*J. Fluid Mech.* (2000), *vol.* 409, *pp.* 223–242. Printed in the United Kingdom



# Hele-Shaw flows and water waves

By DARREN G. CROWDY

Department of Mathematics, 2-335, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(Received 15 December 1998 and in revised form 22 November 1999)

By adapting a new mathematical approach to the problem of steady free-surface Euler flows with surface tension recently devised by the present author, it is demonstrated that exact solutions for steady, free-surface multipole-driven Hele-Shaw flows with surface tension can be constructed using similar methods. Moreover, a (one-way) mathematical transformation between exact solutions to the two distinct free-boundary problems is identified: known exact solutions for free-surface Euler flows with surface tension are shown to automatically generate steady quadrupolar-driven Hele-Shaw flows (with non-zero surface tension) existing in exactly the same domain with the same free surface. This correspondence highlights the essential dynamical differences between the two physical problems. Using the transformation, the exact Hele-Shaw analogues of all known exact solutions for free-surface Euler flows (including Crapper's classic capillary water wave solution) are catalogued thereby producing many previously unknown exact solutions for steady Hele-Shaw flows with capillarity. In particular, this paper reports what are believed to be the first known exact solutions for Hele-Shaw flows with surface tension in a doubly-connected fluid region.

## 1. Introduction

The study of Hele-Shaw flows and of water waves are two areas in which Philip Saffman has made many remarkable contributions throughout his career. Water wave theory involves the study of solutions of the two-dimensional Euler equations and it is well-known that Euler flows and Hele-Shaw flows have some unexpected similarities. For example, it is by now a standard textbook fact (e.g. Batchelor 1967; Acheson 1990) that Hele-Shaw cells can be used as an apparatus for visualizing the streamline pattern of two-dimensional Euler flows of an ideal fluid past an obstacle. Indeed, such a visualization constitutes the very first photograph in Van Dyke's *Album of Fluid Motion* (Van Dyke 1982) and it is annotated with the comment that 'It is at first sight paradoxical that the best way of producing the unseparated pattern of plane potential flow past a bluff object, which would be spoiled by separation in a real fluid of even the slightest viscosity, is to go to the opposite extreme of creeping flow in a narrow gap, which is dominated by viscous forces.'

These unexpected similarities between the two very different physical problems do not seem quite so unexpected when one writes down the mathematical problem statement in each case. Indeed, after the various approximations and assumptions, in the bulk fluid both Euler flow and Hele-Shaw flow reduce to a Laplacian field equation for a velocity potential $\phi$ with the fluid velocity given as $\boldsymbol{u} = \nabla\phi$, i.e. the problems are kinematically equivalent. The essential (and only) difference between the problems is dynamical in nature – in the (suitably non-dimensionalized) Hele-Shaw problem, the pressure $p(x, y)$ in the fluid is linearly related to the potential function via

$$p(x, y) = -\phi(x, y) \tag{1.1}$$

while for steady potential flows, integration of the Euler equations yields Bernoulli's theorem which states that the fluid pressure $p(x, y)$ is given as a nonlinear function of the velocity potential via the condition that

$$p + \frac{|\boldsymbol{u}|^2}{2} \equiv p + \frac{\phi_x^2 + \phi_y^2}{2} \tag{1.2}$$

is constant on streamlines.

For this reason, as soon as one starts to consider free-surface problems in which the fluid pressure usually enters the boundary conditions explicitly, one no longer anticipates any kind of connection, either physical or mathematical, between Euler and Hele-Shaw flows. It is the purpose of this paper to illustrate that, in fact, there continues to exist some rather surprising mathematical similarities and connections.

More specifically, it is shown herein that there is a mathematical connection between the problem of finding steady free-surface Euler flows with surface tension on the free boundaries, and the problem of finding steady quadrupole-driven free-surface Hele-Shaw flows with surface tension. By generalizing, in a natural way, an approach recently developed in Crowdy (1999*a*) for identifying exact solutions to the problem of Euler flows with free capillary surfaces, it is demonstrated that the problem of free-surface Hele-Shaw flows with surface tension and driven by multipoles can be tackled using similar ideas, and that exact solutions to the latter problem can thus be found.

While there exist many known exact solutions for multipole-driven free-surface Hele-Shaw flows with zero surface tension, few exact solutions for the much more challenging problem with non-zero surface tension are known, even in the steady case. Vasconcelos & Kadanoff (1991) identified perhaps the first exact solutions for a steady Hele-Shaw flow in the presence of surface tension, but despite their mathematical significance, the resulting solutions are rather artificial and appear to be of limited physical interest. Later, using Schwarz function theory, Entov, Etingof & Kleinbock (1993) found a class of exact solutions for steady Hele-Shaw flows in a radial (i.e. blob or bubble) geometry driven by a single quadrupole either at the origin (in the case of a simply-connected blob) or at infinity (in the case of a single bubble). To the best of the author's knowledge, these two studies represent the only known exact solutions to steady free-surface Hele-Shaw flows with surface tension.

The present method of solution of the free boundary problem (and that of Crowdy 1999*a*) relies on several important mathematical ideas – in particular, the consideration of an analytically continued boundary condition, and the determination of the singularity structure of the conformal mapping function by writing this analytically-continued equation in such a way that an identification with a well-studied differential equation can be established. These same mathematical ideas have proved remarkably versatile and have been effective in tackling a wide range of free-surface problems, e.g. free-surface capillary flows (Crowdy 1999*a–c*; Tanveer 1996) and deep water gravity waves (Tanveer 1991) as well as the problem of unsteady Hele-Shaw flows with small non-zero surface tension (e.g. Tanveer 1993). This, we believe, demonstrates the importance of the mathematical ideas. We remark that the present paper provides an alternative approach to the Schwarz function method presented by Entov *et al.* (1993). The present approach is, however, particularly flexible and readily generalizes to unveil previously unknown exact solutions for flows in different flow geometries and, in particular, to flows in doubly-connected fluid regions (where the extension of a Schwarz function approach is not obvious).

The general technique for solving the general free-boundary problem is illustrated explicitly in § 2 in a 'water-wave geometry'. While this new approach provides a flexible method which can readily be adapted to find exact solutions to a given multipole-driven Hele-Shaw flow in various geometries and in domains of various connectivities, we choose not to employ this method to generate any new exact solutions (although this is easily done). Rather, we prefer to generate illustrative exact solutions to the Hele-Shaw problem by pointing out an intriguing mathematical transformation that automatically produces an exact solution to a quadrupole-driven Hele-Shaw flow with surface tension corresponding to known exact solutions to the problem of free-surface Euler flow with surface tension. This transformation is presented in § 4. The exact solutions produced by this transformation are surprisingly instructive in illustrating the essential dynamical differences between the two distinct free-boundary problems because the two 'corresponding' (yet physically-distinct) flows take place in the same flow region, bounded by the same free-surface. This transformation thus provides an explicit demonstration of the physical differences between the two classes of flows by showing the streamline distribution for each flow necessary to sustain exactly the same free-surface force distribution (due to surface tension) in a state of equilibrium. A number of 'corresponding' solutions to each problem are plotted and juxtaposed in this paper.

It is important to emphasize that the mathematical transformation only works in one direction. This is because, as just seen, the pressure condition for the problem of free-surface Euler flow with surface tension is significantly more nonlinear in the field variable $\phi$, and therefore much more difficult to solve, than the steady Hele-Shaw problem. It is only to be expected that it is not possible to automatically generate exact solutions to a more difficult problem by solving an easier one, even though the converse is indeed possible, as demonstrated herein.

## 2. Solution of the free-boundary problem

### 2.1. *Hele-Shaw flows with surface tension*

We illustrate the general method by finding solutions to a steady quadrupole-driven Hele-Shaw free-surface flow in a 'water-wave' geometry. Suppose there exists a regular, spatially-periodic array of quadrupoles separated by distance $\lambda = 2\pi/k$ at some distance beneath the free surface of a layer of viscous fluid of infinite depth in a Hele-Shaw cell. Uniform surface tension forces act on the free surface. It is reasonable to expect that there might exist solutions for the steady shape of the free surface that are also periodic with spatial period $\lambda$. We now seek such solutions. In this case, it is enough to seek a conformal mapping from the cut unit circle in figure 1 to one period of the free surface. Such a map will therefore have the general form

$$z(\zeta) = \frac{2\pi}{k} + \frac{\mathrm{i}}{k}(\log\zeta + g(\zeta)) \tag{2.1}$$

where, for smooth waves with no corners or cusps, $g(\zeta)$ is analytic everywhere inside $|\zeta| \leqslant 1$. The point $\zeta = 0$ maps to the point at infinite vertical depth, while the two sides of the logarithmic branch cut (taken along the positive real axis) map to the two vertical sides (separated by a horizontal distance $\lambda$) of the fluid region of interest.

Since the velocity potential is harmonic inside the fluid (except at the quadrupole)

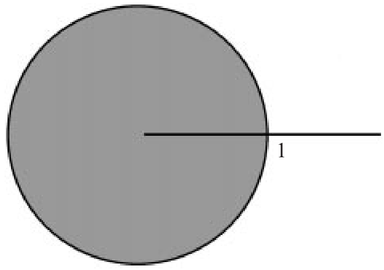


FIGURE 1. Parametric $\zeta$-plane.

we can define a composed complex potential $W(\zeta)$ for this problem:

$$W(\zeta) = w(z(\zeta)) \tag{2.2}$$

where $w(z) = \phi + i\psi$ is the usual complex potential. Defining $\zeta_q$ as the point inside the unit circle such that

$$z_q = z(\zeta_q) \tag{2.3}$$

is the physical location of the quadrupole, it is clear that the Laurent expansion of $W(\zeta)$ about $\zeta_q$ has the form

$$W(\zeta) = \frac{Q_{-2}}{(\zeta-\zeta_q)^2} + \frac{Q_{-1}}{(\zeta-\zeta_q)} + \cdots \tag{2.4}$$

for some constants $Q_{-2}$ and $Q_{-1}$. For a flow driven by a single quadrupole, $\zeta_q$ is the only allowable singularity of $W(\zeta)$ inside the unit circle. We remark, however, that the general approach presented here readily extends to incorporate any distribution of multipole singularities (of any order $n \geqslant 2$).

The kinematic condition on the free-surface $|\zeta| = 1$ is that it should be a streamline, i.e. $\psi = 0$. In complex notation, this condition is given by

$$\mathrm{Im}\,[W(\zeta)] = 0. \tag{2.5}$$

The symmetry constraints also imply that this same condition also holds on the real axis between $0 < \zeta < 1$. The (non-dimensionalized) pressure condition on the free surface is that the pressure $\phi(x, y)$ is exactly balanced by the surface tension due to curvature, i.e.

$$-\phi = \kappa. \tag{2.6}$$

Since $W(\zeta) = \phi + i\psi$ and since $\psi = 0$ on $|\zeta| = 1$, the pressure condition is equivalent to

$$-W(\zeta) = \kappa = \frac{1}{z_\zeta}\frac{d}{d\zeta}\left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2} \quad \text{on } |\zeta| = 1 \tag{2.7}$$

or equivalently

$$-\frac{d}{d\zeta}\left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2} - W(\zeta) z_\zeta(\zeta) = 0. \tag{2.8}$$

From the fact that there is just a single quadrupole inside the fluid region at the point $\zeta_q$, the most general admissible form of the composed complex potential $W(\zeta)$ can now be deduced using the boundary condition (2.5), specifically

$$W(\zeta) = Q\left(\frac{(\zeta-\eta_1)(\zeta^{-1}-\bar{\eta}_1)(\zeta-\eta_2)(\zeta^{-1}-\bar{\eta}_2)}{(\zeta-\zeta_q)^2(\zeta^{-1}-\bar{\zeta}_q)^2}\right). \tag{2.9}$$

$Q$ is related to the strength of the quadrupole, and is expected to be specifiable. This strength will be specified in a natural way later in the analysis.

Define the function $q_1(\zeta)$ as follows:

$$q_1(\zeta) \equiv -\frac{W(\zeta)\bar{z}_\zeta(\zeta^{-1})}{\zeta^2}. \tag{2.10}$$

Note that outside the unit circle (i.e. for $|\zeta| > 1$), the only singularity of $q_1(\zeta)$ is a single second-order pole at $\bar{\zeta}_q^{-1}$. We also define the function $\Psi(\zeta)$ as follows:

$$\Psi(\zeta) = \left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2}. \tag{2.11}$$

Note that exactly the same function is used to develop the theory of Crowdy (1999*a*). Then the pressure condition on the fluid boundary implies that

$$-\frac{\mathrm{d}\Psi(\zeta)}{\mathrm{d}\zeta} - W(\zeta)z_\zeta(\zeta) = 0. \tag{2.12}$$

Equivalently

$$-\frac{\mathrm{d}\Psi(\zeta)}{\mathrm{d}\zeta} + q_1(\zeta)\Psi^2(\zeta) = 0. \tag{2.13}$$

Equation (2.13) has the analytical form of a Riccati equation (or, more precisely, a Bernoulli equation – not to be confused with the free-surface Bernoulli pressure condition!). For references, see Hille (1976). The coefficient function $q_1(\zeta)$ is analytic everywhere outside the unit circle except for the known fixed singularity at $\bar{\zeta}_q^{-1}$. It can be shown – either by a linearization-type procedure (see Crowdy 1999*a*) or by adapting the standard results of Painleve (see the chapter on Riccati equations in Hille 1976) – that the movable singularities of $\Psi$ at the regular points of equation (2.13) in $|\zeta| > 1$ are generically simple poles. From the definition of $\Psi(\zeta)$, this is clearly also true of $[\zeta z_\zeta(\zeta)]^{1/2}$ since $[\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})]^{-1/2}$ is known *a priori* to be analytic everywhere in $|\zeta| > 1$.

The above result does not hold, however, at any fixed singularities of equation (2.13). To determine the behaviour of $\Psi$ in the neighbourhood of the fixed singularity $\bar{\zeta}_q^{-1}$ it is appropriate to exploit the well-known fact that Bernoulli equations can be linearized: defining $M = \Psi^{-1}$ it is clear that

$$M_\zeta + q_1(\zeta) = 0. \tag{2.14}$$

Note that (2.14) is not, of course, linear because the coefficient function $q_1(\zeta)$ is itself a nonlinear function of the dependent variable $M(\zeta)$. However, the important point is that, outside the unit circle, the solutions to equation (2.14) for $M$ share all the analytic properties of solutions of a linear equation with meromorphic coefficients having known, fixed singularities. It can be immediately deduced that the only singularity of $M(\zeta)$ outside the unit circle is at $\bar{\zeta}_q^{-1}$ which implies (from the structure of $M$) that $[\zeta z_\zeta]^{1/2}$ necessarily has zeros at these points. By association with this linear-type equation, we conclude that $M$ has no other movable singularities in $|\zeta| > 1$. We also note in passing that this 'linearization' of the Bernoulli-type equation (2.13) can be used to deduce the result (already established above) that the singularities of $\Psi$ are generically simple poles.

Next, note that the tangent $T$ to the free surface can be written in complex form as

$$T = x_s + \mathrm{i}y_s = z_s \tag{2.15}$$

where, in terms of $\zeta$ on the unit circle, it is known that

$$z_s = \frac{\mathrm{i}\zeta z_\zeta}{|z_\zeta|}. \tag{2.16}$$

But from this equation it can be seen that

$$\Psi = \frac{\zeta z_\zeta^{1/2}(\zeta)}{\bar{z}_\zeta^{1/2}(\zeta^{-1})} = -\mathrm{i}z_s. \tag{2.17}$$

By the assumed periodicity of the free surface of the fluid (and therefore the periodicity of its tangent as one goes around the unit $\zeta$-circle), it can be concluded that $\Psi$ (and hence $M$) is a single-valued function on the unit circle $|\zeta| = 1$. This, in turn, implies that the singularity of $M$ at $\bar{\zeta}_q^{-1}$ outside the unit circle must be a simple pole. Thus we conclude that $M$ is meromorphic everywhere in $|\zeta| > 1$.

However, it can easily be verified that $M$ satisfies the following functional equation:

$$\overline{M}(\zeta^{-1}) = \frac{1}{M(\zeta)} \tag{2.18}$$

which provides the analytic continuation of $M(\zeta)$ inside the unit circle and, in particular, shows that $M$ is a meromorphic function everywhere in the plane (including at infinity). $M$ is therefore a rational function.

From this, it is readily deduced that $[\zeta z_\zeta(\zeta)]^{1/2}$ is also rational, analytic in $|\zeta| \leqslant 1$ (with no zeros there) and with known zeros (i.e. there is only one in this case – at $\bar{\zeta}_q^{-1}$) in $|\zeta| > 1$. It has also been deduced that $[\zeta z_\zeta(\zeta)]^{1/2}$ has movable simple pole singularities in $|\zeta| > 1$ although precisely how many remains to be determined. However, this can now be done using the well-known test-power test (see Hille 1976). Since $[\zeta z_\zeta(\zeta)]^{1/2}$ is rational, it is necessarily either analytic as $\zeta \to \infty$ or, at worst, has a polar singularity there. Hypothesizing that

$$[\zeta z_\zeta(\zeta)]^{1/2} \sim \zeta^\delta \tag{2.19}$$

as $\zeta \to \infty$ for some integer $\delta$ and applying the test-power test to (2.13) leads conclusively to the fact that $[\zeta z_\zeta(\zeta)]^{1/2}$ is necessarily analytic at the point at infinity. A local analysis of the equation shows that, in fact, $[\zeta z_\zeta(\zeta)]^{1/2}$ tends to a constant function as $\zeta \to \infty$. Thus, we conclude that

$$z_\zeta(\zeta) = \frac{A}{\zeta}\left(\frac{\zeta - \bar{\zeta}_q^{-1}}{\zeta - \zeta_1}\right)^2 \tag{2.20}$$

for some $\zeta_1$ – a single movable pole.

The admissible functional form of $[\zeta z_\zeta(\zeta)]^{1/2}$ has now been deduced, to within a (finite number) of unknown parameters. The question of whether there exists an equilibrium solution is now reduced to the question of the existence of parameters $\{A, \zeta_1, \zeta_q\}$ (appearing in (2.20)) and $\{Q, \eta_1, \eta_2\}$ (appearing in (2.9)) for which equation (2.13) is an identity for all $\zeta$. Since both terms in equation (2.13) are known to be rational functions which are real on the unit circle with second-order poles at $\zeta_q$ and the inverse point $\bar{\zeta}_q^{-1}$, the finite system of nonlinear equations to be solved result from equating the principal parts of both terms at the single second-order pole at $\zeta_q$ (the principal part at the inverse point $\bar{\zeta}_q^{-1}$ will automatically vanish by reflection). This involves three conditions to be satisfied by the six parameters. It is clear that there remains three degrees of freedom in this finite system. It is most natural, perhaps, to specify (i) $\zeta_q$ which can be thought of as specifying the position $z_q$ of the quadrupole (see equation (2.3)), (ii) $Q$ which corresponds to specifying the strength of the quadrupole and, finally, (iii) the parameter $A$ which corresponds to specifying the wavelength of the surface disturbance (and hence the inter-quadrupole spacing $\lambda$). If a solution exists, the three conditions necessary for (2.13) to be an identity then determine the three remaining parameters $\zeta_1, \eta_1$ and $\eta_2$.

The above prescription is now straightforward. Using it, we might well expect to find either no solution, a single solution or indeed multiple solutions for $\{\zeta_1, \eta_1, \eta_2\}$ given any specified values of $\{\zeta_q, A, Q\}$. The above analytical arguments have reduced the full nonlinear free-boundary problem to the problem of solving a (consistent) finite system of nonlinear algebraic equations for a finite set of parameters.

## 3. Capillary waves

The methodology described in the previous section can be used to construct exact solutions to the free-surface problem of Hele-Shaw flows in a fluid of infinite depth (with surface tension) driven by a periodic array of quadrupoles. Rather than use the above methodology however, we choose to produce new exact solutions to this Hele-Shaw problem by a more intriguing, and theoretically interesting, route. There are many general similarities in the above methodology and that presented in Crowdy (1999*a*) to address the problem of inviscid Euler flows with surface tension. The problem of inviscid Euler flows with surface tension is, in fact, a significantly more difficult problem to solve – the pressure condition on the interface being more nonlinear (in the conformal mapping function) than in the analogous Hele-Shaw problem. However, in § 4 it will be demonstrated that to every exact solution to the more difficult problem of free surface potential flow with capillarity, an exact quadrupolar-driven Hele-Shaw analogue can be found.

First, in this section, we review the Crapper (1957) exact solution for pure capillary waves on deep water as retrieved recently in Crowdy (1999*a*). Unlike Crapper's original paper, Crowdy (1999*a*) employs conformal mapping considerations. The form of the conformal mapping for a spatially periodic wave of wavelength $\lambda = 2\pi/k$ (relative to a frame moving with the wave at a speed $c$ to the right) is taken to be

$$\mathscr{Z}(\zeta) = \frac{2\pi}{k} + \frac{\mathrm{i}}{k}(\log\zeta + f(\zeta)) \tag{3.1}$$

where $f(\zeta)$ is taken to be analytic in the unit circle $|\zeta| = 1$.

The complex potential is defined as

$$\hat{w}(\mathscr{Z}) = \phi(x, y) + \mathrm{i}\psi(x, y) \tag{3.2}$$

where $\phi(x, y)$ is a velocity potential and $\psi(x, y)$ is the streamfunction. Infinitely far from the free surface, the flow is uniform. The kinematic boundary condition on the fluid interface is

$$\mathrm{Im}\,[\hat{w}] = \psi = 0 \tag{3.3}$$

on the boundary. These requirements dictate the functional form for the composed function $\mathscr{W}(\zeta) = \hat{w}(\mathscr{Z}(\zeta))$. In particular, for the water wave problem considered here we take

$$\mathscr{W}(\zeta) = \frac{2\pi c}{k} + \frac{\mathrm{i}c}{k}\log\zeta \tag{3.4}$$

where $c$ is the velocity of the fluid at large depths (with respect to a frame of reference in which the fluid boundary is stationary). The non-dimensionalized Bernoulli condition on the fluid interface can be written

$$-\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\zeta\mathscr{Z}_\zeta(\zeta)}{\zeta^{-1}\overline{\mathscr{Z}}_\zeta(\zeta^{-1})}\right]^{1/2}+\Gamma\mathscr{Z}_\zeta-\frac{\mathscr{W}_\zeta(\zeta)\overline{\mathscr{W}}_\zeta(\zeta^{-1})}{2\overline{\mathscr{Z}}_\zeta(\zeta^{-1})}=0. \tag{3.5}$$

For water waves on a fluid of infinite depth, $\Gamma = c^2/2$. Note that (3.5) must hold globally by analytic continuation. By direct inspection and comparison of (3.5) and (2.8) it is seen that it is likely to be much harder to find functions $\mathscr{Z}(\zeta)$ satisfying (3.5) than it is to find functions $z(\zeta)$ satisfying (2.8).

### 3.1. *Crapper's* (*1957*) *solution for capillary waves*

The approach to this problem presented in Crowdy (1999*a*) obviates the need for many of the unwieldy algebraic manipulations of the original paper, Crapper (1957). In Crowdy (1999*a*), Crapper's solution is retrieved in the following form:

$$\mathscr{Z}(\zeta)=\mathrm{i}A\left(\log\zeta-\frac{4\zeta_1}{\zeta-\zeta_1}\right) \tag{3.6}$$

and

$$\mathscr{W}(\zeta)=\frac{2\pi c}{k}+\frac{\mathrm{i}c}{k}\log\zeta \tag{3.7}$$

with

$$\Gamma A^2=\frac{c^2}{2k^2}, \tag{3.8}$$

$$\Gamma A=\frac{1}{2}\frac{|\zeta_1|^2-1}{|\zeta_1|^2+1}. \tag{3.9}$$

It is important for what follows to note, from (3.6), that

$$\mathscr{Z}_\zeta(\zeta)=\frac{\mathrm{i}A}{\zeta}\left(\frac{(\zeta+\zeta_1)}{(\zeta-\zeta_1)}\right)^2. \tag{3.10}$$

## 4. Mathematical transformation

We now make the following identification between the conformal maps and complex potentials $(\mathscr{Z}(\zeta),\mathscr{W}(\zeta))\mapsto(z(\zeta),W(\zeta))$ of the respective physical problems:

$$z(\zeta)=\mathscr{Z}(\zeta), \tag{4.1}$$

$$W(\zeta)=\Gamma-\frac{\mathscr{W}_\zeta(\zeta)\overline{\mathscr{W}}_\zeta(\zeta^{-1})}{2\mathscr{Z}_\zeta(\zeta)\overline{\mathscr{Z}}_\zeta(\zeta^{-1})}, \tag{4.2}$$

and make the important observation that if $[\zeta\mathscr{Z}_\zeta(\zeta)]^{1/2}$ happens to be a rational function (i.e. as in Crapper's solution above – see (3.10)) with simple zeros outside the unit circle, then the corresponding $W(\zeta)$ also turns out to be a rational function with second-order poles at the inverse points to these zeros inside the unit circle. Thus, $W(\zeta)$ (as given by the above transformation) corresponds to physically-admissible complex potential for a quadrupole-driven Hele-Shaw flow, the number of driving quadrupoles being exactly equal to the number of zeros of the derivative of the conformal map $\mathscr{Z}$ of the 'corresponding' free-surface Euler flow problem. In fact, it follows from the general theory (Crowdy 1999*a*) that $[\zeta\mathscr{Z}_\zeta(\zeta)]^{1/2}$ is generically a meromorphic function. Exact solutions to this problem correspond to the case where

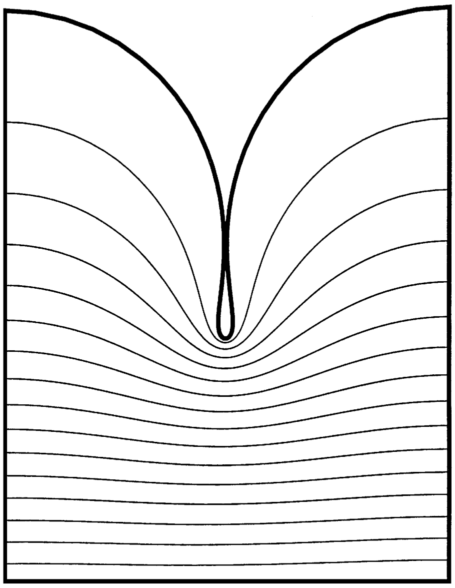

FIGURE 2. Crapper's (1957) solution.

this meromorphic function is a rational function (we refer the interested reader to Crowdy 1999*a* for more details) and therefore has a finite set of zeros outside the unit circle. In the context of the transformation (4.1)–(4.2) it would therefore seem that, in general, any exact solution of the problem of free-surface Euler flows with surface tension will generate an exact solution to a quadrupole-driven Hele-Shaw flow with surface tension. Moreover, because the conformal maps are the same, the corresponding Hele-Shaw flow takes place in exactly the same domain as the Euler flow, and has exactly the same free surface.

In figures 2 and 3, the streamline plots for the special limiting form of Crapper's water wave (i.e. the critical 'pinching' case where two sides of the interface draw together to form an enclosed bubble) are shown for both Crapper's solution and the analogous Hele-Shaw flow taking place in exactly the same domain. The latter solution is new. Note that while the free surface shapes for these very different physical problems are exactly the same, the streamline distribution (i.e. the flow) keeping that free surface in equilibrium is very different in each case. This provides a rather explicit illustration of the differing dynamics of the two kinematically-equivalent problems.

In the light of the well-known fact that Hele-Shaw cells can be used as an apparatus to realize ideal Euler flows past fixed obstacles, another view of our results is to show that it is possible to use a Hele-Shaw cell to realize the shape of pure capillary wave profiles (by suitably placing a quadrupole, of appropriate strength, beneath an infinite free surface in such a cell). Note, however, that while the free-surface streamline is exactly the same as for pure capillary waves, the internal streamline distributions are clearly very different due to the intrinsic dynamical differences between the two problems.

## 5. Hele-Shaw flows in fluid layers

Given the mathematical connection between Crapper's (1957) solution, and the newly-derived solutions for quadrupolar-driven Hele-Shaw flows, it is natural to ask

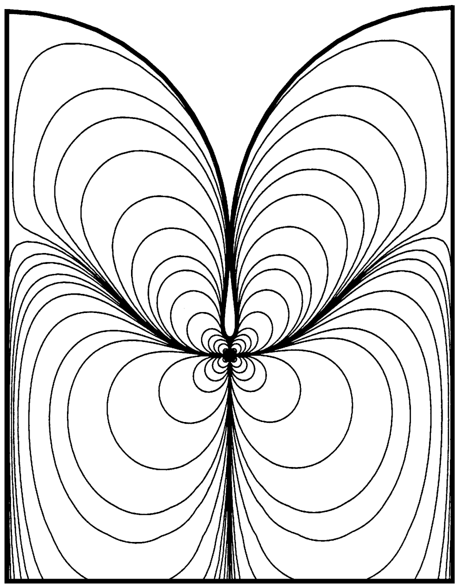

FIGURE 3. Hele-Shaw analogue of Crapper's (1957) solution.

whether a steady, exact Hele-Shaw solution can be found in a layer of fluid of finite thickness. The same question was asked by Kinnersley (1976) who explicitly identified finite-depth extensions of Crapper's solutions.

### 5.1. *Quadrupole-driven Hele-Shaw flows in a fluid layer*

First, we indicate briefly the natural extension of the theory in § 2, i.e. we outline how to solve the free-boundary problem in the general case when the layer of fluid in a Hele-Shaw cell has finite thickness. The presence of two disjoint free surfaces requires some non-trivial (but natural) modifications of the analysis. Consider an infinitely long sheet of fluid of finite thickness in a Hele-Shaw cell containing a spatially-periodic array of quadrupole singularities. Surface tension forces act on both fluid interfaces. We will seek symmetric solutions where the shape of both the upper and lower fluid interfaces are spatially periodic in $x$ with wavelength $\lambda \equiv 2\pi/k$. Under this assumption, it is enough to consider the structure of a conformal map from a standard parametric region (in a $\zeta$-plane) to a window of the fluid sheet of length $2\pi/k$. Therefore, consider the conformal mapping function $z(\zeta)$ from the annulus $\rho < |\zeta| < 1$ (as shown in figure 4) in a parametric conformal mapping plane ($\zeta$-plane) to a one-period window of the finite sheet of fluid.

Without loss of generality, it can be assumed that the upper interface of the fluid sheet is the image of the circle $|\zeta| = 1$. We assume that the second free surface of the fluid sheet maps from $|\zeta| = \rho$ where $0 < \rho < 1$. It is clear that in this case the conformal mapping function can be written in the form

$$z(\zeta) = \frac{2\pi}{k} + \frac{\mathrm{i}}{k}(\log \zeta + f(\zeta)) \tag{5.1}$$

where $f(\zeta)$ is analytic everywhere in the annulus $\rho < |\zeta| < 1$. The branch cut is taken along the positive real axis. Note that it is assumed that $z_\zeta$ vanishes nowhere in $\rho \leqslant |\zeta| \leqslant 1$. This corresponds to seeking smooth surface waves with no corners or cusps.

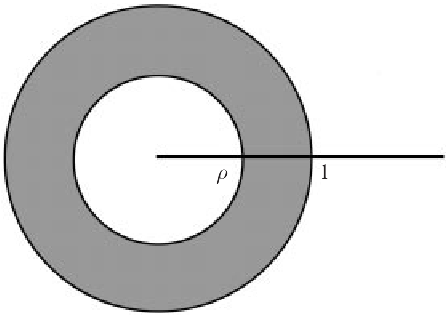


FIGURE 4. Parametric $\zeta$-plane.

The complex potential is defined to be

$$w(z) = \phi(x, y) + \mathrm{i}\psi(x, y). \tag{5.2}$$

The kinematic conditions on the boundaries are satisfied if the complex potential $W(\zeta)$ satisfies

$$\overline{W}(\zeta^{-1}) = W(\zeta) \quad \text{on } |\zeta| = 1, \tag{5.3}$$

$$\overline{W}(\rho^2\zeta^{-1}) = W(\zeta) \quad \text{on } |\zeta| = \rho. \tag{5.4}$$

For symmetric solutions, it is also necessary that the real axis $\rho < \zeta < 1$ corresponds to a streamline. Furthermore, if $\zeta_q$ is the point inside the annulus $\rho < |\zeta| < 1$ corresponding to the pre-image of the quadrupole, i.e. $z_q = z(\zeta_q)$, then the only singularity of $W(\zeta)$ inside $\rho < |\zeta| < 1$ is a second-order pole at $\zeta_q$.

On $|\zeta| = 1$, the condition of constant pressure is taken to be

$$-\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\zeta z_\zeta}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2} - W(\zeta)z_\zeta = 0. \tag{5.5}$$

Assuming the pressure on both sides of the sheet of fluid to be the same, the pressure condition on the other free surface becomes

$$+\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\zeta z_\zeta}{\rho^2\zeta^{-1}\bar{z}_\zeta(\rho^2\zeta^{-1})}\right)^{1/2} - W(\zeta)z_\zeta = 0. \tag{5.6}$$

Note the change of sign in the curvature terms. Steady solutions only exist provided that (5.5) and (5.6) are mutually consistent. This is seen to require that the conformal mapping $z(\zeta)$ satisfies the following functional equation:

$$[\rho^2\zeta z_\zeta(\rho^2\zeta)]^{1/2} = -[\zeta z_\zeta(\zeta)]^{1/2} \tag{5.7}$$

for all $\zeta \neq 0$. The problem is to find a function $z(\zeta)$, analytic everywhere in the annulus $\rho < |\zeta| < 1$ such that (5.6) and (5.7) hold globally.

The analysis of §2 can be readily generalized to this new geometrical situation. The analysis follows in essentially the same way that the analysis of Crowdy (1999*a*) is extended to retrieve Kinnersley's symmetric sheet wave solutions in Crowdy (1999*c*). By a natural extension of the approach in Crowdy (1999*a*), Crowdy (1999*c*) rederived Kinnersley's symmetric sheet wave solutions and found a much simpler representation for them. The formulation utilizes the theory of *loxodromic functions* (Valiron 1947), and the fundamental region of interest (as in the present case) becomes the annulus $\rho^2 < |\zeta| < 1$. A loxodromic function $g(\zeta)$ is a function that is meromorphic for all $\zeta \neq 0$ and which satisfies the multiplicative periodicity condition

$$g(\rho^2\zeta) = g(\zeta), \quad \zeta \neq 0. \tag{5.8}$$

It turns out that one can deduce that steady solutions of the problem of quadrupole-driven Hele-Shaw flows in a finite fluid sheet are such that $[\zeta z_\zeta]^{1/2}$ is meromorphic everywhere in the sub-annulus $\rho^2 < |\zeta| < \rho$ with just simple pole singularities there (it is necessarily analytic in the sub-annulus $\rho < |\zeta| < 1$) and that the function $\zeta z_\zeta$ is, in fact, a loxodromic function. Note that it is clear from (5.7) that $\zeta z_\zeta$ must satisfy the functional equation (5.8). Loxodromic function theory has recently been found to be useful in a number of different free-boundary problems (see, for example, Crowdy & Tanveer 1998; Richardson 1996; Crowdy 1999*c*, *d*). For purposes of brevity, we simply refer the interested reader to the analysis of Crowdy (1999*c*) for an idea of how to make the appropriate generalization of the theory in § 2 when the conformal mapping for the Hele-Shaw problem is from the annulus shown in figure 4. The necessary steps should be clear from a comparison of Crowdy (1999*c*) and given the detailed analysis of § 2.

While the earlier mathematical arguments can indeed be extended to derive a systematic approach to solving the above free-boundary problems we again choose here to produce new exact solutions by exploiting the mathematical transformation described in § 4. This time we establish the 'Hele-Shaw analogue' of Kinnersley's symmetric sheet wave solutions.

## 6. Capillary waves on fluid sheets

We briefly describe the formulation of the problem of pure capillary waves on fluid sheets as presented in Crowdy (1999*c*). Denote the conformal map by $\mathscr{Z}(\zeta)$ and the complex potential by $\mathscr{W}(\zeta)$. The non-dimensionalized Bernoulli condition $\zeta$ on $|\zeta| = 1$ gives

$$\frac{\mathscr{W}_\zeta(\zeta)\overline{\mathscr{W}}_\zeta(\zeta^{-1})}{2\bar{\mathscr{Z}}_\zeta(\zeta^{-1})} = -\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\zeta\mathscr{Z}_\zeta(\zeta)}{\zeta^{-1}\overline{\mathscr{Z}}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma\mathscr{Z}_\zeta. \tag{6.1}$$

Assuming the same value of the surface tension on both free-surfaces, the Bernoulli condition on $|\zeta| = \rho$ can be written in a similar fashion:

$$\frac{\mathscr{W}_\zeta(\zeta)\overline{\mathscr{W}}_\zeta(\rho^2\zeta^{-1})}{2\bar{\mathscr{Z}}_\zeta(\rho^2\zeta^{-1})} = +\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\zeta\mathscr{Z}_\zeta(\zeta)}{\rho^2\zeta^{-1}\overline{\mathscr{Z}}_\zeta(\rho^2\zeta^{-1})}\right]^{1/2} + \Gamma\mathscr{Z}_\zeta. \tag{6.2}$$

Note further that $\Gamma = c^2/2$.

### 6.1. *Kinnersley's (1976) symmetric sheet waves*

We now present Kinnersley's symmetric sheet wave solutions using the much-simplified representation recently derived by Crowdy (1999*c*).

The conformal map $\mathscr{Z}(\zeta)$ and corresponding complex potential $\mathscr{W}(\zeta)$ are given by

$$\mathscr{Z}(\zeta) = \int_1^\zeta \frac{\mathrm{i}A}{\zeta'}\left(\frac{P(-\zeta'\zeta_1^{-1})}{P(\zeta'\zeta_1^{-1})}\right)^2 \mathrm{d}\zeta', \tag{6.3}$$

$$\mathscr{W}(\zeta) = \frac{2\pi}{k} + \frac{\mathrm{i}c}{k}\log\zeta, \tag{6.4}$$

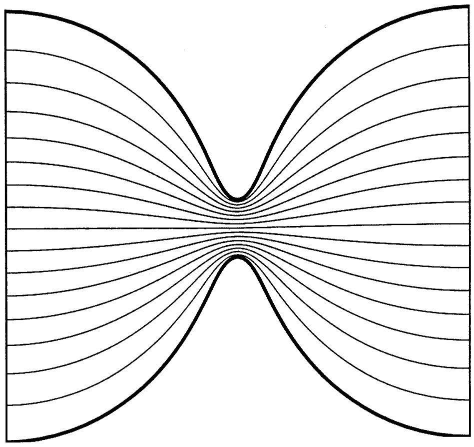

FIGURE 5. Kinnersley's (1976) solution ($\rho = 0.1, \zeta_1 = 3.11$).

where $A$ and $\zeta_1$ are real parameters. Specifying the wavelength of the waves leads to the condition that $A = 1/k$. The parameter $\zeta_1$ controls the wave amplitude. Further details of this solution can be found in Crowdy (1999*c*). The function $P(\zeta)$ is defined via the infinite product expansion

$$P(\zeta) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{6.5}$$

It is straightforward to check that $P(\zeta)$ satisfies the following functional equations for all $\zeta \neq 0$:

$$P(\zeta^{-1}) = -\zeta^{-1}P(\zeta) = P(\rho^2\zeta). \tag{6.6}$$

It is also clear from (6.3) that

$$[\zeta \mathscr{Z}_\zeta]^{1/2} \propto \frac{P(-\zeta'\zeta_1^{-1})}{P(\zeta'\zeta_1^{-1})} \tag{6.7}$$

so that using (6.6), it is easy to check that this solution satisfies the functional equation (5.7) required for consistency.

### 6.2. *Corresponding Hele-Shaw solutions*

As was done in previous sections, we now make the identification between the complex potentials for the two physical problems and their respective conformal mappings. These are exactly as given in (4.1)–(4.2).

In this way, we automatically derive a two-parameter family of exact solutions to the problem of finding the equilibrium shapes of waves on a sheet of fluid of finite thickness in a Hele-Shaw flow containing a periodic array of quadrupoles. Some typical streamline plots for the water wave and Hele-Shaw problems are shown in figures 5 and 6 for the arbitrary choice of parameters $\zeta_1 = 3.11, \rho = 0.1$. Intuitively, $\zeta_1$ provides a measure of the wave amplitude and $\rho$ a measure of the thickness of the fluid sheet.

Finally we remark that, although they have not been explicitly presented here, we anticipate that the anti-symmetric sheet wave solutions of Kinnersley also have an anti-symmetric Hele-Shaw analogue.

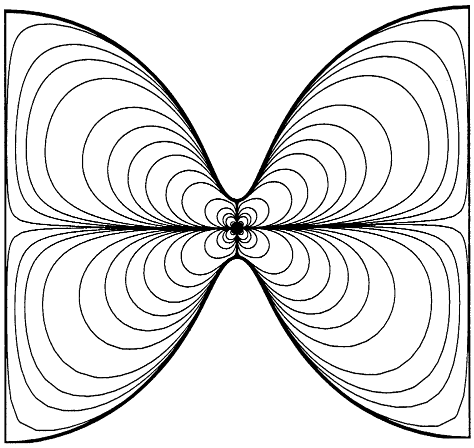

FIGURE 6. Hele-Shaw analogue of Kinnersley's (1976) solution.

## 7. Other exact solutions in different geometries

The mathematical transformation described in § 4 which generates exact solutions for free-surface Hele-Shaw flows driven by quadrupoles is clearly applicable, in principle, to any known exact solution for free-surface Euler flows with surface tension. Unfortunately, because of its high degree of nonlinearity, exact solutions to the latter problem are rare. In this section, we record the 'Hele-Shaw analogues' (via this transformation) of the only other exact solutions for free-surface Euler flows with surface tension known to the present author. These are (i) the exact solution of McLeod (1955) for uniform flow past a bubble, (ii) the solutions found recently by Crowdy (1999*b*) for circulation-induced shape deformations of free drops and bubbles, and (iii) the solutions found by Crowdy (1999*c*) for steady capillary waves on a fluid annulus induced by an irrotational swirling flow inside the annulus. It is emphasized, however, that while the transformation (4.1)–(4.2) immediately generates a subclass of exact solutions for Hele-Shaw flows with surface tension, the solution schemes outlined in § 2 and § 5 can be extended in a straightforward fashion to each of the following flow configurations to find much more general classes of solution. All the Hele-Shaw solutions which follow are new and have not been previously reported in the literature.

### 7.1. *Hele-Shaw analogue of McLeod's* (*1955*) *solution*

In McLeod (1955) an exact solution for a very special choice of Bernoulli constant was identified for the shape of a steadily-translating bubble with surface tension on its boundary. This exact result is, in fact, a non-generic situation as illustrated in some detail in Tanveer (1996) and Crowdy (1999*a*). This solution is plotted in figure 7 along with some typical streamlines. The analogous exact solution for a quadrupolar-driven Hele-Shaw flow with surface tension is shown in figure 8. It is clear that exactly two quadrupoles, of appropriate strength and suitably positioned, are required to maintain the McLeod (1955) free surface in equilibrium in a Hele-Shaw cell. We note that this solution, discovered originally by McLeod (1955), was later rediscovered by Shankar (1992) using power series methods. See also the even earlier work of Vanden-Broeck & Keller (1980). We also note that for any other (generic) choice of Bernoulli constant, the corresponding Hele-Shaw solution is not exact and corresponds to a flow driven

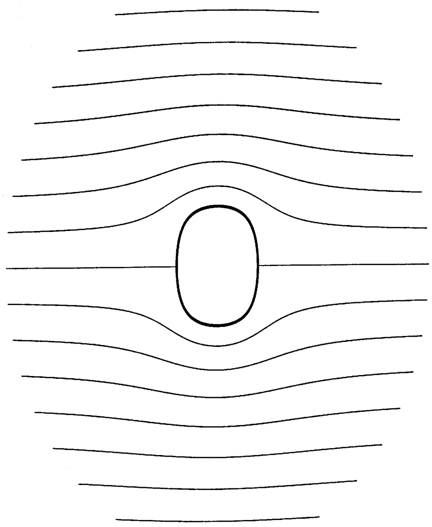

FIGURE 7. McLeod's (1955) solution.

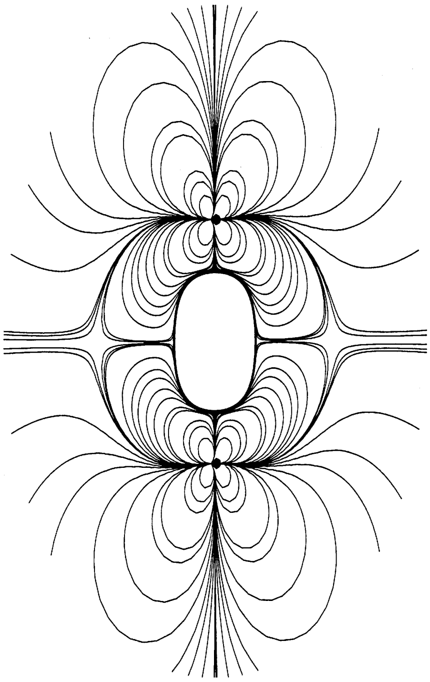

FIGURE 8. Hele-Shaw analogue of McLeod's (1955) solution.

by an infinity of quadrupole singularities forming a cluster-point essential singularity at $\zeta \to \infty$.

### 7.2. *Hele-Shaw analogues of solutions of Crowdy* (1999*b*)

Crowdy (1999*b*) has recently studied the problem of circulation-induced shape deformations of capillary drops and bubbles by considering some simple two-dimensional models which incorporate the essential physics. Such paradigmatic problems are of

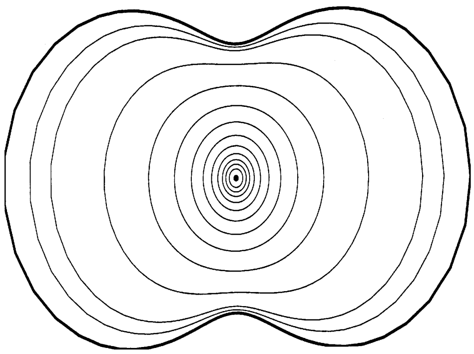

FIGURE 9. Crowdy (1999*b*) solution (blob).

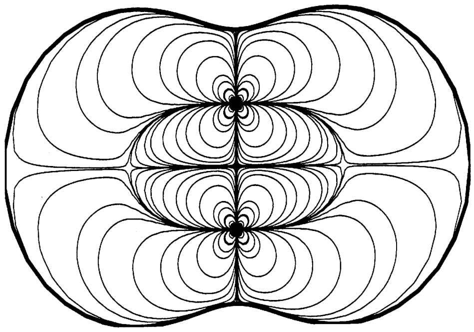

FIGURE 10. Hele-Shaw analogue of Crowdy (1999*b*) solution (blob).

great interest in various chemical engineering processes and in the study of the heat and mass transport properties of liquid dispersions. In the model of a droplet with circulation, a circulatory flow inside a droplet of fluid is modelled by placing a line vortex inside a droplet, the free surface of which is assumed to have a non-zero surface tension. Apart from the line vortex, the flow is assumed otherwise irrotational and inviscid. Using methods similar to those described in § 2, this free-boundary problem was solved for its equilibrium configurations. The problem is found to admit a one-parameter family of exact solutions. A typical solution is plotted in figure 9 and has the Hele-Shaw analogue consisting of flow in a simply-connected blob of fluid driven by two quadrupoles with surface tension on the free-surface. The Hele-Shaw flow corresponding to figure 9 is given in figure 10. Crowdy (1999*b*) also derives a one-parameter family of exact solutions for irrotational swirling flow (due to a point vortex at infinity) outside a single constant-pressure bubble with surface tension on its boundary. A typical streamline plot is given in figure 11 and the Hele-Shaw flow corresponding to it is plotted in figure 12. In this case, the corresponding Hele-Shaw flow keeping the same free surface in equilibrium consists of two suitably-placed quadrupoles driving the fluid outside the constant-pressure bubble.

### 7.3. *Hele-Shaw analogues of solutions of Crowdy* (1999*c*)

In the same way that Kinnersley extended Crapper's deep-water exact solution to find a class of solutions for waves on fluid sheets (where there are two free capillary surfaces), the analysis of Crowdy (1999*b*) can similarly be extended to a (doubly-connected) annular fluid region. In this vein, Crowdy (1999*c*) identified a

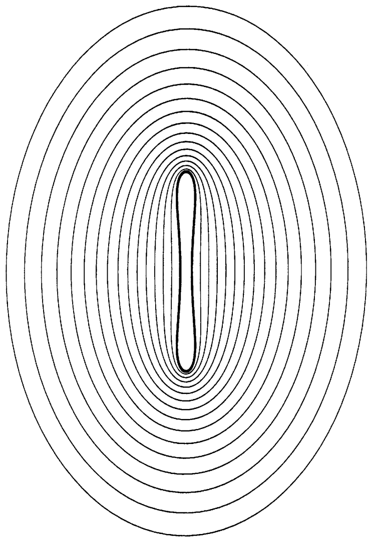

FIGURE 11. Crowdy (1999*b*) solution (bubble).

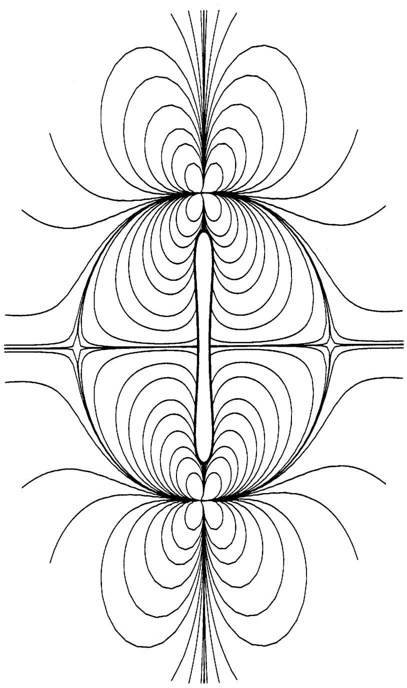

FIGURE 12. Hele-Shaw analogue of Crowdy (1999*b*) solution (bubble).

two-parameter family of exact solutions for steady capillary waves on a fluid annulus induced by an irrotational swirling flow inside it. A typical solution is plotted in figure 13. These also have a Hele-Shaw analogue (although, in this case, the values of the constant surface tension parameter on each surface are different). The Hele-Shaw analogue (of figure 13) is plotted in figure 14 and consists of flow in an annulus driven by two quadrupoles. We remark, however, that it is straightforward to apply

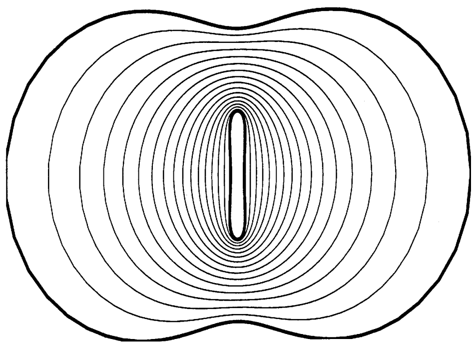

FIGURE 13. Crowdy (1999*c*) solutions ($\zeta_1 = 1.6, \rho = 0.2$).

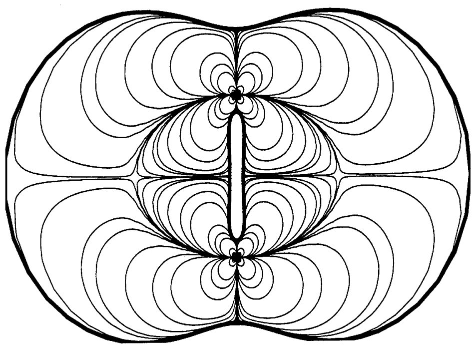

FIGURE 14. Hele-Shaw analogue of Crowdy (1999c) solutions.

the general method of solution to the free boundary problem (as outlined in § 5) to find a it much wider class of exact solutions to this doubly-connected Hele-Shaw problem driven by two quadrupoles, including solutions where the value of the surface tension parameter on each boundary is the same. These solutions (an example is plotted in figure 14) are believed to be the first-known examples of exact Hele-Shaw flows with non-zero surface tension taking place in a doubly-connected fluid domain. On the subject of Hele-Shaw flows in doubly-connected fluid regions, it is interesting to point out that Richardson (1996) has recently identified exact solutions for time-evolving Hele-Shaw flows (with zero surface tension) in a fluid annulus driven by a distribution of sources and sinks, while Crowdy (1999*d*) has recently identified exact solutions for the evolution of a fluid annulus (again with no surface tension) under the effects of centrifugal forces in a rotating Hele-Shaw cell.

## 8. Summary and discussion

In this paper, a new approach to solving the free-boundary problem of steady Hele-Shaw flows with surface tension driven by a periodic array of quadrupoles in a water-wave geometry has been presented. The mathematical approach to finding a conformal map corresponding to a given flow hodograph $W(\zeta)$ is described. The methodology is related to a new mathematical approach to the problem of free-surface irrotational Euler flow with capillarity presented in Crowdy (1999*a*). While the particular problem considered has been quite specific, we emphasize that the approach is flexible enough to be generalized to (*a*) other geometries, (*b*) other distributions of multipoles of order $n \geqslant 2$, and ($c$) genuinely doubly-connected fluid domains. In addition, by introducing a mathematical transformation between Euler and Hele-Shaw flows, it has been possible to illustrate the dynamical differences between the two kinematically equivalent problems in an unexpectedly direct way. Thus, while it is well-known (e.g. Batchelor 1967) that the streamline distribution for steady Hele-Shaw flow past a circular cylindrical obstacle is exactly the same as the streamline distribution for steady irrotational Euler flow past the same obstacle, the presence of free surfaces forces this analogy between the two types of flow to break down in a rather dramatic way, as our figures show.

Perhaps one of the most important reasons why knowledge of exact solutions is useful is that they can be used to check the accuracy of numerical codes written to solve more difficult problems where exact results are not available. Indeed, there has been much recent numerical interest in the problem of Hele-Shaw flows with surface tension, and as is often the case, knowledge of special classes of exact solutions is valuable in providing checks on numerical codes as well as being of theoretical interest. For example, Kelly & Hinch (1997$a, b$) recently solved an initial value problem in a radial geometry, and time-evolved the evolution equations for a quadrupolar-driven flow in an initially circular blob of fluid to examine whether the solutions eventually relax into the exact steady solutions found by Entov *et al.* (1993). It would be of some interest to examine similar questions here: will an initially flat interface driven by a periodic array of quadrupoles relax to the steady states derived in § 4 of this paper? Similarly, will a layer of fluid of finite thickness in a Hele-Shaw flow containing a periodic array of quadrupoles relax to the steady solutions found in § 6? A study of the basins of attraction of the new exact solutions might help in understanding more complicated Hele-Shaw flows with surface tension. We also remark that Ceniceros, Hou & Si (1999) and Nie & Tian (1998) have recently performed numerical studies of the evolution of Hele-Shaw interfaces driven by sinks and with small but non-zero surface tension. The Hele-Shaw solutions found here take place in a variety of different geometries and, although they represent quadrupole-driven flows which are perhaps not as physically relevant as, say, sink-driven flows, the solutions are expected to be of use in checking numerical codes written to resolve more physically-realistic Hele-Shaw flows taking place in the same geometry.

Finally, we note that the analysis of this paper has interesting ramifications for the 'inverse problem' of finding a multipole distribution leading to a particular steady state. Indeed, it is easy to generate arbitrary exact solutions for steady boundary shapes driven by some distribution of multipoles (i.e. determined *a posteriori* as opposed to specified *a priori*) (Crowdy & Hill 1999).

The author acknowledges many useful discussions with Dr David Hill, and partial financial support from the National Science Foundation (Grant Numbers DMS-9803167 DMS-9803358). The author would like to especially thank Philip Saffman for his personal support and generosity over the last few years.

## REFERENCES

Acheson, D. J. 1990 *Elementary Fluid Dynamics*. Clarendon.

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Ceniceros, H. D., Hou, T. Y. & Si, H. 1999 A numerical study of Hele-Shaw flow with suction. *Phys. Fluids* **11**, 2471–2486.

Crapper, G. D. 1957 An exact solution for progressive capillary waves of arbitrary amplitude. *J. Fluid Mech.* **2**, 532–540.

Crowdy, D. G. 1999*a* A new approach to free surface Euler flows with capillarity. *Stud. Appl. Maths* (to appear).

Crowdy, D. G. 1999*b* Circulation-induced shape deformations of drops and bubbles: exact two-dimensional models. *Phys. Fluids* **11**, 2836–2845.

Crowdy, D. G. 1999*c* Exact solutions for steady capillary waves on a fluid annulus. *J. Nonlinear Sci.* **9**, 615–640.

Crowdy, D. G. 1999*d* Theory of exact solutions for the evolution of a fluid annulus in a rotating Hele-Shaw cell. *Q. Appl. Maths* (submitted).

Crowdy, D. G. & Hill, D. 1999 Exact solutions for Hele-Shaw flow with surface tension (in preparation).

Crowdy, D. G. & Tanveer, S. 1998 A theory of exact solutions for annular viscous blobs. *J. Nonlinear Sci.* **8**, 375–800.

Entov, V. M., Etingof, P. I. & Kleinbock, D. Y. 1993 Hele-Shaw flows with a free boundary produced by multipoles. *Eur. J. Appl. Maths* **4**, 97–120.

Hille, E. 1976 *Ordinary Differential Equations in the Complex Plane*. Wiley-Interscience.

Kelly, E. D. & Hinch, E. J. 1997*a* Numerical simulations of sink flow in the Hele-Shaw cell with small surface tension. *Eur. J. Appl. Maths.* **8**, 533–550.

Kelly, E. D. & Hinch, E. J. 1997*b* Numerical solution of Hele-Shaw flows driven by a quadrupole. *Eur. J. Appl. Maths* **8**, 551–566.

Kinnersley, W. 1976 Exact large amplitude capillary waves on sheets of fluid. *J. Fluid Mech.* **77**, 229–241.

McLeod, E. B. Jr. 1955 The explicit solution of a free boundary problem involving surface tension. *J. Rat. Mech. Anal.* **4**, 557–567.

Nie, Q. & Tian, F. R. 1998 Singularities in Hele-Shaw flows. *SIAM J. Appl. Maths* **58**, 34–54.

Richardson, S. 1996 Hele-Shaw flows with time-dependent free boundaries involving a concentric annulus. *Phil. Trans. R. Soc. Lond.* **353**, 2513.

Shankar, P. N. 1992 On the shape of a two-dimensional bubble in uniform motion. *J. Fluid Mech.* **244**, 187–200.

Tanveer, S. 1991 Singularities in water waves and Rayleigh–Taylor instability. *Proc. R. Soc. Lond.* A, **435**, 137–158.

Tanveer, S. 1993 Evolution of Hele–Shaw interface for small surface tension. *Phil. Trans. R. Soc. Lond.* A, **343**, 155–204.

Tanveer, S. 1996 Some analytical properties of solutions to a two dimensional steadily translating inviscid bubble. *Proc. R. Soc. Lond.* A **452**, 1397–1410.

Valiron, G. 1947 *Cours d'Analyse Mathematique*, *Theorie des Fonctions*, 2nd Edn. Masson et Cie.

Vanden-Broeck, J. & Keller, J. B. 1980 Deformation of a drop or bubble in a uniform flow. *J. Fluid Mech.* **101**, 673–686.

Van Dyke, M. 1982 *An Album of Fluid Motion*. Parabolic Press.

Vasconcelos, G. L. & Kadanoff, L. P. 1991 Stationary solutions for the Saffman–Taylor problem with surface tension. *Phys Rev Lett.* **44**, 6490–6495.