A Note on the Linear Stability of Burgers Vortex

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A two-parameter family of analytical solutions of the linearized equations for axially dependent disturbances to the three-dimensional base strain field associated with the well-known axisymmetric Burgers vortex is presented. The solutions are valid asymptotically at large axial distances from the stagnation point. By a formal perturbation analysis, perturbative solutions are also found for disturbances to the Burgers vortex for small Reynolds numbers. The solutions are believed to provide important insights into the nature of the as-yet-unsolved problem of the linear stability of Burgers vortex to axially varying disturbances.

1. Introduction

The axisymmetric Burgers vortex represents one of the few known exact solutions to the full Navier–Stokes equations; however, very little has been deduced about its stability properties since its discovery nearly 50 years ago [1]. Given the extensive use of the vortex as a model of the fine-scale structure of turbulence, its stability properties are of great importance. The vortex consists of a pure swirl flow superposed on an irrotational base strain flow. The flow is incompressible. In cylindrical coordinates (in which lengths

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are nondimensionalized with respect to the Burgers length scale $\sqrt{\nu/a}$ and times with respect to a^{-1}, where a is the strain rate of the background flow field) the solution can be written

$$\mathbf{u}(r,\theta,z) = (-r, V_{\mathrm{B}}(r), 2z), \tag{1.1}$$

where

$$V_B(r) = \frac{\Gamma}{2\pi r}(1-e^{-r^2/2}) \equiv \frac{\Gamma}{2\pi}\tilde{V}_{\mathrm{B}}(r) \tag{1.2}$$

and Γ is the circulation. We define the Reynolds number to be

$$\mathrm{Re} = \frac{\Gamma}{2\pi\nu},$$

where ν is the viscosity of the fluid.

Robinson and Saffman [2] and, more recently, Prochazka and Pullin [3] have investigated the linear stability of the vortex to a general disturbance in the plane perpendicular to the axial straining direction and found it to be stable, at least for moderately high Reynolds numbers. Leibovich and Holmes [4] analyzed the global stability of the vortex and showed it to be globally unstable for all Reynolds numbers. These results say nothing about the linear stability of the vortex to the important class of z-dependent disturbances, and no study of this, either analytical or numerical, seems to have been carried out before. This is probably due to the difficulty in even *formulating* the linear stability problem—the classical notion of "wavenumber" typically associated with Fourier-mode eigenfunctions is not available owing to the lack of translational symmetries of the base strain flow on which the Burgers vortex is superposed. Rather than viewing this as a drawback, this note *exploits* the nonautonomous nature of the linearized disturbance equations to glean important analytical information on the large-z behavior of solutions. We also note that the two-dimensional linear stability of the related Burgers vortex layer has recently received attention [5] but again, the important question of the its three-dimensional linear stability was not broached.

To elucidate our approach, consider the procedure for analyzing the linear stability of a two-dimensional Blasius boundary layer [6]. Suppose x is the coordinate along the wall and y is the coordinate perpendicular to the wall. In this case the linearized equations are also not autonomous in x, but by use of a parallel-mean flow assumption [6], the equations can be approximated by an autonomous set (especially at large Re—see [6]). The approximate equations then admit the following eigenfunctions for the streamfunction

$$\psi(x,y,t) = f(y)e^{ikx}e^{-i\omega t}. \tag{1.3}$$

Fitting the boundary conditions for $f(y)$ (i.e., on the wall at $y=0$ and at $y \to \infty$) then provides an eigenvalue relation between k and ω. For Burgers vortex, owing to the nonautonomous nature of the linearized partial differential equations, eigenfunctions analogous to the Fourier modes above are not available in general. However, in this note, we explicitly find a two-parameter family of self-consistent large-z asymptotic solutions of the linearized partial differential equations for small Reynolds numbers. For $\mathrm{Re}=0$ the solutions have an algebraic dependence on z as $z \to \infty$. Fitting the appropriate boundary conditions at $r=0$ and $r \to \infty$ then provides the eigenvalue relation between the frequency and the exponent of z in the asymptotic solutions as expected by analogy with the Blasius boundary-layer analysis.

The principal aim of this note is to present our analytical observations on the structure of a class of solutions of the linearized disturbance equations about the Burgers vortex for small Reynolds numbers. However, we go further and conjecture some possible implications of these observations. In the discussion in Section 4 we use the evidence of the explicit large-z solutions found here to put the case for a *spatial mode analysis* of the linear stability of the Burgers vortex to axially varying disturbances and conjecture the possible role played in such an analysis by the solutions found here. In particular, a spatial mode analysis would essentially involve causing a general oscillatory disturbance at some z-station near the stagnation point at the origin and observing whether the disturbances grow spatially as they are convected with the flow to $z \to \infty$. Clearly, the possible behavior of solutions as $z \to \infty$ is then of crucial interest and is fundamental to understanding the linear stability problem. In Section 2 we argue that, for perturbations to the base strain field with no vortex (corresponding to $\mathrm{Re}=0$), there are two fundamental behaviors of solutions as $z \to \infty$: some solutions grow exponentially with z, while the remainder have milder (less-than-exponential) behavior as $z \to \infty$. The existence of the latter class of solutions is demonstrated by explicit construction, and the subset of such solutions found here is shown to have an *algebraic* dependence on z as $z \to \infty$. It is argued that it is these solutions (and not the exponentially growing solutions) that are relevant to the linear stability analysis. By a formal perturbation procedure, similar explicit large-z asymptotic solutions can be found for perturbations to weak Burgers vortices (small Re). This is done in Section 3. It does not seem to be possible to derive explicit analytic forms for the exponential solutions.

Because the family of solutions presented here is not derived in any systematic way (so that there may well be other behaviors at infinity that we have not identified) it is not possible to make any *definite* statements on the linear (spatial) stability of the Burgers vortex for small Reynolds numbers, but some informed speculations on how to formulate a numerical treatment of the problem can now at least be made on the basis of this analysis.

2. Large-z solutions for Re = 0

It is clearly sufficient to consider the half-space $r \in [0,\infty)$, $z \in [0,\infty)$. The solution method is straightforward: an ansatz for large-z asymptotic solutions to the linearized disturbance equations is made. Assuming the ansatz, certain terms in the linearized equations are shown to be asymptotically negligible at large z. Solutions of the resulting asymptotic equations (satisfying appropriate boundary conditions at $r = 0$ and $r \to \infty$) are then explicitly found having the form assumed initially. Thus, such solutions are *consistent* large-z solutions of the original equations. This is a standard dominant balance argument [7]. The velocity field is written

$$\mathbf{u}(r,z,t) = (-r + u(r,z,t), V_{\mathrm{B}}(r) + v(r,z,t), 2z + w(r,z,t)) \quad (2.1)$$

and the pressure field

$$P(r,z,t) = P_{\mathrm{B}}(r,z) + p(r,z,t), \quad (2.2)$$

where $u(r,z,t), v(r,z,t), w(r,z,t)$, and $p(r,z,t)$ represent the perturbation quantities to be determined. $P_{\mathrm{B}}(r,z)$ represents the pressure field associated with the steady Burgers vortex solution. For simplicity (and through lack of an analogue to Squire's theorem for this case) we simply assume that the solutions have no azimuthal dependence. This is permissible by the axisymmetry of the Burgers vortex and strain field. Substituting into the Navier-Stokes equations and linearizing, the nondimensionalized evolution equations become

$$\frac{\partial u}{\partial t} - u - r\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial z} - \mathrm{Re}\left(\frac{2\tilde{V}_{\mathrm{B}}(r)}{r}\right)v$$

$$= -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \quad (2.3)$$

$$\frac{\partial v}{\partial t} - v - r\frac{\partial v}{\partial r} + 2z\frac{\partial v}{\partial z} + Re\left(\frac{\partial \tilde{V}_{\mathrm{B}}}{\partial r} + \frac{\tilde{V}_{\mathrm{B}}}{r}\right)u = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \tag{2.4}$$

$$\frac{\partial w}{\partial t} + 2w - r\frac{\partial w}{\partial r} + 2z\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \tag{2.5}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0. \tag{2.6}$$

Note that the solution structure is most clearly seen by working with the above equations. Thus we have deliberately avoided the alternative stream-function-vorticity formulation. Any solutions for u, v, and w must be regular at $r = 0$. Since we seek perturbations to the Burgers vortex where the vorticity decays exponentially as $r \to \infty$, we require all components of the *perturbation vorticity* also to decay exponentially as $r \to \infty$. A *sufficient* (but not necessary) condition is that u, v, and w decay exponentially. Thus, for our purposes, we impose the boundary condition on the perturbation velocities that they decay exponentially as $r \to \infty$.

We make the following ansatz for large-z asymptotic solutions:

$$\begin{aligned} u(r,z,t) &\sim \bar{u}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}}, & v(r,z,t) &\sim \bar{v}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}}, \\ w(r,z,t) &\sim \bar{w}(r)e^{-\mu t}\frac{1}{z^{\sigma}}, & p(r,z,t) &\sim \bar{p}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}}, \end{aligned} \tag{2.7}$$

where μ, σ are some (generally complex) parameters. It is understood throughout that the *real part* of all functions should be taken to obtain a physical solution—by linearity this can always be done. *Assuming* the ansatz, it is clear that as $z \to \infty$ the $\partial^2/\partial z^2$ terms in Equations (2.3)–(2.5) can be consistently neglected with respect to the other terms, as can the $\partial p/\partial z$ in Equation (2.5). All neglected terms in each equation are $O(1/z^2)$ (i.e., small for $z \gg 1$) compared to the terms retained. Note also that since z has been nondimensionalized with respect to the Burgers length scale, the asymptotic solutions are valid for (dimensional) $z \gg \sqrt{\nu/a}$. Note that the alternative of balancing z-advection with z-diffusion in any of the momentum equations (2.3)–(2.5), e.g.,

$$+2z\frac{\partial u}{\partial z} \sim +\frac{\partial^2 u}{\partial z^2} \tag{2.8}$$

is likely to lead to perturbation velocities *growing exponentially* with z as $z \to \infty$.

Substituting the above ansatz for $w(r,z,t)$ into the *asymptotic* version of (2.5) yields the following ordinary differential equation (o.d.e.) for $\overline{w}(r)$

$$\frac{d^2\overline{w}}{dr^2} + \left(r + \frac{1}{r}\right)\frac{d\overline{w}}{dr} + (\mu - 2 + 2\sigma)\overline{w} = 0. \tag{2.9}$$

In principle, if $\overline{w}(r)$ (satisfying the boundary conditions) can be determined from (2.9), (2.6) must be solved for a $\overline{u}(r)$, which also satisfies the boundary conditions. If an appropriate $\overline{u}(r)$ can be found, the asymptotic version of (2.4) then provides an o.d.e. for $\overline{v}(r)$. Finally, if a suitable $\overline{v}(r)$, satisfying the boundary conditions, can be found, the asymptotic version of (2.3) can be directly integrated to give the corresponding $\overline{p}(r)$. It remains to see if appropriate solutions to the o.d.e.'s can be determined. Equation (2.9) can be identified with a confluent hypergeometric equation and, in the notation of [8], the solution can be written

$$\overline{w}(r) = M\left[\frac{(\mu - 2 + 2\sigma)}{2}; 1; -\frac{r^2}{2}\right], \tag{2.10}$$

where M is the confluent hypergeometric function, regular at the origin. The general asymptotic behavior of this function as $r \to \infty$ is

$$\begin{aligned} M\left[a; b; -r^2/2\right] = {} & \left(-\frac{r^2}{2}\right)^{-a} e^{ia\pi}\frac{\Gamma(b)}{\Gamma(b-a)} \\ & \times\left[\sum_{m=0}^{P-1}\frac{[a]_m[1-a-b]_m}{m!}\left(\frac{r^2}{2}\right)^{-m} O\left(\frac{1}{r^{2P}}\right)\right] \\ & + e^{-r^2/2}\left(-\frac{r^2}{2}\right)^{a-b}\frac{\Gamma(b)}{\Gamma(a)} \\ & \times\left[\sum_{m=0}^{Q-1}\frac{[b-a]_m[1-a]_m}{m!}\left(-\frac{r^2}{2}\right)^{-m} + O\left(\frac{1}{r^{2Q}}\right)\right], \end{aligned} \tag{2.11}$$

where $[a]_m = a(a+1)\ldots(a+m-1)$. The requirement of *exponential* decay as $r \to \infty$ gives the eigenvalue condition $\Gamma(b-a)^{-1} = 0$, i.e., $(b-a) = -k$, where $k = 0, 1, 2\ldots$ (using well-known properties of $\Gamma(z)$). Using the parameters in (2.10), the eigenvalue condition is

$$\mu = 2k - 2\sigma + 4. \tag{2.12}$$

Using Kummer's transformation to identify the solutions in terms of the generalized Laguerre polynomials $L_k^{(n)}(r^2/2)$, to within normalization,

$$\overline{w}(r) = e^{-r^2/2} L_k^{(0)}\left(r^2/2\right), \qquad k = 0, 1, 2 \ldots, \tag{2.13}$$

with μ as in (2.12). The function $\bar{u}(r)$ must now be deduced from (2.6). It is easily shown that

$$\bar{u}(r) = \sigma U(r) \qquad \text{where } U(r) \equiv \frac{1}{r}\int_0^r \tilde{r}\overline{w}(\tilde{r})\, d\tilde{r}. \tag{2.14}$$

In general, for arbitrary choices of function $\overline{w}(r)$, regular at $r = 0$ and exponentially decaying as $r \to \infty$, the function obtained by integration as in (2.14) clearly cannot be expected to be exponentially decaying. However, we now illustrate that this is *not* the case for $\overline{w}(r)$ having the special form (2.13) provided $k \geq 1$. Substituting from (2.13) in (2.14) yields

$$U(r) = \frac{1}{r}\int_0^r \tilde{r} e^{-\tilde{r}^2/2} L_k^{(0)}\left(\tilde{r}^2/2\right) d\tilde{r} = \frac{1}{r}\sum_{j=0}^{k} \frac{b_j}{2^j} I_j(r), \tag{2.15}$$

where we denote the coefficients of the kth-order Laguerre polynomial $L_k^{(0)}(r^2/2)$ by $\{b_j \mid j = 0..k\}$ so that

$$L_k^{(0)}\left(r^2/2\right) = \sum_{j=0}^{k} \frac{b_j}{2^j} r^{2j} \tag{2.16}$$

and we define

$$I_j(r) = \int_0^r \tilde{r}^{2j+1} e^{-\tilde{r}^2/2} d\tilde{r}. \tag{2.17}$$

It can be shown using integration by parts that

$$I_j(r) = f_j(r^2) e^{-r^2/2} + 2^j j! \qquad \forall j \geq 0 \tag{2.18}$$

for some polynomial $f_j(r^2)$. Thus

$$U(r) = \frac{1}{r}\sum_{j=0}^{k} \frac{b_j}{2^j} f_j(r^2) e^{-r^2/2} + \frac{1}{r}\sum_{j=0}^{k} b_j j! \tag{2.19}$$

from which, without further inspection, it might be concluded that $U(r) \sim 1/r$ as $r \to \infty$. But remarkably,

$$b_j = \frac{(-1)^j}{j!j!} \frac{k!}{(k-j)!}, \tag{2.20}$$

which implies

$$\sum_{j=0}^{k} b_j j! = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \equiv 0 \qquad \forall k \geq 1, \tag{2.21}$$

which then implies that $U(r)$ is indeed exponentially decaying as $r \to \infty$ (for all $k \geq 1$) as required to satisfy the boundary conditions. Also, it is clear that $\bar{u}(r)$ is regular at $r = 0$. It still remains to establish that an appropriate $\bar{v}(r)$ can be found. Substituting the ansatz for $v(r, z, t)$ and $u(r, z, t)$ into the asymptotic form of (2.4) yields

$$\frac{d^2 \bar{v}}{dr^2} + \left(r + \frac{1}{r} \right) \frac{d\bar{v}}{dr} + \left(\mu + 3 + 2\sigma - \frac{1}{r^2} \right) \bar{v} = (\mathrm{Re}) e^{-r^2/2} \bar{u}. \tag{2.22}$$

Even more remarkably, it can be shown that the spectrum of the self-adjoint linear differential operator (LDO) in (2.9) is a subset of the spectrum of the self-adjoint LDO on the left-hand side of (2.22). Therefore, if (2.12) holds, the solution to the *homogeneous* equation (2.22), which satisfies the boundary condition at $r = 0$ and $r \to \infty$, is in fact given by

$$\bar{v}(r) = r M \left[\frac{\mu + 4 + 2\sigma}{2}; 2; -\frac{r^2}{2} \right] \tag{2.23}$$

or, again using Kummer's transformation (to within normalization),

$$\bar{v}(r) = r e^{-r^2/2} L_{k+2}^{(1)}(r^2/2). \tag{2.24}$$

Thus, for a $\bar{v}(r)$ (satisfying the boundary conditions) to exist, a Fredholm alternative compatibility condition will have to be satisfied by the inhomogeneous term of (2.22), namely,

$$\langle (\mathrm{Re}) e^{-r^2/2} \bar{u}(r), r e^{-r^2/2} L_{k+2}^{(1)}(r^2/2) \rangle = 0, \tag{2.25}$$

where angle brackets denote the inner product defined by

$$\langle f(r), g(r)\rangle = \int_0^\infty f(r)g^*(r)e^{r^2/2}rdr \tag{2.26}$$

associated with the self-adjoint LDO on the left-hand side of (2.22). In general, for $\mathrm{Re} \neq 0$, (2.25) will not hold and there will be *no* solution for $\bar{v}(r)$ satisfying the boundary conditions and hence *no* solution having the form of the assumed ansatz. However, for $\mathrm{Re} = 0$, the equation for $\bar{v}(r)$ can be solved and is given (to within normalization) by (2.24). The corresponding $\bar{p}(r)$ then follows immediately from integration of the asymptotic form of (2.3). We have therefore succeeded in finding a family of consistent large-z solutions (for $\mathrm{Re} = 0$) parametrized by integers $k \geq 1$ and the complex parameter σ, with μ given by the eigenvalue relation (2.12) having the form originally hypothesized in the ansatz (2.7) and satisfying the required boundary conditions at $r = 0$ and $r \to \infty$.

3. Perturbation theory for small Re

Despite the string of fortuitous circumstances that led to the identification of the above two-parameter family of solutions, it is not expected that these represent isolated solutions that exist only for $\mathrm{Re} = 0$. Indeed we expect to be able to find perturbative solutions about the $\mathrm{Re} = 0$ results valid for small nonzero Re, although they will clearly not have the simple form, given in (2.7), as already noted by the failure of such solutions to satisfy the secularity condition (2.25). The relevant perturbation analysis is now outlined in this section. The analysis is not only interesting as an example of a tractable perturbation analysis on a system of linear partial differential equations (an analysis with some very interesting properties—in particular the eigenvalue relation for small Re can be determined to all orders and summed), but it also provides valuable insights into how the large-z solutions found in Section 2 change when a weak Burgers vortex is superposed on the base strain field. The zeroth-order solution is taken to be given by

$$\begin{aligned}
w(r,z,t) &= \overline{w}(r)e^{-\mu_0 t}\frac{1}{z^\sigma} \\
v(r,z,t) &= \bar{v}(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\
u(r,z,t) &= \sigma U(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\
\mu_0 &= 2k - 2\sigma + 4,
\end{aligned} \tag{3.1}$$

where we define

$$\overline{w}(r) = e^{-r^2/2}L_k^{(0)}(r^2/2)$$

$$\bar{v}(r) = re^{-r^2/2}L_{k+2}^{(1)}(r^2/2)$$

$$U(r) = \frac{1}{r}\int_0^r \tilde{r}e^{-\tilde{r}^2/2}L_k^{(0)}(\tilde{r}^2/2)d\tilde{r}.$$

It is taken to be understood that the solutions sought are *asymptotic* solutions valid at large z, although we use = rather than ~ throughout. We now seek to continue these solutions for small nonzero Re. In the following analysis, special care must be taken to ensure that we always find a perturbation to the above zeroth-order solution for given k and σ, and that we do not add onto the perturbed solution any contributions from neighboring solutions. This will ensure uniqueness of the perturbed solution.

We now attempt to solve the *same* large-z asymptotic equations as in Section 2—in other words, the same dominant balance is expected to be good for solutions for small Re. First it is observed that the large-z asymptotic equation for $w(r,z,t)$ is independent of Reynolds number. Thus the perturbed solution for $w(r,z,t)$ must also have the form

$$w(r,z,t) = e^{-r^2/2}L_k^{(0)}(r^2/2)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}}}, \tag{3.2}$$

where we take

$$\mu = \mu_0 + \mathrm{Re}\,\mu_1 + \mathrm{Re}^2\,\mu_2 + \dots. \tag{3.3}$$

Note also that the eigenvalue condition continues to be

$$\mu = 2k - 2\tilde{\sigma} + 4, \qquad k \geq 1. \tag{3.4}$$

Thus we immediately conclude that

$$\tilde{\sigma} = \sigma - \mathrm{Re}\frac{\mu_1}{2} - \mathrm{Re}^2\frac{\mu_2}{2} - \dots. \tag{3.5}$$

It is seen that the perturbed expression for $w(r,z,t)$ has exactly the same functional form as the zeroth-order solution but with perturbed parameters. By continuity, the same is true of the perturbed $u(r,z,t)$, which can be written

$$u(r,z,t) = \left(\frac{\tilde{\sigma}}{r}\int_0^r \tilde{r}e^{-\tilde{r}^2/2}L_k^{(0)}(\tilde{r}^2/2)d\tilde{r}\right)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}+1}}. \tag{3.6}$$

This is again fortuitous because it means that exactly the same arguments as used before ((2.15)–(2.21)) can be applied to the perturbed $u(r,z,t)$ to demonstrate that it is a function decaying exponentially as $r \to \infty$. It remains to determine $\mu_1, \mu_2 \ldots$, which are derived from solvability conditions for the perturbed $v(r,z,t)$. We note at this point that while solving for $v(r,z,t)$, care is taken not to add into the solution any terms having the following form

$$\frac{C\mathrm{Re}^j \log z\bar{v}(r)}{z^{\sigma+1}}e^{-\mu t} \tag{3.7}$$

for any integer j and any constant C. This is clearly the $O(\mathrm{Re}^j)$ term in a small Reynolds number expansion of

$$\frac{\bar{v}(r)}{z^{\hat{\sigma}+1}}e^{-\mu t} \qquad \text{where } \hat{\sigma} = \sigma - C\,\mathrm{Re}^j. \tag{3.8}$$

It is straightforward to show that adding in any such solution would correspond to altering the parameter σ in the zeroth-order solution; however, it is assumed that a particular value of σ is specified *a priori*.

As a convenient shorthand we define the following linear operators:

$$\begin{aligned}
M(r,z) &\equiv \frac{\partial^2}{\partial r^2} + \left(r + \frac{1}{r}\right)\frac{\partial}{\partial r} + \left(\mu + 1 - \frac{1}{r^2}\right) - 2z\frac{\partial}{\partial z} \\
M_0(r,z) &\equiv \frac{\partial^2}{\partial r^2} + \left(r + \frac{1}{r}\right)\frac{\partial}{\partial r} + \left(\mu_0 + 1 - \frac{1}{r^2}\right) - 2z\frac{\partial}{\partial z} \\
M_0(r) &\equiv \frac{d^2}{dr^2} + \left(r + \frac{1}{r}\right)\frac{d}{dr} + \left(\mu_0 + 1 - \frac{1}{r^2}\right) + 2(\sigma + 1).
\end{aligned} \tag{3.9}$$

Note that the solution of the ordinary differential equation

$$M_0(r)\Omega(r) = 0 \tag{3.10}$$

is

$$\Omega(r) = A\bar{v}(r) \tag{3.11}$$

for some constant A.

Note also how the operator $M_0(r,z)$ acts on functions such as $\Omega(r)(\log^p z/(z^{\sigma+1}))$ (p an integer):

$$M_0(r,z)\left(\Omega(r)\,\frac{\log^p z}{z^{\sigma+1}}\right) = \left(\frac{\log^p z}{z^{\sigma+1}}\right)M_0(r)\,\Omega(r) - 2p\left(\frac{\log^{(p-1)} z}{z^{\sigma+1}}\right)\Omega(r). \tag{3.12}$$

We now write

$$v(r,z,t) = \hat{v}(r,z)e^{-\mu t}, \qquad u(r,z,t) = \hat{u}(r,z)e^{-\mu t}, \tag{3.13}$$

where it is already known from (3.6) that

$$\hat{u}(r,z) = \tilde{\sigma}\frac{U(r)}{z^{\tilde{\sigma}+1}}, \tag{3.14}$$

with $\tilde{\sigma}$ given by (3.5). In terms of this notation, the equation for $\hat{v}(r,z)$ can be written

$$M(r,z)\hat{v}(r,z) = (\mathrm{Re})e^{-r^2/2}\hat{u}(r,z). \tag{3.15}$$

This is a partial differential equation for $\hat{v}(r,z)$ with an $O(\mathrm{Re})$ forcing depending on $\hat{u}(r,z)$. Expanding (3.14) for small Re gives

$$\hat{u}(r,z) = \frac{\sigma U(r)}{z^{\sigma+1}}\left(1+\mathrm{Re}\left(\frac{\mu_1 \log z}{2} - \frac{\mu_1}{2\sigma}\right)\right.$$
$$\left. + \mathrm{Re}^2\left(\frac{{\mu_1}^2\log^2 z}{8} + \log z\left(\frac{\mu_2}{2} - \frac{{\mu_1}^2}{4\sigma}\right) - \frac{\mu_2}{2\sigma}\right)\right). \tag{3.16}$$

We now write

$$\hat{v}(r,z) = v_0(r,z) + \mathrm{Re}\,v_1(r,z) + \mathrm{Re}^2\,v_2(r,z) + \dots. \tag{3.17}$$

Substituting the expansions for μ, $u(r,z,t)$, and $v(r,z,t)$ into (3.15) gives

$$\big(M_0(r,z) + \mathrm{Re}\,\mu_1 + (\mathrm{Re}^2)\,\mu_2 + \dots\big)$$
$$\times\big(v_0(r,z) + (\mathrm{Re})\,v_1(r,z) + (\mathrm{Re})^2 v_2(r,z) + \dots\big)$$
$$= \sigma(\mathrm{Re})e^{-r^2/2}\left(\frac{U(r)}{z^{\sigma+1}}\right)\left(1 + (\mathrm{Re})\left(-\frac{\mu_1}{2\sigma} + \frac{\mu_1 \log z}{2}\right) + \dots\right). \tag{3.18}$$

The leading-order equation obviously gives the zeroth-order solution

$$v_0(r,z) = \frac{\bar{v}(r)}{z^{\sigma+1}}. \tag{3.19}$$

At first order in Re we get

$$M_0(r,z)v_1(r,z) = -\frac{\mu_1\bar{v}(r)}{z^{\sigma+1}} + \frac{\sigma e^{-r^2/2}U(r)}{z^{\sigma+1}}. \tag{3.20}$$

To solve this, we try $v_1(r,z)=\bar{v}_1(r)/(z^{\sigma+1})$ yielding

$$M_0(r)\bar{v}_1(r) = -\mu_1\bar{v}(r) + \sigma e^{-r^2/2}U(r). \tag{3.21}$$

This is an ordinary differential equation for the function $\bar{v}_1(r)$ and by the self-adjointness of the operator $M_0(r)$, a solution for $\bar{v}_1(r)$ satisfying the boundary conditions exists only provided that a Fredholm alternative condition is satisfied by the inhomogenous term in (3.21), namely,

$$\langle \mu_1\bar{v}(r) + \sigma e^{-r^2/2}U(r), \bar{v}(r)\rangle = 0 \tag{3.22}$$

yielding the result

$$\mu_1 = \frac{\langle \sigma e^{-r^2/2}U(r), \bar{v}(r)\rangle}{\langle \bar{v}(r), \bar{v}(r)\rangle}. \tag{3.23}$$

Given this solvability condition, $\bar{v}_1(r)$ can, in principle, be computed as an expansion in the complete set of eigenfunctions $\{rL_p^{(1)}(r^2/2)\exp(-r^2/2) \mid p = 0,1..\}$ if needed. Observe that $\bar{v}_1(r)$ seems to be determined only to within an arbitrary multiple of $\bar{v}(r)$, but adding any amount of the function $\bar{v}(r)$ simply alters the normalization of the zeroth-order solution, which is assumed fixed a priori. At second order in Re we obtain

$$M_0(r,z)v_2(r,z) = -\frac{\mu_1\bar{v}_1(r)}{z^{\sigma+1}} - \frac{\mu_2\bar{v}(r)}{z^{\sigma+1}} - \frac{\mu_1U(r)e^{-r^2/2}}{2z^{\sigma+1}} + \frac{\sigma\mu_1U(r)e^{-r^2/2}\log z}{2z^{\sigma+1}}. \tag{3.24}$$

We try

$$v_2(r,z) = \left(\frac{\log^2 z}{z^{\sigma+1}}\right)\bar{v}_2^{(2)}(r) + \left(\frac{\log z}{z^{\sigma+1}}\right)\bar{v}_2^{(1)}(r) + \left(\frac{1}{z^{\sigma+1}}\right)\bar{v}_2^{(0)}(r) \quad (3.25)$$

Substitution and use of (3.10)–(3.12) yield

$$\left(\frac{\log^2 z}{z^{\sigma+1}}\right)M_0(r)\bar{v}_2^{(2)}(r) + \left(\frac{\log z}{z^{\sigma+1}}\right)M_0(r)\bar{v}_2^{(1)}(r) + \left(\frac{1}{z^{\sigma+1}}\right)M_0(r)\bar{v}_2^{(0)}(r)$$

$$= \left(\frac{\log z}{z^{\sigma+1}}\right)\left(4\bar{v}_2^{(2)}(r) + \frac{\sigma\mu_1 U(r)e^{-r^2/2}}{2}\right) - \frac{\mu_1\bar{v}_1(r)}{z^{\sigma+1}} - \frac{\mu_2\bar{v}(r)}{z^{\sigma+1}}$$

$$- \frac{\mu_1 U(r)e^{-r^2/2}}{2z^{\sigma+1}} + \frac{2\bar{v}_2^{(1)}(r)}{z^{\sigma+1}}. \quad (3.26)$$

Using linearity and equating coefficients of the three different functions of z,

$$M_0(r)\bar{v}_2^{(2)}(r) = 0; \quad (3.27)$$

from this we deduce that

$$\bar{v}_2^{(2)}(r) = A\bar{v}(r) \quad (3.28)$$

for some constant A. Also,

$$M_0(r)\bar{v}_2^{(1)}(r) = 4A\bar{v}(r) + \frac{\sigma\mu_1 U(r)e^{-r^2/2}}{2}. \quad (3.29)$$

The Fredholm alternative condition that an appropriate $\bar{v}_2^{(1)}(r)$ should exist yields the value of the constant A. We can then solve for the function $\bar{v}_2^{(1)}(r)$ as an eigenfunction expansion to within an arbitrary multiple of $\bar{v}(r)$, the kernel function. As previously discussed, if we were to add any of the kernel function to $\bar{v}_2^{(1)}(r)$, we would be adding a term of the form (3.7), which we have disallowed for reasons discussed earlier. This requirement implies that $\bar{v}_2^{(1)}(r)$ is uniquely determined. Finally

$$M_0(r)\bar{v}_2^{(0)}(r) = -\mu_1\bar{v}_1(r) - \mu_2\bar{v}(r) - \frac{\mu_1 U(r)e^{-r^2/2}}{2} + 2\bar{v}_2^{(1)}(r). \quad (3.30)$$

By construction, the first and fourth terms on the right-hand side are orthogonal to $\bar{v}(r)$ but solvability for $\bar{v}_2^{(0)}(r)$ uniquely gives the value of μ_2, i.e.,

$$\mu_2 = -\frac{\langle \mu_1 U(r) e^{-r^2/2}, \bar{v}(r)\rangle}{2\langle \bar{v}(r), \bar{v}(r)\rangle} = -\frac{\mu_1^2}{2\sigma}. \tag{3.31}$$

Again, $v_2^{(0)}(r)$ can be written as an eigenfunction expansion if required. The perturbation calculation was carried out to $O(\mathrm{Re}^3)$ and it became clear that the procedure could in principle be carried out indefinitely. Indeed, the perturbation analysis reveals a particularly interesting structure, which in fact allows the eigenvalue relation to be not only determined to all orders, but also summed. This is primarily a result of the fact that the form of the forcing in (3.15) is known at all orders in Re. It becomes clear that

$$\mu_n = -\frac{\mu_{n-1}}{2}\frac{\langle e^{-r^2/2} U(r), \bar{v}(r)\rangle}{\langle \bar{v}(r), \bar{v}(r)\rangle} = -\frac{\mu_1}{2\sigma}\mu_{n-1} \qquad \forall n \geq 2 \tag{3.32}$$

$$v_n(r,z) = \frac{1}{z^{\sigma+1}} \sum_{j=0}^{n} (\log z)^j v_n^{(j)}(r) \qquad \forall n \geq 2, \tag{3.33}$$

where $v_n^{(j)}(r)$ are some functions of r that can be determined as eigenfunction expansions from the perturbation analysis. The results (3.32) and (3.33) can be formally proved by induction. Using (3.32) in (3.5) it becomes clear that

$$\begin{aligned}
\tilde{\sigma} &= \sigma - \frac{1}{2}\left(\mathrm{Re}\,\mu_1 + \mathrm{Re}^2\,\mu_2 + \mathrm{Re}^3\,\mu_3 + \ldots\right) \\
&= \sigma - \frac{\mu_1 \mathrm{Re}}{2}\left(1 - \mathrm{Re}\frac{\mu_1}{2\sigma} + \mathrm{Re}^2\left(\frac{\mu_1}{2\sigma}\right)^2 + \cdots\right) \\
&= \sigma\left(1 - \frac{\mathrm{Re}\,\mu_1}{2\sigma + \mathrm{Re}\,\mu_1}\right).
\end{aligned} \tag{3.34}$$

The final perturbed solution for the velocity field can therefore be written

$$w(r,z,t) = \bar{w}(r) e^{-\mu t} \frac{1}{z^{\tilde{\sigma}}}$$

$$
\begin{aligned}
v(r,z,t) &= \frac{e^{-\mu t}}{z^{\sigma+1}}\Big(\bar{v}(r) + \mathrm{Re}\,\bar{v}_1(r) \\
&\qquad + Re^2\Big(v_2^{(2)}(r)(\log z)^2 + v_2^{(1)}(r)\log z + v_2^{(0)}(r)\Big) + \ldots\Big) \\
u(r,z,t) &= \tilde{\sigma} U(r) e^{-\mu t}\frac{1}{z^{\tilde{\sigma}+1}} \qquad (3.35) \\
\tilde{\sigma} &= \sigma\left(1 - \frac{\mathrm{Re}\,\mu_1}{2\sigma + \mathrm{Re}\,\mu_1}\right) \\
\mu &= 2k + 4 - 2\tilde{\sigma},
\end{aligned}
$$

with μ_1 in (3.23) and where the functions $\{v_n^{(j)}(r) | 0 \le j \le n, n \ge 2\}$ can be found as eigenfunction expansions if needed.

Although the radius of convergence of the expansion in Re is not known, we do not anticipate any problems with convergence. The fact that the eigenvalue can be found to all orders and has a finite sum lends credence to this. It is straightforward to see that the corresponding perturbation pressure will have the large-z asymptotic form

$$
\begin{aligned}
p(r,z) &= \frac{\bar{p}(r)}{z^{\sigma+1}} + \mathrm{Re}\left[\frac{p_1^{(0)}(r)}{z^{\sigma+1}} + p_1^{(1)}(r)\frac{\log z}{z^{\sigma+1}}\right] \\
&\quad + \mathrm{Re}^2\left[\frac{p_2^{(0)}(r)}{z^{\sigma+1}} + p_2^{(1)}(r)\frac{\log z}{z^{\sigma+1}} + p_2^{(2)}(r)\frac{\log^2 z}{z^{\sigma+1}}\right] + O(\mathrm{Re}^3),
\end{aligned}
\qquad (3.36)
$$

where the functions of r appearing in (3.36) can be obtained by direct integration of the asymptotic form of (2.3).

Finally, note that as Re gets larger it is not expected that the large-z asymptotic assumptions made to simplify the equations will continue to be valid and a numerical study of the full equations will probably be needed to see how these solutions continue for larger Re. However, for *small* Re, we have shown by explicit construction of self-consistent solutions that the asymptotic assumptions were the correct ones to make to find those solutions. The two important results of this section are to note that when $\mathrm{Re} \neq 0$ the solutions become more complicated and do not take the simple separable form as given in (2.7), and also to note how the eigenvalue relation changed for small Re, i.e., to first-order

$$
\mu = 2k + 4 - 2\sigma + \mathrm{Re}\,\sigma\frac{\langle e^{-r^2/2}U(r), \bar{v}(r)\rangle}{\langle \bar{v}(r), \bar{v}(r)\rangle}. \qquad (3.37)
$$

4. Discussion

We now discuss the possible relevance of these solutions to the linear stability problem of Burgers vortex to axially varying perturbations. Using the results of this note we now argue the case for a *spatial mode analysis* (see [9] and references therein). Such analyses are usually more appropriate than a *temporal mode analysis* in stability problems where there is an overall mean flow direction (the z-direction in this case). A suggested stability problem is to find the large-z asymptotic behavior of disturbances forced by a general localized oscillatory perturbation near the stagnation point (cf. the oscillating Schubauer ribbon experiment in boundary-layer stability analysis [6]). In classical spatial mode analyses for flow problems allowing the usual Fourier-mode decomposition (cf. (1.3)), the eigenvalue relation is interpreted as a relation giving the (generally complex) wavenumber k as a function of the real frequency ω, rather than a relation for the (generally complex) frequency ω as a function of the real wavenumber k (temporal mode analysis). The existence of spatially growing modes (e.g., a mode with $\mathrm{Im}[k] > 0$ for some real ω in (1.3)) then implies spatial instability *provided* the group velocity of the spatially growing modes is such that the waves travel *downstream* of the excitation. We conjecture that the proposed exponentially growing modes suggested by the balance in (2.8) propagate *towards* the stagnation point *from* infinity and thus would be discounted physically using some generalized radiation condition. Note that it is clear that if a temporal mode analysis was being carried out, some form of boundary condition at $z \to \infty$ would be needed. In that case, it is not at all clear what form this boundary condition should take. We conjecture that the appropriate boundary condition should be to discount *exponentially* growing solutions, although the reason for this choice is more easily understood (if the radiation condition conjecture is correct) from a spatial mode perspective. Thus ruling out solutions that grow *exponentially* with z as physically irrelevant, then naively inverting the eigenvalue relation (2.12) for σ setting $\mu = i\omega$ for the class of large-z solutions found explicitly in Section 2 yields

$$\sigma = k + 2 - \frac{i\omega}{2}, \qquad k = 1,2\dots. \tag{4.1}$$

Since $\mathrm{Re}[\sigma] = k + 2 > 0$ for $k = 1,2\dots$, implying algebraic *decay* as $z \to \infty$ of *all* the solutions of the form (2.7) forced by a purely oscillatory excitation of frequency ω (note that the analysis of Section 2 made no a priori assumptions on the sign of $\mathrm{Re}[\sigma]$), this *suggests* spatial stability for $\mathrm{Re} = 0$, but since we have not been able to systematically find all modes, no such comprehensive statement can be made. However, it can further be *specu-*

lated that a possible solution (after transients) for $\mathrm{Re} = 0$ to an initial value problem (IVP) with, say, no initial disturbance in $z \geq 0$ and forced by an *appropriate* excitation of single-frequency ω at some z-station near the stagnation point could be written

$$w(r,z,t) \sim \mathrm{Real}\left[e^{-i\omega t} \sum_{k=1}^{\infty} A_k(\omega) \frac{e^{-r^2/2} L_k^{(0)}(r^2/2)}{z^{k+2-i\omega/2}} \right] \qquad \text{as } z \to \infty \quad (4.2)$$

for some $\{A_k(\omega) \mid k = 1,2..\}$, with similar expressions for u and v. By the term "appropriate" we mean a specially manufactured disturbance that will excite (at large z) only those modes that we have explicitly found. Since we have not systematically found *all* modes, we cannot hope to write down the most general large-z asymptotic solution generated by a *general* oscillatory of frequency ω. Note that for $\mathrm{Re} \neq 0$, again naively inverting (3.37) for σ with $\mu = i\omega$ implies that

$$\sigma = \frac{2k + 4 - i\omega}{2 - \mathrm{Re}\Big(\langle e^{-r^2/2} U(r), \bar{v}(r) \rangle / (\langle \bar{v}(r), \bar{v}(r) \rangle) \Big)}. \qquad (4.3)$$

It is then seen from (3.35) (setting $\mu = i\omega$) that by introducing a weak Burgers vortex, the real part of the exponent of algebraic decay of the perturbation swirl velocity v is seen to increase or decrease according to whether the sign of the real quantity

$$\frac{\langle e^{-r^2/2} U(r), \bar{v}(r) \rangle}{\langle \bar{v}(r), \bar{v}(r) \rangle} \qquad (4.4)$$

is positive or negative, while that of u and w remains the same. Thus we might say that the particular $\mathrm{Re} = 0$ solutions given in (3.1) become *more* or *less* spatially stable by the addition of a weak Burgers vortex according to whether the real quantity in (4.4) is positive or negative (note that the quantity in (4.4) depends implicitly on the integer k).

Even if we had systematically found all possible asymptotic behaviors, it would still be necessary to determine, using perhaps some generalized notion of group velocity, which modes propagate downstream of the excitation, i.e., *toward* $z \to \infty$, and in particular that the proposed exponentially growing modes can be genuinely discounted for the physical reasons just conjectured. In general, given the complexity of the equations, a numerical solution of the full IVP will probably be needed to verify or disclaim these conjectures. This would constitute a somewhat formidable undertaking, especially if perturbations with azimuthal dependence are also included, and this is left for future study. In any event, it will be of great interest to see precisely what role the explicit asymptotic solutions found here play in the linear stability problem of Burgers vortex to axially varying perturbations for small Reynolds numbers.

In summary, the analytical observations presented in Sections 2 and 3 throw light on the structure at large axial distances of a certain class of solutions of the linearized disturbance equations about the Burgers vortex. Given the complexity of the equations, it is remarkable that any such explicit analytical insights can be made at all. In this section it has further been argued that these observations are important for providing clues for the formulation of the linear stability problem for the vortex to general three-dimensional disturbances. At the very least, the results allow some definite mathematical questions to be asked, which a future numerical treatment of the linear stability problem might attempt to answer. Certainly they suggest that allowance should be made in any numerical treatment for a *continuous* spectrum associated with the z-direction and a *discrete* spectrum associated with the r-direction (some collocation method using the complete set of Laguerre polynomials found above seems appropriate). The solutions might also provide a useful check for a numerical code. Finally we remark that the results also suggest the possible use of some form of Mellin transform technique as a tool in the numerical study of this problem.

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References

1. J. M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech.* 1:171–199 (1948).
2. A. C. Robinson and P. G. Saffman, Stability and structure of stretched vortices, *Stud. App. Math.* 70:163–181 (1984).
3. A. Prochazka and D. I. Pullin, On the 2-dimensional stability of the axisymmetrical Burgers vortex, *Phys. Fluids* 7:1788–1790 (1995).
4. S. Leibovich and P. Holmes, Global stability of Burgers vortex, *Phys. Fluids* 24:548–549 (1981).

5. K. N. Beronov and S. Kida, Linear two-dimensional stability of a Burgers vortex layer, *Phys. Fluids* 8(4) (1996).

6. M. Gaster, On the generation of spatially growing waves in a boundary layer, *J. Fluid Mech.* 22:433–441 (1965).

7. C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw–Hill, (1978).

8. M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards.

9. P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge Univ. Press, (1981).

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