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Exact Solutions for Steady Capillary Waves on a Fluid Annulus

D. G. Crowdy

Department of Mathematics, Massachusetts Institute of Technology 2-335, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

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Summary. Exact solutions for steady capillary waves on an annulus of swirling irrotational fluid are presented. The solutions have an intimate mathematical connection with the finite amplitude waves on fluid sheets identified by Kinnersley [8]. This mathematical connection is made explicit by first retrieving the solutions of Kinnersley using an extension of a new approach to free surface potential flows with capillarity recently devised by the present author (Crowdy [3]). A much-simplified representation of Kinnersley's original solutions results from the reformulation. The method is then generalized to identify the exact solutions for steady capillary waves on an annulus.

1. Introduction

The fundamental problem of understanding the interaction of inviscid hydrodynamic pressure forces with capillary effects on a free surface is classical—Lord Rayleigh, for example, studied the small amplitude oscillations of a spherical globule of fluid in a dynamically inactive ambient and zero gravity held together by surface tension [9]. The literature concerned with the study of the free surface dynamics of free blobs and bubbles is extensive (see [4] for a list of references). Recently, the author [4] considered some mathematical models of the physical problem of finding the steady-state equilibria of free drops (in zero gravity, surrounded by a dynamically inactive ambient and with surface tension on the boundary) where the shape deformations of the free surface are induced by internal circulations. An understanding of the free surface dynamics in this situation represents an important theoretical paradigm in modelling a large number of physical processes [4].

In a recent paper [3], a new theoretical approach to some two-dimensional free-surface potential flow problems with surface tension was developed. The new approach was used to elucidate the mathematical connection between the classical exact solutions for deep water capillary waves originally found by Crapper [2] and some more recent results by Tanveer [12] which revealed that conformal mapping solutions for a steadily translating bubble have a very specific polar decomposition. In [4], the present author used the same theoretical reformulation to unveil the existence of some new exact solutions involving simply connected flow regions. These include solutions for the steady-state shapes of a bubble in a simple vortical flow, and the steady-state shapes of a droplet of fluid containing a single point vortex.

In this paper, the doubly connected continuation of some of the solutions in [4] is found. To the best of the author's knowledge, these exact solutions do not appear to have been reported before in the literature. Furthermore, in the same way that the new solutions found in [4] have intimate mathematical connections with the deep water capillary wave solutions of Crapper [2], the solutions presented here are intimately related to the extension of Crapper's solutions originally expounded in detail by Kinnersley [8] for finite amplitude waves on fluid sheets.

As a natural first step, in Section 2 of the paper, it is indicated how to generalize the methods presented in [3] to retrieve Kinnersley's symmetric sheet wave solutions. The theory of loxodromic functions is naturally applicable to this case. The solution method is very different from the original scheme used by Kinnersley and leads naturally to a remarkably simple representation of the solutions. A subsidiary result of the present paper therefore is that we have found a much-simplified representation of Kinnersley's sheet wave solutions that is not apparent from the original paper [8].

In Section 3, the ideas of Section 2 are developed to unveil a two-parameter family of exact solutions for steady capillary waves on a fluid annulus which represents a genuinely doubly-connected flow configuration. The solutions can be written in closed form. Typical steady-state shapes of the fluid annulus are plotted, as well as typical flow streamlines.

The new solutions are presented here as a contribution to the mathematical theory of nonlinear free surface problems and while interesting as an example of a paradigmatic Laplace-type free surface problem that admits exact solutions, possible physical applications of the solutions remain to be determined. The solutions certainly enhance our understanding of the way in which hydrodynamic pressure forces interact nonlinearly with free capillary surfaces to produce equilibrium configurations for fluid droplets containing enclosed air bubbles and, as such, the results might be useful as a basis for some "mean-field" model of large-scale dispersive systems containing large numbers of blobs and/or bubbles in the same way that well-known paradigmatic exact vortex solutions (e.g., Burgers vortex, Hill's spherical vortex) have formed the basis for numerous models of the fine-scale structure of turbulence [11]. Moreover, there exists in the literature very few known solutions for free boundary problems involving *two* free surfaces interacting with eachother – this paper presents new examples of such a situation. The solutions bear the virtue of mathematical exactness, which therefore makes them a useful tool for checking numerical codes written to resolve more complicated free boundary problems in the same (doubly connected) geometry when additional physical effects are included and exact results are not available. Further, not only are the solutions exact, but they are interesting in that they are related to well-known classical exact solutions [2] [8] in a different ("water wave") geometry where the physical relevance of the solutions is more easily seen.

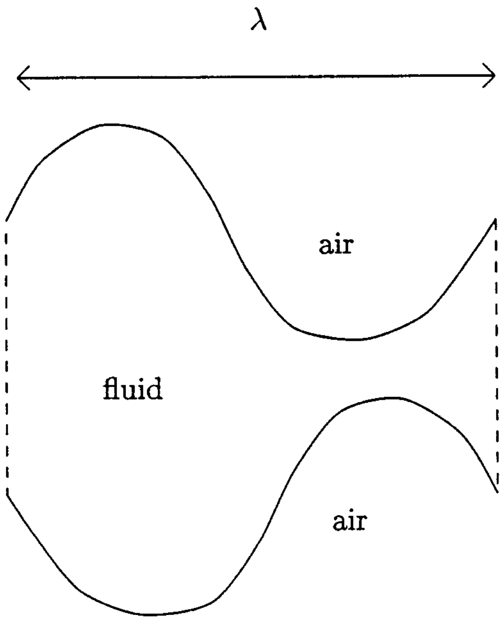

Fig. 1. Symmetric sheet waves.

2. Symmetric Sheet Waves

By a direct extension of the analysis of Crapper [2], Kinnersley [8] found two physically distinct types of steady solutions for waves on sheets of fluid described in [8] as waves of types Ia, Ib, IIa, and IIb. Waves of type Ia and Ib are completely equivalent except for a relabelling of streamlines, as are waves of type IIa and IIb. Type I waves are symmetric, type II waves antisymmetric.

In this section, we show how to generalize the new methodology presented in [3] to the problem of waves on fluid sheets. Kinnersley's symmetric (type I) waves will be systematically retrieved.

First, we assume that the shape of both the upper and lower fluid interfaces are spatially periodic in x with wavelength $\lambda \equiv \frac{2\pi}{k}$. It is well known that this potential flow problem can be reformulated as the problem of finding a conformal mapping function $z(\zeta)$ describing the fluid region. Under this assumption, it is enough to consider the structure of a conformal map from a standard parametric region (in a ζ-plane) to a window of the fluid sheet of length $\frac{2\pi}{k}$. Therefore, consider the conformal mapping function $z(\zeta)$ from the cut annulus $\rho < |\zeta| < 1$ (as shown in Figure 2) in a parametric conformal mapping plane (ζ-plane) to a window of the finite sheet of fluid as shown in Figure 1. Without loss of generality, it can be assumed that the upper interface of the fluid sheet is the image of the circle $|\zeta| = 1$. We assume that the second free surface of the fluid sheet maps from $|\zeta| = \rho$ where $0 < \rho < 1$. It is clear that in this case the

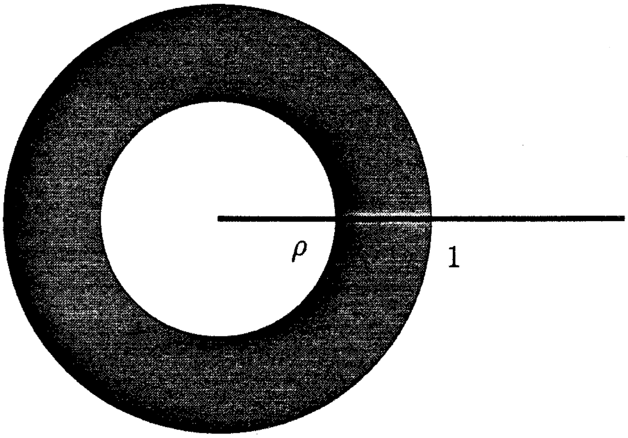

Fig. 2. Parametric ζ-plane.

conformal mapping function can be written in the form

$$z(\zeta) = \frac{2\pi}{k} + \frac{i}{k}\left(\log \zeta + f(\zeta)\right), \tag{1}$$

where $f(\zeta)$ is analytic everywhere in the annulus $\rho < |\zeta| < 1$. The branch cut associated with the logarithm is taken along the positive real axis. Note that it is assumed that z_ζ vanishes nowhere in $\rho \leq |\zeta| \leq 1$. This corresponds to seeking smooth waves with no corners or cusps.

The complex potential is defined to be

$$w(z) = \phi(x, y) + i\psi(x, y). \tag{2}$$

The fluid velocity field (u, v) is then given by $\frac{dw}{dz} = u - iv$. In the steady case, the kinematic boundary condition on each of the fluid interfaces is equivalent to specifying that

$$Im[w] \equiv \psi = \text{constant}, \tag{3}$$

on each fluid boundary. It is clear that, for symmetric waves, the form of $W(\zeta) \equiv w(z(\zeta))$ is given by

$$W(\zeta) = i\frac{c}{k}\log \zeta + \frac{2\pi c}{k}. \tag{4}$$

Note that this form satisfies the kinematic requirement (3) and implies that $\text{Re}[W] =$ const. on the vertical sides of the physical window in Figure 1.

The nondimensionalized Bernoulli (or pressure) condition on each fluid interface can be written

$$\frac{1}{2}\left|\frac{dw(z)}{dz}\right|^2 = \kappa + \Gamma, \tag{5}$$

where κ is the curvature of the interface and Γ is the Bernoulli constant. Rewriting this in terms of the conformal mapping variable ζ on $|\zeta| = 1$ gives

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = -\frac{z_\zeta^{1/2}(\zeta)}{\bar{z}_\zeta^{1/2}(\zeta^{-1})}\mathrm{Re}\left[1 + \frac{\zeta z_{\zeta\zeta}}{z_\zeta}\right] + \Gamma z_\zeta. \tag{6}$$

This can be written more conveniently as

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = -\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma z_\zeta. \tag{7}$$

Assuming the same value of the surface tension parameter on both free surfaces, the Bernoulli condition on $|\zeta| = \rho$ can be written in a similar fashion:

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\rho^2\zeta^{-1})}{2\bar{z}_\zeta(\rho^2\zeta^{-1})} = +\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\rho^2\zeta^{-1}\bar{z}_\zeta(\rho^2\zeta^{-1})}\right]^{1/2} + \Gamma z_\zeta. \tag{8}$$

It is assumed in what follows that the complex potential $W(\zeta)$ is completely specified and also that Γ is given. The corresponding $z(\zeta)$ will then be found. Since $\Gamma = \frac{c^2}{2}$ and from (4), it is clear that these requirements imply that the wavespeed c and the wavenumber k of the solutions are specified. We therefore seek solutions for $z(\zeta)$ corresponding to these given values of c and k.

We define some functions which will be important in the subsequent analysis:

Definition. Define the function $R(\zeta)$ as follows:

$$R(\zeta) \equiv \left(\zeta z_\zeta(\zeta)\right)^{1/2}. \tag{9}$$

Definition. Define the function $S(\zeta)$ as follows:

$$S(\zeta) \equiv -\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma z_\zeta(\zeta). \tag{10}$$

Definition. Define the function $\Psi(\zeta)$ as follows:

$$\Psi(\zeta) \equiv \left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2}. \tag{11}$$

The annulus $\rho < |\zeta| < 1$ will be referred to as C_0, while the annulus $1 < |\zeta| < \rho^{-1}$ will be referred to as C_1. These annuli are drawn in Figure 3. Note finally that the quantity

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = \frac{c^2}{2k^2\bar{z}_\zeta(\zeta^{-1})} \tag{12}$$

is *analytic* everywhere in C_1.

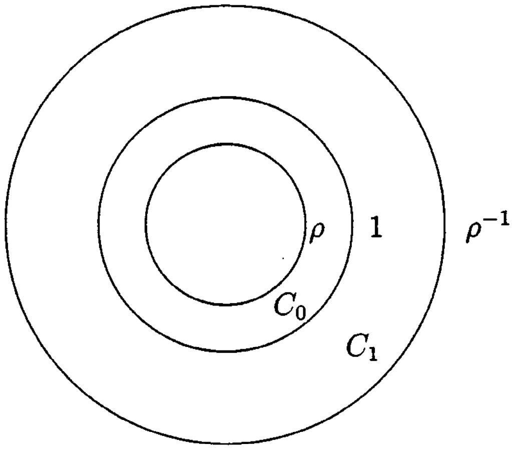

Fig. 3. Definition of annuli.

2.1. Singularity Structure

Theorem 2.1. *In the problem of finding steady symmetric solutions for finite fluid sheets, everywhere in the annulus C_1 the function $\Psi(\zeta)$ satisfies a Riccati-type equation of the form*

$$-\frac{d}{d\zeta}\Psi(\zeta)+q_1(\zeta)\Psi^2(\zeta)=q_2(\zeta), \tag{13}$$

where $q_1(\zeta)$ and $q_2(\zeta)$ are analytic everywhere in C_1. Specifically,

$$q_1(\zeta)=\frac{\Gamma\bar{z}_\zeta(\zeta^{-1})}{\zeta^2}, \tag{14}$$

and

$$q_2(\zeta)=\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})}. \tag{15}$$

Proof. The proof of this theorem is immediate using analytic continuation of the Bernoulli condition (7) into C_1 and by substituting the definition (11) for $\Psi(\zeta)$. □

By viewing the equation for the analytically continued mapping function in an appropriate way, the rigorous results of Painlevé (e.g., [7]) can be employed to make some important deductions about the singularity structure of the analytically continued conformal mapping.

Theorem 2.2. *In the annulus C_1 the singularities of $\Psi(\zeta)$ are necessarily simple poles.*

Proof. From the previous theorem, it is known that $\Psi(\zeta)$ satisfies a Riccati-type equation everywhere in C_1 with coefficient functions $q_1(\zeta)$ and $q_2(\zeta)$ which are known a priori to be analytic everywhere in C_1. Painlevé established [7] that *generically*, for an equation of the form (13) (with analytic coefficients) the movable singularities in C_1 are simple poles. The theory also establishes that (nongeneric) higher order poles can only possibly occur at zeros of $q_1(\zeta)$; however since, by the conformality requirements, $\bar{z}_\zeta(\zeta^{-1})$ (and hence $q_1(\zeta)$) does not vanish in C_1, this event does not occur. Thus the theorem follows. □

Remark 1. Note that since $[\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})]^{-1/2}$ is analytic in C_1, it is clear that $\Psi(\zeta)$ and $R(\zeta)$ have the same singularity structure in C_1. By squaring, we deduce that the only possible singularities of $\zeta z_\zeta(\zeta)$ in C_1 are second-order poles. It is then clear that $[\zeta z_\zeta(\zeta)]^{1/2}$ is single-valued everywhere in the annulus $\rho < |\zeta| < \rho^{-1}$.

2.2. Loxodromic Mapping Functions

In this section, we indicate how the complex variable formulation leads to the consideration of the subclass of possible mapping functions such that ζz_ζ is a *loxodromic function* with multiplicative period ρ^2. Loxodromic functions are intimately related to elliptic functions (via an exponential transformation) [10] [15]. Richardson [10] provides a useful summary of the general properties of these functions based on a presentation given in Valiron [15]. For recent applications of these special functions in different physical problems, see Richardson [10] and Crowdy and Tanveer [5].

Definition. A function $H(\zeta)$ will be described as possessing the *loxodromic property* with multiplicative period ρ^2 if it satisfies the following functional relation for all $\zeta \neq 0$:

$$H(\rho^2\zeta) = H(\zeta). \tag{16}$$

Definition. A function $H(\zeta)$ will be described as possessing the *antiloxodromic property* with multiplicative period ρ^2 if it satisfies the following functional relation for all $\zeta \neq 0$

$$H(\rho^2\zeta) = -H(\zeta) \tag{17}$$

Remark 2. Note that the previous two definitions are not generally acknowledged terminology but are made for convenience for the purposes of this paper. The following definition is, however, standard [15]:

Definition. A function $H(\zeta)$ will be said to be a *loxodromic function* if it is both *meromorphic* and possesses the loxodromic property.

Remark 3. We emphasize the distinction between a function possessing the loxodromic property and a loxodromic function. A function possessing the loxodromic property need not necessarily be meromorphic.

Now consider the two Bernoulli conditions (7) and (8). These two functional relations hold on $|\zeta| = 1$ and $|\zeta| = \rho$ respectively but, by analytic continuation, they also hold off these two contours. There can only possibly exist steady solutions provided that these two functional relations are compatible (i.e., mutually consistent). Assuming the single-valuedness of $[\zeta z_\zeta(\zeta)]^{1/2}$ everywhere in $\rho < |\zeta| < \rho^{-1}$ (this assumption is consistent with the arguments on the singularity structure of $R(\zeta)$ deduced in the previous section) and by the uniqueness of analytic continuation, it is clear that the consistency of the two Bernoulli conditions (7) and (8) requires that the function $R(\zeta) \equiv \left(\zeta z_\zeta\right)^{1/2}$ satisfies the functional equation

$$R(\rho^2\zeta) = -R(\zeta); \tag{18}$$

i.e., $R(\zeta)$ must possess the antiloxodromic property. Squaring (18), it is seen that $\zeta z_\zeta(\zeta)$ then possesses the loxodromic property with multiplicative period ρ^2, i.e.,

$$\rho^2\zeta z_\zeta(\rho^2\zeta) = \zeta z_\zeta(\zeta). \tag{19}$$

In summary, for symmetric waves, $R(\zeta) \equiv \left(\zeta z_\zeta(\zeta)\right)^{1/2}$ possesses the antiloxodromic property with multiplicative period ρ^2. This in turn implies that $\zeta z_\zeta(\zeta)$ possesses the loxodromic property with fundamental annulus $\rho < |\zeta| < \rho^{-1}$. Moreover, it has been established that ζz_ζ is meromorphic in this annulus. Thus we conclude that it is a loxodromic function. *For the remainder of this section on sheet waves, it is assumed that consideration is restricted to this class of functions*—i.e., functions $z(\zeta)$ such that $R(\zeta)$ satisfies (18) and has only simple poles in C_1.

2.3. Exact Solutions (Sheet Waves)

Theorem 2.3. *The condition that $S(\zeta)$ is analytic everywhere in C_1 is equivalent to Bernoulli conditions (for some c and k) holding on the boundaries of the fluid sheet.*

Proof. First assume that Bernoulli conditions for symmetric sheet waves (for some c and k) hold on both surfaces of the fluid sheet. By the antiloxodromic property of $R(\zeta)$ from the discussion above, this implies that

$$S(\zeta) = \frac{c^2}{2k^2\bar{z}_\zeta(\zeta^{-1})} \tag{20}$$

on $|\zeta| = 1$ and also everywhere else, by analytic continuation (in particular, on $|\zeta| = \rho$ where (20) is equivalent to a Bernoulli condition). (20) immediately implies that $S(\zeta)$ is analytic everywhere in C_1.

Conversely, suppose that $S(\zeta)$ is analytic everywhere in C_1. Thus, $\zeta S(\zeta)$ is also analytic everywhere in C_1. Moreover, from the analytical structure of $S(\zeta)$, and by the antiloxodromic property of $R(\zeta)$ (which in turn implies that $\zeta z_\zeta(\zeta)$ possesses the loxodromic property), it can be seen that $\zeta S(\zeta)$ also possesses the loxodromic property with multiplicative period ρ^2, i.e.,

$$\rho^2\zeta S(\rho^2\zeta) = \zeta S(\zeta). \tag{21}$$

Thus, since the function $[\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})]^{-1}$ possesses the loxodromic property and is analytic in C_1, $\zeta S(\zeta)$ can be written in the following form:

$$\zeta S(\zeta) = \frac{\zeta H(\zeta)}{\bar{z}_\zeta(\zeta^{-1})}, \tag{22}$$

for some function $H(\zeta)$ (to be determined) which is analytic everywhere in C_1 and which also possesses the loxodromic property.

We now observe the important fact that on $|\zeta| = 1$

$$\overline{S(\zeta)\bar{z}_\zeta(\zeta^{-1})} = S(\zeta)\bar{z}_\zeta(\zeta^{-1}). \tag{23}$$

This can be seen after some manipulation and depends crucially on the functional structure of $S(\zeta)$. This implies that $S(\zeta)\bar{z}_\zeta(\zeta^{-1})$ is **real** on $|\zeta| = 1$. From (22) this in turn implies that $H(\zeta)$ is **real** on $|\zeta| = 1$. $H(\zeta)$ is necessarily analytic everywhere in C_1 and possesses the loxodromic property with multiplicative period ρ^2, which means that it has a fundamental annulus that is the union of C_1 and C_0 (i.e., the annulus $\rho < |\zeta| < \rho^{-1}$). However, the reality of $H(\zeta)$ on the unit circle implies

$$H(\zeta) = \overline{H}(\zeta^{-1}), \tag{24}$$

a functional equation that furnishes the analytic continuation of $H(\zeta)$ into C_0 and, in particular, implies that $H(\zeta)$ must be analytic *everywhere* in its fundamental annulus (since it is necessarily analytic in C_1). By Liouville's theorem for loxodromic functions [15], it follows that $H(\zeta)$ must be a (real) constant function, i.e.,

$$H(\zeta) = \frac{c^2}{k^2}, \tag{25}$$

for some constant $\frac{c^2}{k^2}$ (say). Substituting this form for $H(\zeta)$ back into (22), it is seen that (22) is then equivalent to Bernoulli conditions holding on both surfaces of the fluid sheet. The theorem is then proved. □

We can now combine these results to deduce a global reformulation of the problem of finding solutions ζz_ζ to the steady fluid sheet problem:

Theorem 2.4 (Main Theorem). *The problem of finding $z(\zeta)$ for steady symmetric waves on a finite fluid sheet is equivalent to finding a function $z(\zeta)$ satisfying the following conditions:*

(i) *$z(\zeta)$ is a holomorphic, univalent map from $\rho \leq |\zeta| \leq 1$ to the fluid domain having the general form*

$$z(\zeta) = \frac{2\pi i}{k}\left(\log\zeta + f(\zeta)\right), \tag{26}$$

where $f(\zeta)$ is analytic in C_0;

(ii) *$[\zeta z_\zeta(\zeta)]^{1/2}$ possesses the antiloxodromic property;*

(iii) *$[\zeta z_\zeta(\zeta)]^{1/2}$ is meromorphic in C_1 with only simple pole singularities;*

(iv) *the function $S(\zeta)$ is holomorphic everywhere in C_1 with*

$$S(1) = \frac{W_\zeta(1)\overline{W}_\zeta(1)}{2\bar{z}_\zeta(1)}. \tag{27}$$

Proof. The proof of this theorem is clear from a combination of the results of all the preceding theorems. The single extra condition (27) simply ensures that the constant function $H(\zeta)$ in the proof of Theorem 2.3 takes the specified value $\frac{c^2}{k^2}$.

It is well known (pp. 105 of [7]) that certain Riccati equations admit solutions that are meromorphic with a *finite* number of poles. We have also now deduced that a solution $(\zeta z_\zeta)^{1/2}$ is necessarily meromorphic in C_1 and satisfies a Riccati-type equation there. We therefore ask the following natural question: does the problem of steady waves on a finite sheet of fluid admit *loxodromic function* solutions of *finite* order N?

The well-known representation theorem for loxodromic functions [10] [15] states that if $H(\zeta)$ is a loxodromic function of finite order $N \geq 2$ with multiplicative period ρ^2, it will necessarily have the general form

$$H(\zeta) = A\frac{\prod_{k=1}^{N} P(\zeta\eta_k^{-1})}{\prod_{j=1}^{N} P(\zeta\zeta_j^{-1})}, \tag{28}$$

where

$$P(\zeta) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)\prod_{j=1}^{\infty}\left(1-\frac{\rho^{2j}}{\zeta}\right), \tag{29}$$

and where

$$\prod_{k=1}^{N}\eta_k = \prod_{j=1}^{N}\zeta_j. \tag{30}$$

The function $P(\zeta)$ has the following important properties:

$$P(\rho^2\zeta) = -\zeta^{-1}P(\zeta) = P(\zeta^{-1}). \tag{31}$$

It is also convenient at this point to define a related function $\hat{P}(\zeta)$ that will be needed later, i.e.,

$$\hat{P}(\zeta) \equiv \prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)\prod_{j=1}^{\infty}\left(1-\frac{\rho^{2j}}{\zeta}\right). \tag{32}$$

It is enough, given the loxodromic nature of $\zeta z_\zeta(\zeta)$ to restrict attention to the *fundamental annulus* $\rho < |\zeta| \leq \rho^{-1}$ since all other annuli are *equivalent* (see [10] [15]). Similarly, by the various conditions on $R(\zeta)$, it can be shown that it has a representation of the form

$$\left(\zeta z_\zeta(\zeta)\right)^{1/2} = A\frac{\prod_{k=1}^{N} P(\zeta\eta_k^{-1})}{\prod_{j=1}^{N} P(\zeta\zeta_j^{-1})}, \tag{33}$$

for $N \geq 1$, where

$$\prod_{k=1}^{N}\eta_k = -\prod_{j=1}^{N}\zeta_j. \tag{34}$$

Note that all the zeros $\{\eta_j\}$ and all poles $\{\zeta_j\}$ must be in the annulus C_1. □

We now combine all the above mathematical information to deduce the following result:

Theorem 2.5. *The problem of steady symmetric waves on fluid sheets of finite thickness admits exact solutions in which ζz_ζ is a loxodromic function of the form*

$$\zeta z_\zeta = iA\left(\frac{\prod_{k=1}^{N} P(\zeta\eta_k^{-1})}{\prod_{j=1}^{N} P(\zeta\zeta_j^{-1})}\right)^2, \tag{35}$$

where $N \geq 1$ and

$$\prod_{k=1}^{N} \eta_k = -\prod_{j=1}^{N} \zeta_j. \tag{36}$$

In particular, the case $N = 1$ corresponds to the type I solutions identified by Kinnersley [8], i.e.,

$$\zeta z_\zeta = iA\left(\frac{P(\zeta\eta_1^{-1})}{P(\zeta\zeta_1^{-1})}\right)^2 \tag{37}$$

for suitable constants A (real), η_1, and ζ_1.

Proof. We prove the theorem by direct construction of a solution for each N. The method of construction is based on the need to satisfy the requirements (i)–(iv) of Theorem 2.4:

Solution for $N = 1$: For $N = 1$, the mapping function has the form

$$z_\zeta(\zeta) = \frac{\hat{A}}{\zeta}\left(\frac{P(\zeta\eta_1^{-1})}{P(\zeta\zeta_1^{-1})}\right)^2. \tag{38}$$

In order to obtain a solution for $z(\zeta)$ with the general form (1), $\hat{A}$ is taken to be purely imaginary, i.e., $\hat{A} = iA$ for some purely real A. The form of $S(\zeta)$ corresponding to (38) is given by

$$S(\zeta) = -i\frac{d}{d\zeta}\left[\frac{P(\zeta\eta_1^{-1})P(\zeta^{-1}\bar{\zeta}_1^{-1})}{P(\zeta\zeta_1^{-1})P(\zeta^{-1}\bar{\eta}_1^{-1})}\right] + \frac{i\Gamma A}{\zeta}\left(\frac{P(\zeta\eta_1^{-1})}{P(\zeta\zeta_1^{-1})}\right)^2. \tag{39}$$

The two equations arising from the vanishing of the principal part of $S(\zeta)$ at ζ_1 are given by

$$\eta_1 = -\zeta_1, \tag{40}$$

$$\Gamma A = \frac{P(|\zeta_1|^{-2})\hat{P}(+1)}{P(-|\zeta_1|^{-2})P(-1)} \tag{41}$$

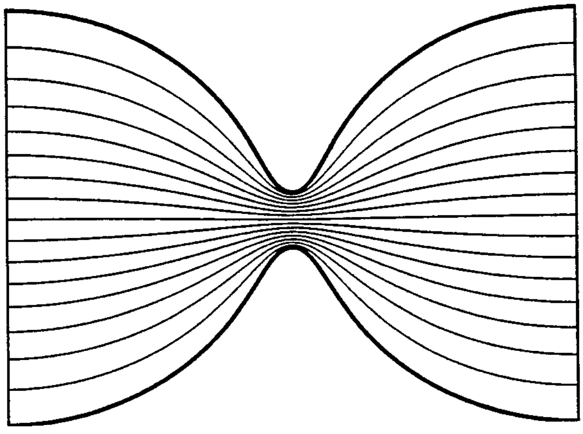

Fig. 4. Kinnersley's sheet waves using (45) with $\zeta_1 = 3.11$, $\rho = 0.1$.

Note that (40) is consistent with the requirement (34) with $N = 1$. This choice of parameters satisfies the two principal part conditions on $S(\zeta)$ at $\zeta = \zeta_1$. Finally, the condition (27) can be shown to be equivalent to the following condition:

$$\frac{c^2}{k^2} = 2A\frac{P(|\zeta_1|^{-2})\hat{P}(+1)}{P(-|\zeta_1|^{-2})P(-1)}. \tag{42}$$

Eliminating between equations (41) and (42) produces a simple relationship between Γ and $\frac{c}{k}$, namely,

$$\Gamma A^2 = \frac{c^2}{2k^2}. \tag{43}$$

This is exactly the same relationship between the parameters as that found in the case of deep water capillary waves [3].

The final form of the exact solution is therefore a three-parameter family of solutions parametrized by A, ζ_1, and ρ, i.e.,

$$z_\zeta = \frac{iA}{\zeta}\left(\frac{P(-\zeta\zeta_1^{-1})}{P(\zeta\zeta_1^{-1})}\right)^2. \tag{44}$$

Integration of this expression yields the final form of the conformal map as

$$z(\zeta) = \int_1^\zeta \frac{iA}{\zeta'}\left(\frac{P(-\zeta'\zeta_1^{-1})}{P(\zeta'\zeta_1^{-1})}\right)^2 d\zeta'. \tag{45}$$

A plot of this conformal map for the typical values $\rho = 0.1$ and $\zeta_1 = 3.11$, along with some typical streamlines, is given in Figure 4.

The concise representation of the solution (45) is exactly equivalent to Kinnersley's type I solutions [8]—a remarkable fact given the rather complicated expression of the solution (in terms of Jacobi elliptic functions) presented in the original paper [8]. It turns out that Kinnersley's original solutions can be greatly simplified. This simplification is given in an appendix.

Solutions for general N: It can similarly be shown that there exists a solution for general N of the following form:

$$\zeta_j = e^{\frac{2\pi i(j-1)}{N}}\zeta_1, \qquad j = 2, \dots N, \tag{46}$$

$$\eta_j = e^{\frac{(2j-1)\pi i}{N}}\zeta_1, \qquad j = 1, \dots N, \tag{47}$$

$$\Gamma A = \hat{P}(+1)\frac{\prod_{j=2}^{N} P(\zeta_1\zeta_j^{-1})}{\prod_{j=1}^{N} P(\zeta_1\eta_j^{-1})}\left(\frac{\prod_{k=1}^{N} P(\zeta_1^{-1}\bar{\zeta}_k^{-1})}{\prod_{k=1}^{N} P(\zeta_1^{-1}\bar{\eta}_k^{-1})}\right). \tag{48}$$

We remark that the solutions for different N do not represent physically distinct solutions. Rather, they represent the Kinnersley solutions with N identical periods of the wave described by the mapping function. □

Remark 4. Note that it is immediately seen that the limit $\rho \to 0$ of the $N = 1$ solution retrieves the exact solutions as obtained in Crowdy [3] (i.e., Crapper's infinite depth capillary wave solutions), i.e., as $\rho \to 0$,

$$z_\zeta = \frac{iA}{\zeta}\left(\frac{P(-\zeta\zeta_1^{-1})}{P(\zeta\zeta_1^{-1})}\right)^2 \to \frac{iA}{\zeta}\left(\frac{(\zeta+\zeta_1)}{(\zeta-\zeta_1)}\right)^2, \tag{49}$$

and that

$$\begin{aligned}\Gamma A = \frac{P(|\zeta_1|^{-2})\hat{P}(+1)}{P(-|\zeta_1|^{-2})P(-1)} &= \frac{P(|\zeta_1|^{-2})}{P(-|\zeta_1|^{-2})}\frac{1}{2}\left(\frac{\prod_{k=1}^{\infty}(1-\rho^{2k})^2}{\prod_{k=1}^{\infty}(1+\rho^{2k})^2}\right)\\ &\to \frac{1}{2}\frac{|\zeta_1|^2-1}{|\zeta_1|^2+1}.\end{aligned} \tag{50}$$

This is exactly Crapper's deep water solution.

Remark 5. Note that there is no assertion of *uniqueness* of the exact, loxodromic function solutions. It is possible that the foregoing mathematical arguments leading to the finite reduction of the full nonlinear free boundary problem could be made fully rigorous, thereby providing a possible means of proof of uniqueness of Kinnersley's symmetric solutions by studying the properties of the finite system to which the problem is reduced. Such a proof of uniqueness of the symmetric solutions using other methods (e.g., functional analysis) does not, to the best of the author's knowledge, yet exist. On the other hand, the solutions might not be unique, and additional distinct physical solutions might well be found using the constructive methods presented above.

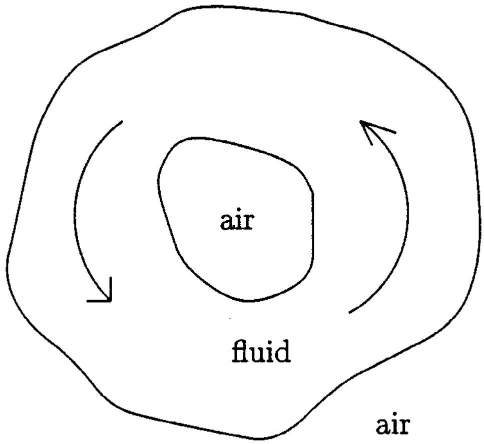

Fig. 5. Irrotational flow in an annulus with surface tension.

3. Capillary Waves on an Annular Blob

The foregoing theoretical reformulation is now extended to unveil a new class of exact solutions. The problem under consideration is that of finding steady state shapes for the two free surfaces of a fluid annulus undergoing an irrotational, swirling motion. The problem is the natural, doubly connected generalization of the exact solutions recently identified by Crowdy [4] concerning the steady motion of a finite fluid drop containing a single point vortex. In the doubly connected case, the point vortex is no longer inside the flow region but can be thought of instead as an image vortex producing an irrotational azimuthal swirl in the fluid annulus. The form of $W(\zeta)$ is again given by

$$W(\zeta) = i\gamma \log \zeta. \tag{51}$$

It will be assumed that γ is specified.

It is expected that steady solutions will exist for particular values of the pressure difference between the pressure inside the bubble enclosed by the annulus and the pressure of the air outside the annulus. This amounts to a statement of the fact that the Bernoulli conditions on each of the fluid interfaces will have *different* Bernoulli constants, which will be called Γ_1 and Γ_ρ, respectively. It expected that the ratio of these two quantities will be given as part of the solution. To keep the analysis as general as possible, we also allow for different values of the (uniform) surface tension parameter on each of the two free surfaces. Denoting the uniform surface tension on the two boundaries as T_1 and T_ρ respectively, the ratio will be denoted by β, i.e.,

$$\beta = \frac{T_\rho}{T_1}. \tag{52}$$

We expect to be able to specify β externally and thus find corresponding solutions. This will be seen to be the case. Physically, a difference in the values of the uniform surface tension on the two interfaces could be brought about by the presence of some surface active agent (i.e., surfactant) on one (or both) of the free surfaces, or perhaps by some other physical mechanism such as a difference in temperature of the air inside the enclosed bubble compared to that outside the blob.

Nondimensionalizing the equations using T_1, on $|\zeta| = 1$ the Bernoulli condition can be written

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = -\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma_1 z_\zeta. \tag{53}$$

The Bernoulli condition on $|\zeta| = \rho$ can be written in a similar fashion:

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\rho^2\zeta^{-1})}{2\bar{z}_\zeta(\rho^2\zeta^{-1})} = +\beta\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\rho^2\zeta^{-1}\bar{z}_\zeta(\rho^2\zeta^{-1})}\right]^{1/2} + \Gamma_\rho z_\zeta. \tag{54}$$

Note that (51) clearly satisfies the kinematic conditions that the two free surfaces be streamlines, i.e.,

$$\psi = \mathrm{Im}[W] = \text{constant}, \qquad \text{on } |\zeta| = \rho \text{ and } 1. \tag{55}$$

3.1. Exact Solutions (Fluid Annulus)

We now choose to seek solutions for which $[z_\zeta(\zeta)]^{1/2}$ satisfies the following functional equation:

$$[z_\zeta(\rho^2\zeta)]^{1/2} = \omega * [z_\zeta(\zeta)]^{1/2}, \tag{56}$$

for some constant ω. Given (53) and (54) it is clear that, in order for the analytically continued Bernoulli conditions to be mutually consistent, it is necessary that

$$\rho = -\frac{1}{\omega\beta} \tag{57}$$

and

$$\Gamma_1 = \rho^2\omega^2\Gamma_\rho. \tag{58}$$

For given β, (57) provides an equation for ρ. (58) then provides the equation relating the two Bernoulli constants Γ_1 and Γ_ρ.

We now state the theorem analogous to Theorem 2.4 that will lead to a constructive method of finding solutions to the above-stated problem:

Theorem 3.1. *The problem of finding solutions to the problem of capillary waves on a fluid annulus (as stated above) is equivalent to finding a conformal mapping $z(\zeta)$ satisfying the following conditions:*

(i) *$z(\zeta)$ is a univalent conformal map from C_0 to the fluid region. $z(\zeta)$ is analytic everywhere in C_0 and z_ζ does not vanish anywhere in C_0;*

(ii) $[z_\zeta(\zeta)]^{1/2}$ *satisfies the functional equation (56);*
(iii) $[z_\zeta(\zeta)]^{1/2}$ *is meromorphic in* C_1 *with only simple pole singularities;*
(iv) $S(\zeta)$ *is analytic everywhere in* C_1 *with*

$$S(1) = \frac{W_\zeta(1)\overline{W}_\zeta(1)}{2\bar{z}_\zeta(1)}. \tag{59}$$

Proof. The proof of this theorem is analogous to the proof of Theorem 2.4 with only minor changes in detail. □

Using the properties (31) it can easily be seen that the following general class of functions satisfy the functional equation (56), i.e.,

$$[z_\zeta(\zeta)]^{1/2} = A\frac{\prod_{k=1}^{N} P(\zeta\eta_k^{-1})}{\prod_{j=1}^{N} P(\zeta\zeta_j^{-1})}, \tag{60}$$

where N is some integer and η_k and ζ_j are some constants satisfying the condition

$$\omega = \frac{\prod_{k=1}^{N} \eta_k}{\prod_{j=1}^{N} \zeta_j}. \tag{61}$$

Note also that, provided ζ_j are in the annulus C_1 and that none are repeated poles, then (60) represents a function satisfying the analyticity constraints imposed by the problem and is therefore a candidate solution. Indeed, using counting arguments resulting from the necessary and sufficient conditions of the previous theorem, the counting problem for any solution of the form (60) is consistent. This indicates at least the *possibility* of finding exact solutions of the form (60) for some N.

An investigation of the finite system of nonlinear equations provided by assuming that $[z_\zeta(\zeta)]^{1/2}$ has the form (60) for some N and then insisting that $S(\zeta)$ is analytic at each of the poles ζ_j and satisfies (59), immediately reveals that a solution with $N = 1$ is impossible. However, a solution with $N = 2$ has been found. We will now present details of the $N = 2$ exact solution.

Remark 6. While we make no claim that the solution with $N = 2$ is the only solution with $[z_\zeta(\zeta)]^{1/2}$ of the form (60), we have not yet been unable to find solutions corresponding to $N \neq 2$, even though the counting problem is consistent for any positive integer $N \geq 2$. The rather intriguing question of the possible non-uniqueness of the new solutions that are about to be presented therefore remains open at this time.

With $N = 2$, the solution has the form

$$[z_\zeta(\zeta)]^{1/2} = A\frac{P(\zeta\eta_1^{-1})P(\zeta\eta_2^{-1})}{P(\zeta\zeta_1^{-1})P(\zeta\zeta_2^{-1})}. \tag{62}$$

The corresponding $S(\zeta)$ is given by

$$S(\zeta) = -\frac{d}{d\zeta}\left(\frac{\zeta P(\zeta\eta_1^{-1})P(\zeta\eta_2^{-1})P(\zeta^{-1}\bar{\zeta}_1^{-1})P(\zeta^{-1}\bar{\zeta}_2^{-1})}{P(\zeta\zeta_1^{-1})P(\zeta\zeta_2^{-1})P(\zeta^{-1}\bar{\eta}_1^{-1})P(\zeta^{-1}\bar{\eta}_2^{-1})}\right) + \Gamma A^2\left(\frac{P(\zeta\eta_1^{-1})P(\zeta\eta_2^{-1})}{P(\zeta\zeta_1^{-1})P(\zeta\zeta_2^{-1})}\right)^2. \quad (63)$$

If this is to be a solution to the free boundary problem stated above, by Theorem 3.1 it is necessary that the five parameters A, η_1, η_2, ζ_1, and ζ_2 satisfy a system of five nonlinear equations. Four of these equations result from the vanishing principal parts of $S(\zeta)$ at both ζ_1 and ζ_2. The fifth equation results from the condition (59).

3.2. Summary of the Exact Solutions (Fluid Annulus)

Using the above prescription, it is readily found that the following choices of parameters yield a solution

$$\begin{aligned} \eta_2 &= -\eta_1, \\ \zeta_2 &= -\zeta_1, \\ \eta_1 &= \alpha\,\zeta_1, \end{aligned} \quad (64)$$

where α is a solution of the nonlinear equation

$$\frac{1}{\alpha}\left(\frac{P'(\alpha^{-1})}{P(\alpha^{-1})} - \frac{P'(-\alpha^{-1})}{P(-\alpha^{-1})}\right) = \frac{1}{2}. \quad (65)$$

This nonlinear equation implicitly determines α as a function of ρ. The solutions for α are found to be purely imaginary, and the *imaginary part* of α is plotted as a function of ρ in Figure 6.

Remarkably, the corresponding form of $z_\zeta(\zeta)$ can be integrated into polar form, giving the meromorphic mapping function:

$$z(\zeta) = A\zeta\left(\frac{P(\zeta/\alpha^2\zeta_1)P(-\zeta/\alpha^2\zeta_1)}{P(\zeta/\zeta_1)P(-\zeta/\zeta_1)}\right). \quad (66)$$

This corresponds to the derivative $z_\zeta(\zeta)$ of (66) given by

$$z_\zeta(\zeta) = AB\left(\frac{P(\zeta/\alpha\zeta_1)P(-\zeta/\alpha\zeta_1)}{P(\zeta/\zeta_1)P(-\zeta/\zeta_1)}\right)^2, \quad (67)$$

where

$$B = \frac{P(\alpha^{-3})P(-\alpha^{-3})P(\alpha^{-1})P(-\alpha^{-1})}{P^2(\alpha^{-2})P^2(-\alpha^{-2})}\left(\frac{1}{2} + \frac{1}{\alpha^3}\left(\frac{P'(\alpha^{-3})}{P(\alpha^{-3})} - \frac{P'(-\alpha^{-3})}{P(-\alpha^{-3})}\right)\right). \quad (68)$$

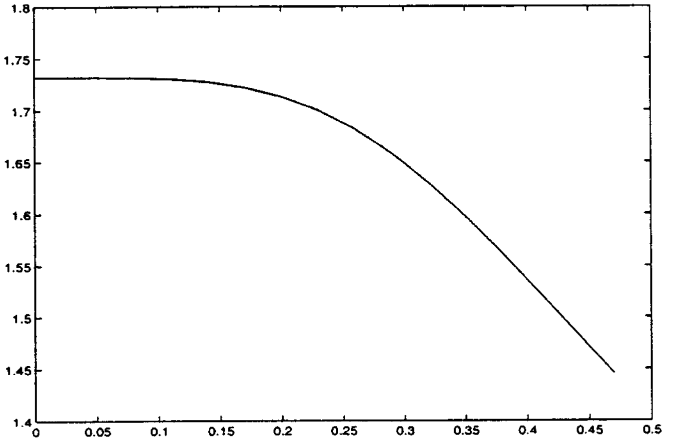

Fig. 6. Plot of Im[α] (vertical axis) as a function of ρ.

The corresponding values of Γ_1 and γ are given by

$$\Gamma_1 = \frac{1}{A^2}\frac{P(\zeta_1^{-2})P(-\zeta_1^{-2})\hat{P}(1)P(-1)}{P(\bar{\alpha}^{-1}\zeta_1^{-2})P(-\bar{\alpha}^{-1}\zeta_1^{-2})P(\alpha^{-1})P(-\alpha^{-1})}, \tag{69}$$

$$\gamma^2 = 2A^2\frac{P(\bar{\alpha}^{-2}\zeta_1^{-2})P(-\bar{\alpha}^{-2}\zeta_1^{-2})\hat{P}(1)P(-1)}{P(\bar{\alpha}^{-1}\zeta_1^{-2})P(-\bar{\alpha}^{-1}\zeta_1^{-2})P(\bar{\alpha})P(-\bar{\alpha})}. \tag{70}$$

This solution corresponds to $\omega = \alpha^2$ so that from (57) and (58) the corresponding values of Γ_ρ and β are provided by

$$\Gamma_\rho = \frac{\Gamma_1}{\rho^2\alpha^4}, \qquad \beta = -\frac{1}{\alpha^2\rho}. \tag{71}$$

Remark 7. Note that it is clearly necessary, in order to avoid negative values of the surface tension parameters, that $\alpha^2 < 0$. Thus, only roots of (65) that are purely imaginary correspond to physically admissible solutions. Such solutions are found to exist.

Remark 8. As $\rho \to 0$ we retrieve the values of the parameters correponding to the solution found in [4] for a single point vortex in a finite fluid blob. In that case, it is easily shown that the solution α to (65) behaves like

$$\alpha \to i\sqrt{3} \qquad \text{as} \quad \rho \to 0. \tag{72}$$

This can be seen explicitly in the plot of Im[$\alpha(\rho)$] given in Figure 6. This implies that

$$\begin{aligned} z(\zeta) &= A\zeta\left(\frac{P(\zeta/\alpha^2\zeta_1)P(-\zeta/\alpha^2\zeta_1)}{P(\zeta/\zeta_1)P(-\zeta/\zeta_1)}\right) \\ &\to \frac{A}{9}\zeta\left(\frac{\zeta^2-9\zeta_1^2}{\zeta^2-\zeta_1^2}\right) = \frac{A}{9}\left(\zeta - \frac{8\zeta_1^2\zeta}{\zeta^2-\zeta_1^2}\right). \end{aligned} \tag{73}$$

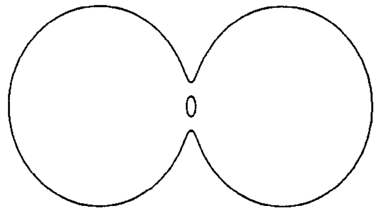

Fig. 7. Shape of the fluid annulus for fixed $\rho = 0.2$ and $\zeta_1 = 1.1$.

The limiting solution is the solution for a simply connected blob containing a single point vortex as found explicitly in Crowdy [4]. This observation justifies an earlier statement that the exact solutions found here represent the analytic continuation into nonzero ρ (i.e., a doubly connected topology) of the simply connected solutions in Crowdy [4].

Remark 9. Note that, using a symbolic manipulator, it is a straightforward matter to verify by direct substitution that the above function (with the corresponding Γ_1, Γ_ρ, γ, and β) satisfies the Bernoulli conditions on the boundaries of C_0. This was done as an explicit check on the exact solutions.

3.3. Results

It is clear that the shape of the interface (i.e., the conformal map) is determined by the three parameters—ζ_1, ρ, and A. Whereas in the derivation of the solutions presented here it has been assumed that Γ_1, β, and γ are specified, in order to study the shape of the interfaces it is more convenient (and natural) to instead specify ζ_1, ρ, and A. Once ζ_1, ρ, and A are specified, the shape of the interface and all the corresponding values of the parameters are then determined—i.e., Γ_1, Γ_ρ, γ, and β. A represents no more than a normalization; it is most conveniently given as a function of ζ_1 and ρ by specifying the area of the fluid annulus. We therefore arbitrarily impose that

$$\pi = \frac{1}{2}\mathrm{Im}\left[\oint_C \bar{z} z_\zeta \, d\zeta\right], \tag{74}$$

where C denotes the contour consisting of the circle $|\zeta| = 1$ traversing anticlockwise and the circle $|\zeta| = \rho$ traversed clockwise. The solutions then represent a two-parameter family of exact solutions parametrized by ζ_1 and ρ.

The results for $\rho = 0.2$ are given in Figures 7–10 for several different values of the parameter ζ_1. For ζ_1 close to the unit circle, the enclosed bubble decreases in size, while the outer boundary tends to pinch in towards it from above and below. As ζ_1 moves away from the unit circle, the enclosed bubble grows in size and becomes vertically elongated, eventually leading to a critical value of ζ_1 where the conformal map loses its univalency. Thus the enclosed bubble is seen to pinch. Some typical streamlines for $\rho = 0.2$ are shown in Figure 15. Note that these plots also provide confirmation of the univalency of the conformal map in the annulus C_0.

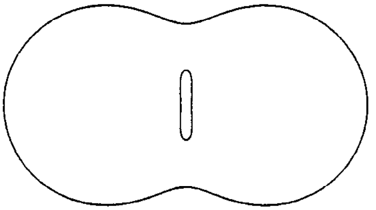

Fig. 8. Shape of the fluid annulus for fixed $\rho = 0.2$ and $\zeta_1 = 1.5$.

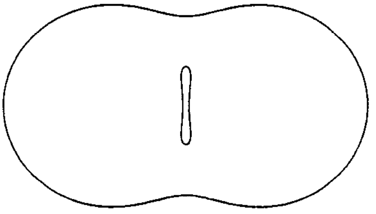

Fig. 9. Shape of the fluid annulus for fixed $\rho = 0.2$ and $\zeta_1 = 1.6$.

The qualitative behavior just described is found to be exactly the same whatever the value of ρ. For comparison, typical shapes for the value $\rho = 0.3$ are shown in Figures 11–14. It is found, however, that while a univalent mapping function $z(\zeta)$ always seems to exist (for any value of ρ) provided ζ_1 is sufficiently close to the unit circle, as ρ increases, the range of ζ_1 over which there exists a univalent map is found to decrease rapidly. Indeed, for $\rho = 0.6$, it is found that ζ_1 had to be of the order of $1 + O(10^{-4})$ if a univalent map is to be found. Note that the value of ζ_1 at which the inner bubble pinches is around $\zeta_1 = 1.70$ for $\rho = 0.2$ (see Figure 10), while it is around $\zeta_1 = 1.225$ for $\rho = 0.3$ (see Figure 14). The "critical value" of ζ_1 at which the enclosed bubble is observed to pinch-off with itself continues to decrease as ρ increases. As ρ gets larger, for a univalent map, ζ_1 has to be closer to the unit circle, meaning that a pole of the mapping draws close to the unit circle. The enclosed bubble becomes very small and the outer boundary exhibits points of high curvature in the cusplike necking regions where it draws close to the inner bubble (cf. Figures 7–11). Because the width of the region

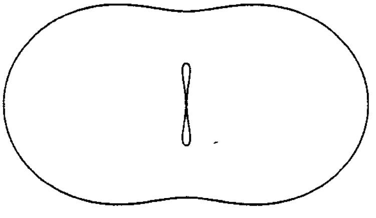

Fig. 10. Shape of the fluid annulus for fixed $\rho = 0.2$ and $\zeta_1 = 1.7$.

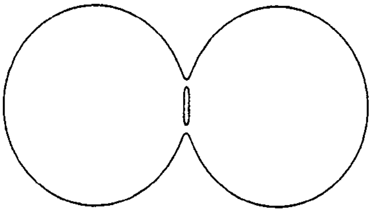

Fig. 11. Shape of the fluid annulus for fixed $\rho = 0.3$ and $\zeta_1 = 1.1$.

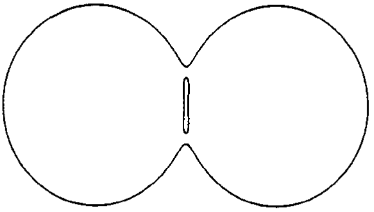

Fig. 12. Shape of the fluid annulus for fixed $\rho = 0.3$ and $\zeta_1 = 1.15$.

between the inner bubble and the outer boundary of the annulus becomes extremely small, the velocity of the fluid in this region becomes very high. Intuitively, a steady solution exhibiting high velocities in a region next to sharp cusps in its free surface is not expected to be stable (and therefore not observable physically); however, a detailed analysis of the stability of the steady solutions found here is left for the future.

Note that from (57), the ratio of the surface tension parameters β on the two interfaces is seen to be just a function of ρ. This is because $\omega = \alpha^2$ and α has been deduced to be a function of ρ as plotted in Figure 6. We note that the value of ρ for which $\beta = 1$ (so that the surface tension parameters on the two interfaces are equal) is given by $\rho = 0.6355$. This is quite a large value of ρ and unfortunately corresponds to a steady solution exhibiting cusplike necking in the outer fluid boundary with a very small enclosed bubble inside the blob. Such solutions are not expected to be physically realizable.

The results for the shapes of the annulus are in line with what might be expected. In [4] the two separate (simply connected) problems of a (constant pressure) bubble of air

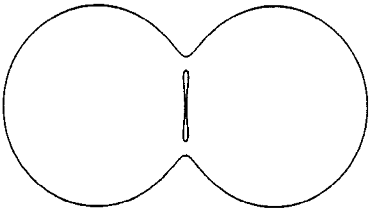

Fig. 13. Shape of the fluid annulus for fixed $\rho = 0.3$ and $\zeta_1 = 1.2$.

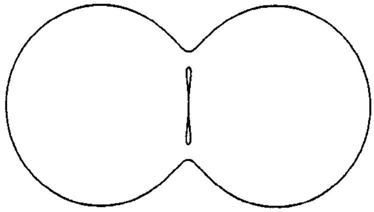

Fig. 14. Shape of the fluid annulus for fixed $\rho = 0.3$ and $\zeta_1 = 1.225$.

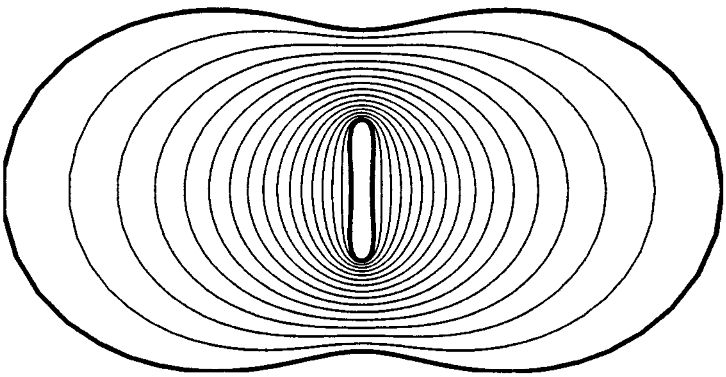

Fig. 15. Typical streamlines: $\rho = 0.2$, $\zeta_1 = 1.6$.

placed in the field of a simple vortex at infinity as well as the problem of a finite blob of fluid containing a single point vortex is considered. Intuitively, the bubble enclosed by the annulus in the present problem might be expected to behave *qualitatively* like a bubble in the flow field of a simple vortex, while the outer boundary of the annulus might be expected to behave qualitatively like a simply connected blob with no air bubble but with an isolated point vortex inside it. This is exactly the qualitative behavior that is observed.

4. Summary

This paper has generalized, to a doubly connected fluid region, a new approach to free-surface potential flows with capillarity recently presented in [3], [4] in the context of simply connected flow domains. The exact finite-amplitude solutions for symmetric waves on fluid sheets originally found by Kinnersley [8] have been retrieved in a novel and simplified fashion. Although we have not done so here, Kinnersley's antisymmetric (type II) solutions are also retrievable using appropriate modifications of the theory. The new method leads naturally to a very simple representation of Kinnersley's symmetric solutions. No unwieldy algebraic manipulations are involved in the derivation.

By extending the theory for sheet waves to a genuinely doubly connected fluid domain, exact solutions have been found for steady capillary waves on a fluid annulus. To the best of the author's knowledge, these solutions are new and have not been previously reported in the literature. As seen from the development in this paper, the new solutions have close mathematical connections with Kinnersley's sheet wave solutions. The new solutions are seen to be a doubly connected continuation of some new simply connected solutions recently identified in [4].

Natural extensions of the present theory include the possibility of generalizing the approach to find exact solutions for Euler flows around two bubbles with surface tension, although preliminary investigations by the author have revealed that while the new formalism and constructive approach is readily extendable, the problem may not admit any exact solutions. Another possible generalization is the problem of pure capillary waves on a fluid sheet of finite depth with a rigid bottom (Kinnersley briefly discussed this problem in [8]). This problem, however, represents a nontrivial extension of the present theory in that the nature of the boundary conditions on the free surface and that on the rigid bottom are now very different in nature, and one can no longer deduce a simple periodicity (or quasi-periodicity) property of the conformal mapping function (cf. (18) and (56)).

Finally, the approach of this paper has centered around the consideration of complex singularities. Complex singularity *dynamics* has led to many theoretical insights into closely related free boundary problems such as unsteady Hele-Shaw flows with small surface tension [14]. The *unsteady* generalization of the results of this paper is not trivial, and while finding exact solutions to the time-dependent problem is unlikely, it is nevertheless expected that the complex singularity dynamics approach is a promising theoretical route to understanding such problems as the dynamics of inviscid capillary pinch-off [6].

A. Simplification of Kinnersley's Solutions

The finite amplitude sheet wave solutions as computed by Kinnersley [8] were, in fact, implicit in Crapper [2]; however, Crapper elected not to compute these solutions stating that the solutions were too complicated. Twenty years later, Kinnersley disputed Crapper's claim that the solutions were complicated and explicitly calculated the solutions in terms of Jacobi elliptic functions. It turns out, however, that the solutions computed by Kinnersley admit an even greater simplification than is apparent from Kinnersley's original paper. Of course, we were led here to this simplification by the need to demonstrate that the very simple representation of solutions as given in (44) (derived in a natural way from our present solution scheme) is in fact equivalent to Kinnersley's original solutions.

In the notation of [8], Kinnersley's type Ia solutions were deduced to be of the form

$$u = c\left(\frac{P-Q}{P+Q}\right)\left(\frac{R^2-S^2}{R^2+S^2}\right), \tag{75}$$

$$v = c\left(\frac{P-Q}{P+Q}\right)\left(\frac{2RS}{R^2+S^2}\right), \tag{76}$$

where

$$P = \mathrm{nd}(\psi, k'), \qquad Q = \mathrm{cd}(\phi, k), \qquad R = \mathrm{sc}(\psi, k'), \qquad S = \mathrm{sn}(\phi, k). \quad (77)$$

c is a constant and $\psi = A\Psi + B$, $\phi = A\Phi$. A and B are constants. k and k' are the (complementary) moduli of the Jacobi elliptic functions [1]. This is the form of the type Ia solutions as given by Kinnersley [8]. Note that, in what follows, we employ the original notation of Kinnersley [8]. We also use Glaisher's notation [1].

We now demonstrate a simplification of the above solution (75)–(77). Combining equations (75) and (76), we deduce

$$\frac{dw}{dz} = u - iv = c\left(\frac{P-Q}{P+Q}\right)\left(\frac{R-iS}{R+iS}\right). \quad (78)$$

Using various properties of Jacobi elliptic functions [1], it can be shown after some manipulation that

$$\left(\frac{R-iS}{R+iS}\right) = \frac{\mathrm{sn}(u,k')\mathrm{dn}(v,k')\mathrm{cn}(u,k')}{\mathrm{sn}(v,k')\mathrm{dn}(u,k')\mathrm{cn}(v,k')}, \quad (79)$$

where $u = \frac{i(\phi+i\psi)}{2}$ and $v = \frac{i(\phi-i\psi)}{2}$. Similarly, some manipulation reveals that

$$\left(\frac{P-Q}{P+Q}\right) = k'^2 \frac{\mathrm{sn}(u,k')\mathrm{sn}(v,k')\mathrm{cn}(u,k')\mathrm{cn}(v,k')}{\mathrm{dn}(u,k')\mathrm{dn}(v,k')}. \quad (80)$$

Combining these implies

$$\frac{dw}{dz} = ck'^2\left(\frac{\mathrm{sn}(u,k')\mathrm{cn}(u,k')}{\mathrm{dn}(u,k')}\right)^2. \quad (81)$$

Now it is appropriate to apply Landen's transformation [1] to enable the singularity structure of these solutions to be seen more clearly. Setting

$$\lambda = \frac{1-k}{1+k}, \qquad M = \frac{1}{1+k}, \quad (82)$$

we deduce that

$$\frac{dw}{dz} = ck'^2 M^2 \mathrm{sn}^2\left(\frac{u}{M};\lambda\right). \quad (83)$$

Since u is a simple linear function of w, this represents a particularly concise statement of the solutions.

Remark 10. It remains, of course, to integrate (83) to find z. Noting that (83) can be equivalently rewritten in the following general form,

$$\frac{dz}{d\hat{w}} = p\,\mathrm{ns}^2(\hat{w};\lambda), \quad (84)$$

for some constant p and some variable $\hat{w}$ that is linearly related to w, then integration of this equation with respect to $\hat{w}$ provides the expression for the free surface in an equally concise fashion:

$$z = x + iy = p\left(\hat{w} - \frac{\mathrm{cn}(\hat{w};\lambda)\mathrm{dn}(\hat{w};\lambda)}{\mathrm{sn}(\hat{w};\lambda)} - E(\hat{w};\lambda)\right), \tag{85}$$

where $E(\hat{w};\lambda)$ is the elliptic integral of the second kind. This provides a parametric representation of the free surface (with $\hat{w}$ as the parameter).

Having presented a simplification of Kinnersley's original solutions, we now briefly indicate how (83) relates to the newly derived representation of the solutions (44) (and hence how (85) relates to the simple form given in (45)). In terms of the notation of the present paper, Kinnersley's solution can be seen to imply

$$R(\zeta) = \left(\zeta z_\zeta(\zeta)\right)^{1/2} \propto \mathrm{ns}\,(a\log\zeta + b;\lambda) \tag{86}$$

for some constants a and b (a can be shown to be real). Recalling the fact that ns [1] is doubly periodic (and meromorphic) with a real period $4K$ and imaginary period $2iK'$ (where K, K' are the appropriate complete elliptic integrals), we identify the imaginary period $2iK'$ with the angular periodicity of $R(\zeta)$ (i.e., it is necessary that if $\zeta \to e^{2\pi i}\zeta$ then $R(\zeta) \to R(\zeta)$ to ensure single-valuedness of the conformal mapping function) so that

$$a * 2\pi i \equiv 2iK', \tag{87}$$

and the real half-period $2K$ with the transformation $\zeta \to \rho^2\zeta$ so that

$$2a\log\rho \equiv 2K. \tag{88}$$

In this way, it becomes apparent how

$$\mathrm{ns}(a\log\zeta + b;\lambda) \propto \frac{P(-\zeta\zeta_1^{-1})}{P(\zeta\zeta_1^{-1})}. \tag{89}$$

The correspondence of (44) with Kinnersley's solution is then clear.

Remark 11. Note that if $\hat{w} \to \hat{w} + 2K$ (corresponding to $\zeta \to \rho^2\zeta$), then $\mathrm{ns}(\hat{w},\lambda) \to -\mathrm{ns}(\hat{w},\lambda)$. This corresponds to the antiloxodromic property of $R(\zeta)$.

Remark 12. Kinnersley's type II solutions (the antisymmetric solutions) can also be simplified in an analogous fashion. To see this directly, note that, as pointed out in [8], waves of type IIa are related to those of type Ia (now simplified to the form (83)) by a simple reciprocal modulus transformation [1].

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