4 and g_2 are some parameters dictated by the imposed far-field conditions. The fluid domains exterior to the various two bubble configurations in Figure 4.7 are doubly-connected, unbounded quadrature domains corresponding to Cauchy transforms of the form

$$C(z) = A_\infty z + \frac{A_0}{z} \tag{27}$$

where the parameters A_∞ and A_0 depend on the far-field parameters in (26). The domains in Figure 4.7 were constructed using conformal maps based on the Schottky-Klein prime function (11) for the genus 1 case.

5. Discussion and future directions

It has been seen that quadrature domain theory arises in a surprisingly broad range of distinct physical contexts, even just within the field of fluid dynamics. It is to be expected that the same will be true of other disciplines (e.g. plane elasticity, electrostatics). As a result of this interpretation in terms of quadrature domain theory, new mathematical results have been found. The abstraction of quadrature domain theory can immediately give invaluable insight into the scope of what is possible mathematically.

There is no doubt that there are many other areas where quadrature domain theory will be found, in future, to have relevance. One unifying observation is that *all* of the time-evolving free boundary problems admitting exact solutions just described, whatever the governing physics, have Cauchy transforms obeying a partial differential equation of the form

$$\frac{\partial C(z,t)}{\partial t} + \frac{\partial I(z,t)}{\partial z} + \sigma_2(z,t)C(z,t) = R(z,t), \; z \notin D(t) \tag{28}$$

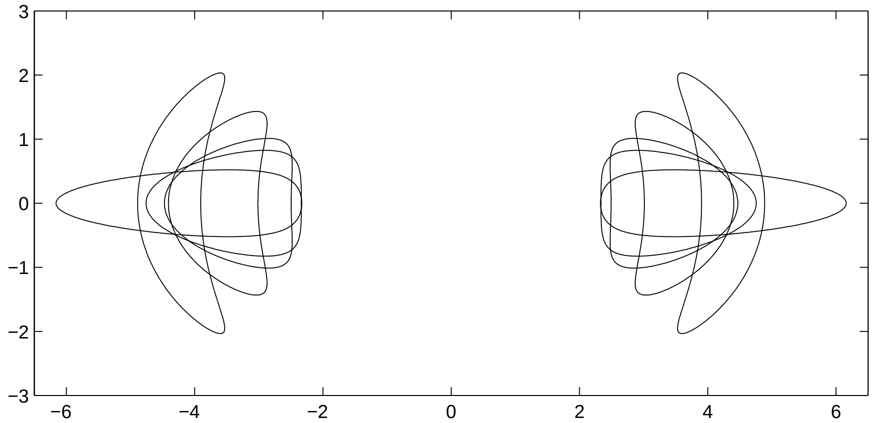

FIGURE 7. Configurations of two steady bubbles in an ambient Stokes flow of the form (26). Different shapes correspond to different choices of the far-field parameters. The domains exterior to the two bubbles are unbounded doubly-connected quadrature domains with Cauchy transforms given by (27).

where

$$I(z,t) = \int\int_{D(t)} \frac{\sigma_1(z',t)dx'dy'}{z'-z}, \tag{29}$$

for various choices of $\sigma_1(z,t)$ and $\sigma_2(z,t)$ which are analytic in $D(t)$ and $R(z,t)$ which is meromorphic in $D(t)$. Crowdy [44] gives details as to why such an equation can be expected to preserve the rational character of $C(z,t)$. The choice $\sigma_1(z,t) = \sigma_2(z,t) = 0$ and $R(z,t) = \sum_{j=1}^{M} \frac{Q_j(t)}{z-z_j}$ gives the case of Hele-Shaw flow driven by sources/sinks; $\sigma_1(z,t) = \frac{b^2\rho g}{12\mu}, \sigma_2(z,t) = 0 = R(z,t)$ gives the case with gravity; $\sigma_1(z,t) = \frac{b^2\rho\omega^2}{12\mu}z$, $\sigma_2(z,t) = R(z,t) = 0$ gives flows in a Hele-Shaw cell rotating with angular velocity ω; the choice $\sigma_1(z,t) = R(z,t) = 0$ and $\sigma_2(z,t) = \frac{\dot{b}}{b}$ gives the case of "squeeze flow" in Hele-Shaw cell where $b(t)$ is the separation of the plates; $\sigma_1(z,t) = -2f(z,t)$, $\sigma_2(z,t) = R(z,t) = 0$ gives the case of viscous sintering; $\sigma_1(z,t) = -2f(z,t)$, $\sigma_2(z,t) = 0$ and

$$R(z,t) = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2g'(z',t)}{z'-z}dz' \tag{30}$$

corresponds to the case of bubbles placed in singular Stokes flows.

Intriguingly, (28) has recently been found to give rise to a tantalizing theoretical link between fixed and free boundary problems. Fokas [45] has introduced a flexible new transform method that is applicable not only to mixed linear boundary value problems in fixed domains but also to integrable nonlinear problems. The method involves the introduction of a differential 1-form which is closed if and only if the governing field equation holds in the domain. The closure of this form implies what Fokas calls a "global relation". If Fokas's algorithmic method is applied to the free boundary problems described herein, the corresponding global relation turns out to be precisely (28). Thus, considering the evolution of $C(z,t)$ in the solution of these problems is no longer arbitrary but becomes a natural *consequence* of a general algorithmic method that applies not only to free boundary problems but to a broad range of applied mathematical problems. This perspective might very well prove to be valuable in future.

Another connection to integrable systems theory has recently arisen in the work of Wiegmann and Zabrodin [46]. They find a theoretical connection between the problem of reconstructing a domain from its harmonic moments and dispersionless integrable hierarchies. The connection applies to very general domains but it is conceivable that quadrature domains have some interpretation as special solutions or reductions of these hierarchies. A tempting connection of quadrature domains with "finite-gap solutions" of nonlinear integrable systems is irresistible but still largely intuitive at present.

Concerning three-dimensional results, there is broad scope for future results there too. These will be far more important for realistic application. The analysts are already paving the way for possible applications by determining what is mathematically possible in higher dimensions. See Shapiro [1] for a discussion of this. As an example, Dritschel and co-workers [47] have recently found a fascinating practical application to the problem of modelling three-dimensional multi-vortex interactions in geostrophic flows of the fact that the exterior potential generated by a uniform ellipsoid is equivalent to that induced by a non-uniform two-dimensional "focal ellipse" [1] [48]. That is, an ellipsoid satisfies a higher-dimensional analogue of (6). Such ideas lie at the heart of quadrature domain theory.

Acknowledgment

The author wishes to sincerely thank Professor Harold Shapiro for his continued interest and support for the author's work over the last few years.

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