# A class of exact multipolar vortices

Darren Crowdy
*Department of Mathematics, 2-335, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

(Received 4 November 1998; accepted 14 May 1999)

A class of exact solutions to the steady Euler equations representing finite area patches of nonuniform vorticity is presented. It is demonstrated that the solutions constitute a special class of steady multipolar vortical structures and have many qualitative similarities with the multipolar equilibria observed in two-dimensional flows at high Reynolds numbers. The results provide insights into the mathematical structure of the two-dimensional Euler equation that, it is argued, underlies the occurrence of such multipolar coherent structures in real physical flows. Moreover, the new solutions possess the interesting feature of being completely ‘‘invisible’’ in that their presence cannot be detected anywhere outside the support of the vorticity. © *1999 American Institute of Physics.* [S1070-6631(99)00409-2]

## I. INTRODUCTION

This paper has two objectives; the principal objective is to present a class of exact mathematical solutions to the two-dimensional steady Euler equations representing finite-area, nonuniform vortex patches. The second objective is to argue that these solutions represent an instructive mathematical paradigm for a large class of multipolar vortical structures observable in many real physical flows at high Reynolds numbers.

### A. Background

Coherent structures are now well-known to constitute a dominant feature of a wide range of two dimensional flows at high Reynolds number.[1–8] The two classes of coherent structure that have received the most attention in the literature are the monopole and the dipole. When isolated monopolar structures become unstable they can be observed to split into two separating dipolar structures. However, it has also been observed that isolated monopolar structures can condense into isolated multipolar structures of higher order. The characteristics of these higher order multipolar structures will be described later in this introduction. The most common case is when a monopolar vortical structure nonlinearly destabilizes into a single tripolar structure with distributed vorticity[5,9,10] characterized by 3 vorticity maxima of alternating polarity. The natural formation of dipoles and tripoles from unstable monopoles was observed in laboratory experiments[3,7] at more or less the same time as they were observed in numerical simulations. It has also been shown that more complex (i.e., higher order) vortex structures than tripoles (i.e., collectively dubbed *multipoles*) can be formed from more strongly perturbed two-dimensional vortices.[8]

A more comprehensive list of references to the experimental and numerical observations of multipolar vortex structures can be found in a recent paper by Morel and Carton[11] who also study numerically the generation, stationary forms and stability of multipolar equilibrium solutions of the two-dimensional Euler equations. Morel and Carton[11] generate multipolar equilibria (numerically) from the instability of monopolar vortical structures called the *two-contour Rankine vortex* and the *three-contour Rankine vortex*. These are both examples of *shielded vortices*, i.e., axisymmetric vortices with zero total circulation with a radial distribution of vorticity consisting of an inner core of uniform vorticity of one sign surrounded by a uniform vorticity region of opposite sign (unshielded monopoles are uninteresting because they are stable by the Rayleigh criterion). Tripole structures generated from a shielded monopole are also found in earlier investigations.[5] A class of steady *quadrupolar* structures can similarly be generated[8] by the destabilization of a shielded monopole.

These studies indicate the importance, in high Reynolds number flows, of isolated multipolar equilibrium solutions which, in the language of dynamical systems theory, appear to be important attractors in the dynamics. In view of this, a thorough understanding of these multipolar equilbria would therefore seem desirable and this is an important active area of research [e.g., Rossi, Lingevitch, and Bernoff,[13] have recently studied the perturbation thresholds (to a Gaussian monopole) separating the domains of attraction of monopolar and tripolar asymptotic solutions]. However, while laboratory[3,4,6] and numerical experiments continue to provide insights into multipolar equilbria, comparatively few theoretical investigations have been performed to try to understand the mathematics behind these coherent structures. Questions of the existence of multipolar equilibrium solutions to the 2D Euler equation, and the associated bifurcation structure, are clearly of great interest. This paper makes a contribution in this direction.

It is important to define what is meant here by a steady multipolar vortex. There are certain gross features which might be said to characterize the multipolar equilibrium solutions observed in practice. We now list six principal attributes of a multipolar vortex:

(i) Multipolar vortices are isolated *finite-area* regions of nonzero vorticity surrounded by irrotational flow;
(ii) The vorticity distribution of a multipolar vortex of

 

order $n$ is characterized by a central *core vortex* of one sign, surrounded by a distribution of $n$ *satellite vortices* all of opposite polarity;

(iii) They are steadily-rotating with a constant angular velocity $\Omega$;

(iv) The approximate shape of a multipolar structure of order $n$ (for $n \geq 3$) is typified by an $n$-polygonal core region with $n$ semicircular satellite (or ''lobe'') regions on each side of the central polygon (see Fig. 1 of Carnevale and Kloosterziel[8]).

(v) The multipolar structures calculated in numerical simulations often have zero total circulation [since they are frequently generated by the instability of ''shielded'' (i.e., zero circulation) monopolar vortices].

(vi) The streamline patterns in the vortical region typically display saddle points joined by separatrix streamlines[8,10] as well as regions of closed streamlines.

Various attempts to model multipolar equilibria have been made in the past. Because of the attendant analytical advantages, most theoretical models rely on hybrid combinations of line vortices and uniform vortex patches of finite area. The most obvious mathematical model of a tripole is the extreme idealization of a line of three point vortices of appropriate strength—such a model is studied by Carton and Legras.[10] They also devise a more sophisticated model in which they replace the central line vortex in the line-vortex tripole model by an elliptical $V$-state thereby allowing the possibility of splitting of the tripole due to instability of the core vortex. The latter model is based on the formalism of Legras and Dritschel[14] and is an approximate solution of the governing equations. Polvani and Carton[15] have modeled the tripole using a linear array of *three* such $V$-states. Even these simplified model solutions usually have to be computed numerically because analytical progress is difficult.

This paper presents a class of exact solutions to the steady Euler equations which can be said to represent multipolar equilibria in that the solutions possess all of the characteristic features (i)–(vi) of multipolar equilibria listed above. For every integer $n \geq 2$, we demonstrate the existence of a continuous one-parameter family (parametrized by a parameter $a$) of finite-area coherent vortical structures. The $n = 2$ case, for example, represents a tripolar structure, the solutions for $n > 2$ provide a class of multipolar equilibria of order $n$. The solutions are *special* cases of multipolar equilibria, however, in that they are non-rotating (i.e., $\Omega = 0$) and the irrotational velocity field surrounding the support of the vorticity is stagnant. Nevertheless, we conjecture herein (see Sec. III) that the new solutions represent a *mathematically exact subclass* of the full class of multipolar equilibrium solutions of the 2-D Euler equations and share all the same qualitative properties of the more general class. Further, it will be seen that the bifurcation structure of the new class of steady solutions appears to be consistent, in many respects, with transitions between equilibria observed in experiments and numerical simulations.

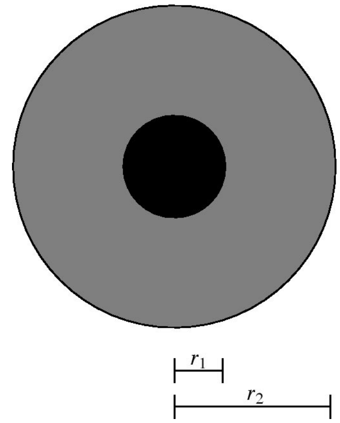


FIG. 1. Two-contour Rankine vortex.

## II. MATHEMATICAL FORMULATION

Consider the well-known uniform, rectilinear, circular vortex filament, also known as the *Rankine vortex*.[16] The uniform Rankine vortex is given by

$$u_\theta = \begin{cases} \dfrac{\omega_0 r}{2} & r < r_0 \\ \dfrac{\omega_0 r_0^2}{2r} & r > r_0 \end{cases}. \tag{1}$$

This represents a circular vortex patch with uniform vorticity $\omega_0$ in $r < r_0$ and a vortex jump at $r = r_0$. Now consider adding a single line vortex at the origin, i.e., consider the flow field given by

$$u_\theta = \begin{cases} \dfrac{\omega_0 r}{2} - \dfrac{\omega_0 r_0^2}{2r} & r < r_0 \\ 0 \quad r > r_0 \end{cases}. \tag{2}$$

This represents a mostly uniform circular vortex patch with a superposed line vortex of opposite polarity placed at its center. The dynamical condition of continuity of velocity (and hence continuity of pressure) is satisfied at the vortex boundary while a kinematic condition requires that the vortex boundary be a streamline, as is clearly the case. Finally, note that this solution is only a consistent solution of the steady Euler equation provided that the line vortex at the origin is steady under the *non-self-induction hypothesis*.[16] This is clearly the case here. The flow (2) therefore represents a self-consistent exact solution of the steady Euler equations. Moreover, since this solution consists of a line vortex with circulation of one sign surrounded by a (finite area) patch of vorticity of opposite polarity such that the combined structure has *zero* total circulation, this solution might well be dubbed a *shielded monopolar vortex*. For convenience in the rest of this paper, the solution (2) will be referred to as the *shielded Rankine vortex*. Note that this shielded Rankine vortex solution is a limit of the *two-contour Rankine vortex*,[11] where the area of the interior vortex patch ($r_1 \rightarrow 0$ in Fig. 1) shrinks to zero with the associated vorticity growing infinite in such a way that the total circulation of the enclosed vortex (in the limit, a line vortex) remains finite.

A natural mathematical question arises; Is it possible to generalize the above solution to find other vortical configurations consisting of finite-area vortical regions with nontrivial shapes containing a finite distribution of line vortices? If so, the resulting solutions might serve as useful models of steady (higher-order) multipolar vortical structures. The answer is in the affirmative.

To generalize solution (2) to other nontrivial vortex shapes with suitable distributions of line vortices, note that the stream function associated with (2) can be written as follows:

$$\psi(x,y)=\frac{\omega_0}{4}z\bar{z}-\frac{\omega_0}{2}\mathrm{Re}\left[\int^z S(z')dz'\right] \tag{3}$$

inside the patch where $z=x+iy$ and $S(z)$ is the function given by

$$S(z)=\frac{r_0^2}{z}. \tag{4}$$

The key observation is that $S(z)$ is exactly the *Schwarz function*[12] of the curve bounding the region of nonzero vorticity, i.e., in this case, the Schwarz function of a circle of radius $r_0$. The Schwarz function of a closed curve $\partial D$ bounding a simply-connected region $D$ is the unique, locally analytic, function $S(z)$ that is equal to $\bar{z}$ everywhere on the curve $\partial D$.

Now consider a *general* closed, analytic nonsingular curve bounding a simply-connected vortical region in the plane. Let the Schwarz function of the curve be denoted by $S(z)$. Since $S(z)$ is analytic in the neighborhood of the bounding curve, its primitive is well-defined there and it is natural to consider a generalized stream function given by

$$\psi(x,y)=\frac{\omega_0}{4}z\bar{z}-\frac{\omega_0}{4}\left(\int^z S(z')dz'+\int^{\bar{z}}\bar{S}(z')dz'\right), \tag{5}$$

which is an expression valid inside the patch of vorticity. We further assume that the vortex is surrounded by stagnant fluid.

Consider the total derivative of $\psi(z,\bar{z})$,

$$d\psi=\psi_z dz+\psi_{\bar{z}}d\bar{z}. \tag{6}$$

Using (5) this becomes

$$d\psi=\frac{\omega_0}{4}(\bar{z}-S(z))dz+\frac{\omega_0}{4}(z-\bar{S}(\bar{z}))d\bar{z} \tag{7}$$

valid inside $D$. Note that, on the boundary $\partial D$, by definition of the Schwarz function $S(z)=\bar{z}$, thus $d\psi=0$ on the boundary and it is therefore a streamline. Furthermore, since $2i\psi_z=u-iv$, the speed of the fluid on $\partial D$ is zero. If the flow outside the patch is stagnant (as has been assumed) then continuity of velocity (and hence of pressure) is ensured at the boundary of the vortex. In this way, both the kinematic and dynamic boundary conditions at the vortex boundary are satisfied. For more details of the theory of vortex patches, see Saffman.[16]

In general, $S(z)$ [and hence $\psi(x,y)$ as given by (5)] will have singularities inside $D$. Now we restrict attention to the special case where the *only singularities* of the Schwarz function $S(z)$ inside the vortical region are simple poles with real residues. *Mathematically*, this corresponds to restricting consideration to a particular class of closed analytic curves. *Physically*, from (5), it is seen that this corresponds to the presence of line vortices inside the vortex patch. It is well-known that, dynamically, line vortices move with the fluid.[16] Thus, for solution (5) to represent a fully self-consistent steady solution of the Euler equation, it is necessary to ensure that each line vortex is stationary under the non-self-induction hypothesis.

### A. Conformal mapping

As just mentioned, making this special choice of the analyticity properties of the Schwarz function $S(z)$ inside the curve $\partial D$ corresponds to specializing to a particular class of closed, analytic, nonsingular curves. It is necessary to be able to characterize, in a convenient way, the class of curves now under consideration. This is done by considering the univalent *conformal map* $z(\zeta)$ from the interior of the unit circle in a parametric $\zeta$-plane to the interior of the vortex patch. Without loss of generality, we can choose

$$z(0)=0. \tag{8}$$

From this perspective, it is possible to exploit a result from the theory of Schwarz functions of analytic curves $\partial D$ which are *meromorphic* everywhere inside a simply-connected region $D$. The important result states that the conformal map from the unit $\zeta$-circle to the vortical region in this special case is *necessarily a rational function of* $\zeta$. This fact will be used to identify an exact conformal representation of the shape of the boundary of the vortex. For more details on the mathematical result just mentioned, we refer the reader to Davis.[12]

### B. Exact solutions with twofold symmetry

Consider the special choice of (rational function) conformal map given by

$$z(\zeta)=R\zeta\left(1+\frac{b}{\zeta^2-a^2}\right), \tag{9}$$

where it is assumed that $a$, $b$ and $R$ are real with $|a|>1$. Note that (9) satisfies (8). The parameters $a$ and $b$ must be such that (9) represents a *univalent* conformal map from the unit circle. Moreover, there are much stronger constraints on $a$ and $b$ as will be seen. It is noted that there is normalization degree of freedom (in the value of $\omega_0$) associated with the magnitude of the vorticity in the patch. For the remainder of this paper, we take $\omega_0=4$.

For this mapping, the Schwarz function is given, as a function of $\zeta$, by

$$S(z(\zeta))=\frac{R}{\zeta}\left(1+\frac{b\zeta^2}{1-a^2\zeta^2}\right). \tag{10}$$

Since $u-iv=2i\psi_z$, the velocity at any point inside the region is given, as a function of $\zeta$, by

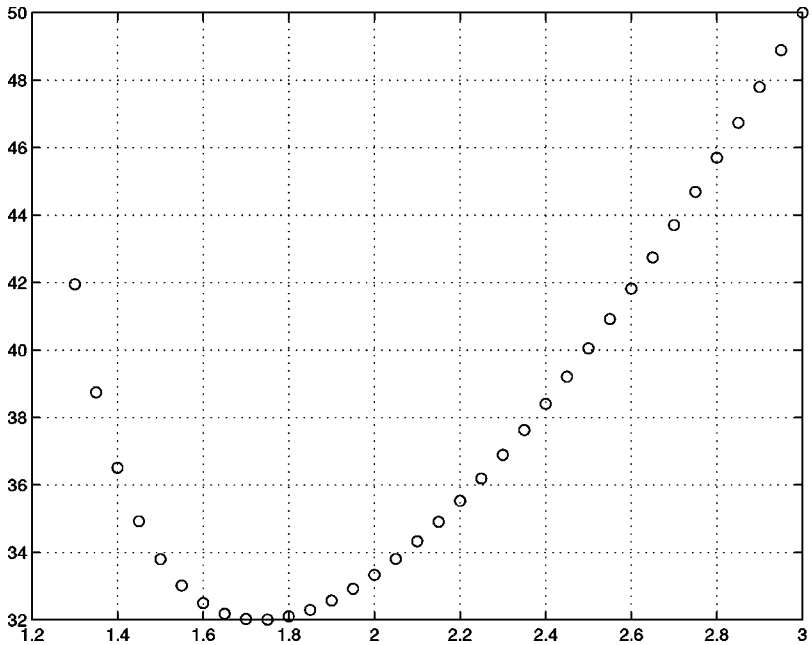


FIG. 2. Plot of $b$ (vertical axis) against $a$ (abscissa) for tripolar solutions.

$$u-iv=2iR\overline{\zeta}\left(1+\frac{b}{\overline{\zeta}^2-a^2}\right)-\frac{2iR}{\zeta}\left(1+\frac{b\zeta^2}{1-a^2\zeta^2}\right). \tag{11}$$

It is clear that this velocity field has three simple pole singularities inside the vortex—at points in the physical plane corresponding to $\zeta=0,\pm(1/a)$. These represent the $\zeta$ preimages of three line vortices in the flow. In order to ensure that the line vortices are not moving in *physical space* it is necessary to find local Laurent expansions (in $z$) of this velocity field about the singularities in the velocity field at $z=0$ and $z=z(\pm 1/a)$. Setting the constant term in the Laurent expansion to zero will ensure that the line vortex is steady. In general, since there are three conditions to be satisfied, one might expect that three conditions will need to be imposed to render each of the three line vortices steady.

Consider first the line vortex at $z=0$. First, rewrite the velocity field as follows:

$$u-iv=-\frac{2iR}{\zeta}-\frac{2iRb\zeta}{1-a^2\zeta^2}+2iR\overline{\zeta}\left(1+\frac{b}{\overline{\zeta}^2-a^2}\right). \tag{12}$$

From the form of the conformal map, it is clear that as $\zeta\to 0$,

$$z(\zeta)=R\zeta\left(1-\frac{b}{a^2}\right)+O(\zeta^3), \tag{13}$$

thus as $z\to 0$,

$$u-iv=-2iR^2\left(1-\frac{b}{a^2}\right)\frac{1}{z}+O(z,\overline{z}), \tag{14}$$

i.e., there are no constant terms in the local velocity field and, under the non-self-induction hypothesis, the line vortex at $z=0$ is steady. Comparing with the formula for the velocity field of a line vortex of circulation $\Gamma_0$ at $z=0$ as given by

$$u-iv=-\frac{i\Gamma_0}{2\pi z}, \tag{15}$$

it is clear that the circulation of the line vortex at $z=0$ is

$$\Gamma_0=4\pi R^2\left(1-\frac{b}{a^2}\right). \tag{16}$$

Now consider the line vortex at $z(a^{-1})\equiv z_a$. Some elementary (but somewhat tedious) algebra yields the following nontrivial condition for the steadiness of the line vortex at $z(1/a)$:

$$\frac{1}{a}+\frac{ba}{1-a^4}-a+\frac{b}{4a}+\frac{b}{4a^2}\,\frac{z_{\zeta\zeta}(a^{-1})}{z_{\zeta}(a^{-1})}=0. \tag{17}$$

Note that (17) is more nonlinear than it seems at first sight; this is because of the dependence of the conformal map $z(\zeta)$ (and its derivatives) on the parameter $b$. By symmetry of the chosen conformal map and the velocity field (11), this is also the condition for the steadiness of the line vortex at $z(-a^{-1})=-z_a$, as is directly confirmed by further algebra. Equation (17) is a real nonlinear algebraic equation for $b$ given the parameter $a$. Given $a$, this equation is solved for $b$ using Newton's method. The results for $b$ as a function of $a$ is plotted in Fig. 2 for $a$ between 1.3 and 3.0. Depending on initial conditions, for a given $a$, the Newton scheme produced multiple roots for $b$, however only the root plotted in Fig. 2 produces a *univalent* conformal mapping (9). Note that the nonlinear Eq. (17) is independent of the value of $R$.

Expanding the velocity field locally in the vicinity of $z=z(1/a)\equiv z_a$ yields

$$u-iv=\frac{iRba^{-2}z_{\zeta}(a^{-1})}{(z-z_a)}+O((z-z_a),(\overline{z}-\overline{z_a})) \tag{18}$$

*provided* condition (17) is satisfied. Comparing with the formula for the velocity field for a line vortex of circulation $\Gamma_a$ at $z=z_a$, i.e.,

$$u-iv=-\frac{i\Gamma_a}{2\pi(z-z_a)}, \tag{19}$$

the circulation of the line vortex at $z_a$ (to be denoted $\Gamma_a$) is found to be

$$\Gamma_a=-2\pi Rba^{-2}z_\zeta(a^{-1}). \tag{20}$$

The line vortex at $z=z(-a^{-1})=-z_a$ is found to have the same strength.

It is seen that fixing an area of the vortex will fix the parameter $R$ as a function of $a$ (now that $b$ is specified as a function of $a$). For example, the area of the patch can be arbitrarily specified to be $\pi$ so that

$$\pi=\frac{1}{2}\mathrm{Im}\left[\oint_{|\zeta|=1}\bar{z}z_\zeta d\zeta\right]. \tag{21}$$

For fixed $\omega_0=4$ and area $\pi$ of the patch, a one-parameter family of exact solutions has been identified (parametrized by the pole position $a>1$), each solution constituting a different shape patch, with a corresponding distribution of three point vortices inside it.

Note the important result that, for any choice of $a$,

$$2\Gamma_a+\Gamma_0=4\pi. \tag{22}$$

This is as expected since it is a statement of the fact that the total circulation of the vortical structure is zero. This follows immediately from the fact that the circulation $\Gamma_C$ around any closed $C$ is given by

$$\Gamma_C=\oint_C \mathbf{u}.d\mathbf{x}=-\mathrm{Im}\left[\oint_C 2\psi_z dz\right], \tag{23}$$

Therefore, taking $C$ as the $\partial D$ (the boundary of the patch) we obtain the following expression for the total circulation $\Gamma_t$ of the structure:

$$\Gamma_t=-\mathrm{Im}\left[\oint_{\partial D}2(\bar{z}-S(z))dz\right]=0. \tag{24}$$

The left-hand side of (22) is clearly the sum of the circulations of the three line vortices, while the right-hand side represents the circulation associated with the finite-area patch of uniform vorticity (of strength $-4$) of total area $\pi$. Equation (22) was used as a check on the calculations of $\Gamma_0$ and $\Gamma_a$.

A typical streamline plot is given in Fig. 3. These have remarkable similarities with the numerically computed streamlines associated with tripoles as shown, for example, in Figs. 7 and 11 of Carton and Legras.[10] Note, in particular, the existence of separatrix streamlines which separate the three finite subregions of closed streamlines. Note also that the outermost streamline in Fig. 3 is also the boundary of the vortex patch.

One of the advantages of possessing exact solutions is that explicit expressions can be simply obtained for various quantities of interest. For the tripolar vortices, plots of $\Gamma_0$ and $\Gamma_a$ as functions of the parameter $a$ are shown in Fig. 4, while a plot of the distance between the central line vortex and each satellite vortex is given in Fig. 5. Note the important fact that $\Gamma_a>0$ for all $a$ while $\Gamma_0<0$ for all $a$ thus, roughly speaking, the ''core'' of the vortical structure has an overall negative circulation while each of the two ''satellite'' regions has an overall positive circulation. To render what we mean by the ''core'' and the ''satellites'' more concrete, it is natural to consider the streamline plots. It seems natural to define the ''core'' region of the vortex as the central region enclosed between the two saddle points and bounded by the separatrix streamlines. This prescription is also suggested in Carton and Legras.[10] Denote the area enclosed in this core region by $A_c$. Since the total area of the patch is $\pi$, it is then natural to take the area of each of the two satellite vortices to be $A_s=\frac{1}{2}(\pi-A_c)$. The total circulation associated with the core region (as defined in this way) is then

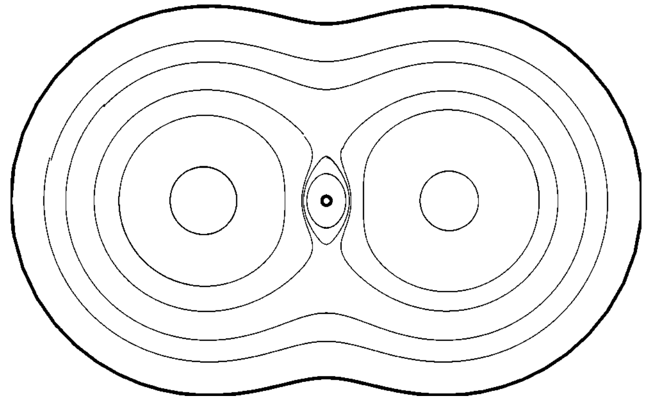

FIG. 3. Typical tripolar streamlines ($n=2$, $a=2.0$); the outermost streamline is also the boundary of the vortex patch.

$$\Gamma_c=\Gamma_0-4A_c \tag{25}$$

since the (constant) uniform value of the vorticity in the patch is $-4$. Similarly, the total circulation of each of the two satellite regions is therefore given by

$$\Gamma_s=\Gamma_a-4A_s=-\frac{\Gamma_c}{2}. \tag{26}$$

Using this prescription, it is clear that the ''core'' and the ''satellite'' regions as defined above always have circulations of opposite polarity. Note also that the ratio of circulations of the three vortical regions is $(-1,2,-1)$.

Finally, we remark that since $z(\zeta)$ is a univalent map, it is in principle possible to invert (9) to find $\zeta(z)$. If one then substitutes $\zeta(z)$ into the expression (11) for the velocity field, a very complicated function of $z$ and $\bar{z}$ results. It is clear that while the present theoretical approach has led to a straightforward parametric representation of the solutions, the resulting solutions for the velocity field are highly nontrivial functions of the spatial variables $x$ and $y$.

### C. Solutions with threefold symmetry

Consider now a conformal mapping having the following rational function form

$$z(\zeta)=R(a)\zeta\left(1+\frac{b(a)}{\zeta^3-a^3}\right). \tag{27}$$

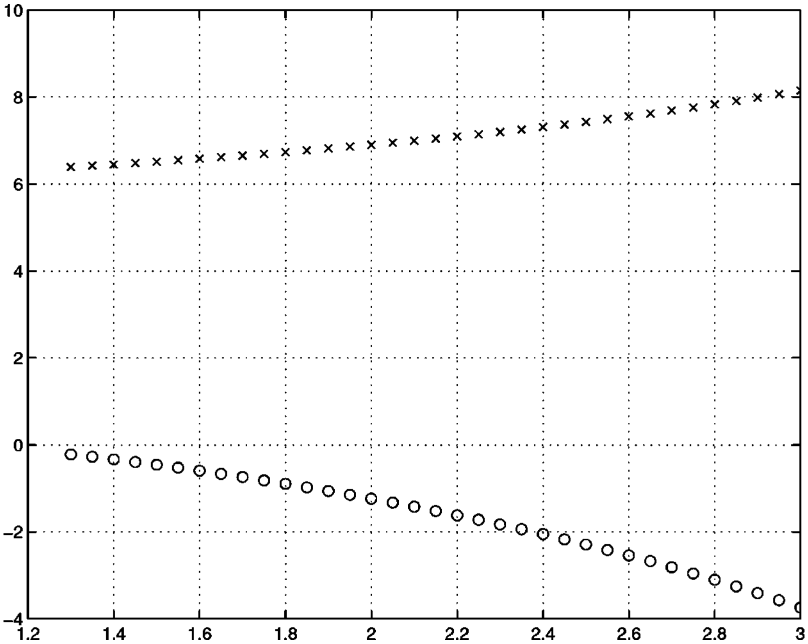


FIG. 4. Plot of $\Gamma_0$ (dots) and $\Gamma_a$ (crosses) against $a$ (abscissa) for tripolar solutions.

Defining $\eta = e^{(2\pi i/3)}$, it is clear that there will be three satellite line vortices at points in the physical plane corresponding to $z(a^{-1})$, $z(\eta a^{-1})$, and $z(\eta^2 a^{-1})$. By the symmetry of the conformal mapping function and the associated velocity field, it can be shown that the condition that *all three* of the satellite line vortices are steady is provided by the single nonlinear algebraic equation relating $b$ and $a$,

$$\frac{1}{a}\left(1+\frac{ba^3}{1-a^6}\right)-a+\frac{b}{3a^2}+\frac{b}{6a^3}\frac{z_{\zeta\zeta}(a^{-1})}{z_{\zeta}(a^{-1})}=0, \tag{28}$$

while the line vortex at the origin turns out to be automatically steady under the non-self-induction hypothesis. Again, given a value of $a$ $(1<a<\infty)$, a Newton solver produced multiple roots for $b$, however only one of these roots produces a univalent conformal mapping function $z(\zeta)$ (as required for the solution to be physical). A plot of the physically admissible $b(a)$ is not given here but is found to be qualitatively similar to Fig. 2. $R(a)$ is again determined from the area condition (21) once $b(a)$ is determined from (28). A typical streamline plot and vortex shape is depicted in Fig. 6. These solutions represent a special (one-parameter) family of quadrupolar vortices. It is instructive to compare the qualitative features of the streamline plots in Fig. 6 with those depicted in Carnevale and Kloosterziel.[8]

### D. Solutions with *n*-fold symmetry

More generally, it can be shown that the formulation admits a generalization to a one-parameter family of solutions with any integer number $n \geq 2$ of line vortices super-

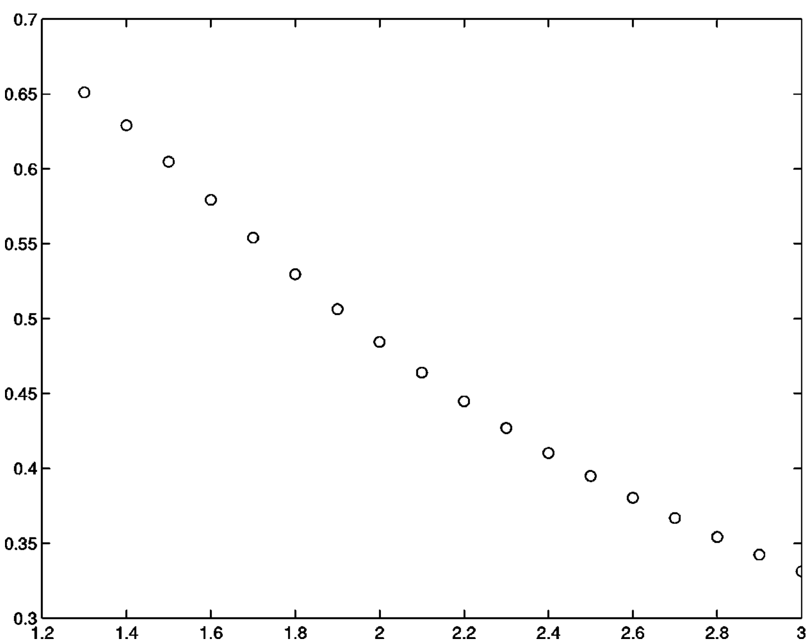


FIG. 5. Distance between central and satellite line vortices as function of parameter $a$ (abscissa) for tripolar solutions.

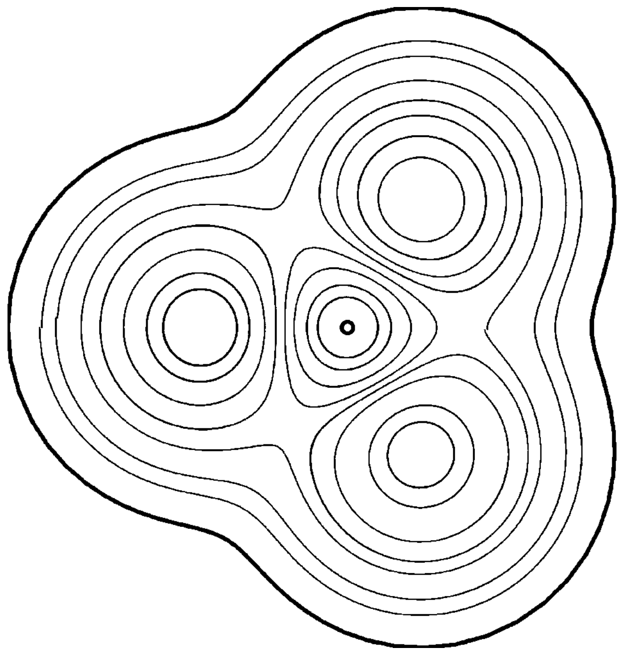

FIG. 6. Quadrupolar streamlines; $n=3$, $a=2.0$ (cf., Fig. 9 of Ref. 8); the outmost streamline is also the boundary of the vortex patch.

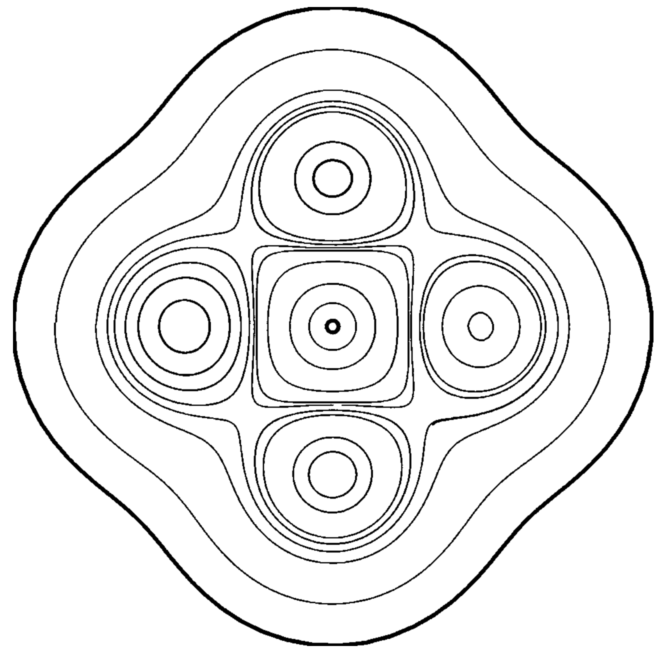

FIG. 7. Pentapolar streamlines; $n=4$, $a=2.0$ (cf., Fig. 12 of Ref. 8); the outermost streamline is also the boundary of the vortex patch.

posed on a uniform vortex patch of appropriate shape. The relevant conformal mapping function is provided by

$$z(\zeta)=R(a;n)\zeta\left(1+\frac{b(a;n)}{\zeta^n-a^n}\right). \tag{29}$$

As before, the line vortex at the origin is automatically steady and there is a single nonlinear equation relating the parameter $b$ to the parameter $a$ which is enough to ensure that all the satellite line vortices are steady and therefore that the solution is a consistent solution of the steady Euler equations. Some algebra reveals that the nonlinear equation, the solution of which implicitly defines the function $b(a;n)$ is

$$\frac{1}{a}\left(1+\frac{ba^n}{1-a^{2n}}\right)-a+\frac{b(n-1)}{2a^{n-1}n}+\frac{b}{2na^n}\,\frac{z_{\zeta\zeta}(a^{-1})}{z_\zeta(a^{-1})}=0. \tag{30}$$

As an example of an order 4 multipolar structure, Fig. 7 shows the streamlines for a typical $n=4$ solution. The outermost streamline corresponds to the boundary of the vortex patch. Again, comparison with Fig. 12 of Carnevale and Kloosterziel[8] shows remarkable similarities in the streamline plots.

### E. Limiting vortex shapes

It is of interest to examine the limiting shapes of the vortical structures as the parameter $a$ tends to the two limits $\infty$ and 1;

*Limiting vortex shapes as* $a\rightarrow\infty$: In the limit $a\rightarrow\infty$ it can be shown that, for all integers $n\geqslant 2$, the shape of the vortical region becomes circular and the $n$ line vortices converge onto the line vortex at the origin. It becomes clear that the shielded Rankine vortex solution is retrieved in the limit as $a\rightarrow\infty$ for all $n$.

*Limiting vortex shapes as* $a\rightarrow 1$: The opposite limit $a\rightarrow 1$ corresponds to the singularities of the conformal mapping function outside the unit $\zeta$-circle drawing closer to the unit circle, leading to more distorted boundary shapes. Figure 8 shows the boundary shapes of vortices corresponding to $n=3$ and $n=4$ for values of $a$ close to 1. These shapes have remarkable similarities with the schematic geometrical drawings of multipolar vortex structures plotted in Fig. 1 of Carnevale and Kloosterziel,[8] i.e., a regular $n$-polygonal core region surrounded by $n$ semicircular satellites.

### F. Nonexistence of isolated dipole

Given the results above, it is natural to ask whether there exists a solution *within the same class* (i.e., nonrotating and surrounded by quiescent fluid) corresponding to an isolated dipolar structure, i.e., a structure characterized by two vorticity maxima. The answer appears to be negative. For a dipolar solution within the present class of solutions, it is straightforward to deduce that the corresponding conformal map must have the following form:

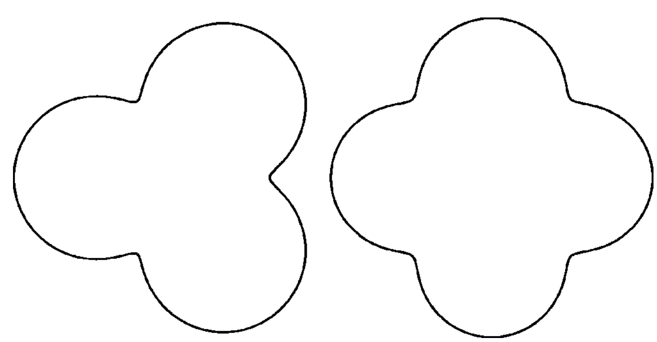

FIG. 8. Limiting vortex shapes as $a\rightarrow 1$ ($n=3,4$, cf., Fig. 1 of Ref. 8).

$$z(\zeta)=R\zeta\left(\frac{1}{\zeta-a_1}+\frac{b}{\zeta-a_2}\right), \tag{31}$$

where $|a_1|,|a_2|>1$. The conditions that the two line vortices be steady gives two (highly nonlinear) algebraic equations that must be satisfied by the parameters $a_1, a_2, b$ (these equations are independent of the normalization $R$). We have so far been unable to find any solutions to this system which do not either (i) reduce to the shielded Rankine vortex solution or (ii) yield nonunivalent conformal mappings (which cannot be admitted physically). This result is not particularly surprising: dipoles are usually characterized by a total nonzero linear or angular momentum (or both) while the present class of solutions clearly admits neither possibility.

## III. DISCUSSION

The exact mathematical solutions just described represent fully self consistent, steady, coherent vortical structures characterized by a finite distribution of vorticity maxima. The structures have zero total circulation and correspond to *finite-area* regions of nonvanishing vorticity surrounded by irrotational flow (that happens to be stagnant). The solutions are steady (i.e., $\Omega=0$). The streamline plots are characterized by the presence of saddle points joined by separatrix streamlines. Furthermore, using the prescription for defining the ''core'' and ''satellite'' regions (using separatrix streamlines) described earlier, it is clear that the approximate shapes of these vortical regions are consistent with those generally observed in practice (and described explicitly in Carnevale and Kloosterziel[8]), i.e., an $n$-polygonal core region with circulation of one sign surrounded by semicircular satellite regions of opposite polarity to the core region. The general features just described are exactly those listed in the introduction as characterizing multipolar equilibria as observed in laboratory experiments and numerical simulations.

An important observation is that in the limit $a\rightarrow\infty$ all solutions tend to the shielded Rankine vortex solution which, as mentioned earlier, is a limit of the *two-contour Rankine vortex* considered in Morel and Carton.[11] The preceding analysis therefore suggests that, for each integer $n\geqslant 2$, there exists a steady solution branch (with $n$th order symmetry and continuation parameter $a$) bifurcating from the shielded Rankine vortex solution. This bifurcation picture is highly reminiscent of physical observation; the formation of higher order multipolar equilibria from the nonlinear destabilization of shielded monopolar structures with zero total circulation.

The many qualitative similarities between the class of exact solutions found in this paper and multipolar vortices lead us to suggest that the new solutions provide important insights into the mathematical structure of the 2D Euler equation which gives rise to the multipolar vortical structures observed in experiment and in numerical simulations. We believe the mathematical solutions found herein represent an instructive theoretical paradigm.

From a mathematical point of view, it is interesting that no *deliberate* attempt has been made to construct solutions which consist of a central core with vorticity of one sign surrounded by $n$ satellite vortices of opposite polarity. Rather, such solutions have been shown herein to arise *naturally* as a mathematical generalization using Schwarz function theory, i.e., (5) of the shielded Rankine vortex solution (3). It is intriguing that this natural *mathematical* relationship exists between these higher-order multipolar vortical structures and the shielded monopolar Rankine vortex given the seemingly natural *physical* relationship that is by now well-known to exist between such structures (i.e., that higher-order multipolar equilibria result from the nonlinear destabilization of shielded monopolar structures).

There are, of course, differences between the mathematical solutions found here and the multipolar vortices observed in practice. In particular, the *detailed* structure of the vorticity distribution of the exact solutions is unrealistic; the multipoles observed in practice are typically ''multicontour'' vortices, the core and satellite vortices often having well-defined boundaries separated by stretches of irrotational fluid. Furthermore, most physically-observable multipoles rotate with a *nonzero* angular velocity $\Omega\neq 0$. On this point, we remark that there is no reason to expect that the present class of mathematical models (i.e., using a finite distribution of line vortices superposed on a uniform vortex patch to model multipolar vortices) cannot be generalized to $\Omega\neq 0$, although such solutions only seem to be available at a price; i.e., the loss of mathematical exactness. Ongoing investigations by the author show that perturbative solutions about the new exact solutions for non-zero $|\Omega|\ll 1$ can be found. In this case, it is necessary to find irrotational $O(\Omega)$ corrections to the velocities interior and exterior to the patch as well as $O(\Omega)$ corrections to the conformal mapping function. These perturbative corrections only seem to be available numerically. A spectral method based on Taylor and Laurent expansions similar to the numerical approach used by Meiron, Saffman, and Schatzmann[17] can be used to find these perturbative solutions. Physically, it is easy to imagine that moving the two satellite line vortices in the tripolar solution slightly inwards towards the central line vortex with a corresponding adjustment of the bounding shape of the patch might lead to a steadily-rotating, self-consistent configuration. We also remark that calculating the perturbative solution for $\Omega\ll 1$ seems to require almost as much numerical effort as solving the general fully-nonlinear $\Omega=O(1)$ problem, and a full numerical study of these model multipolar solutions for general $\Omega$ will be presented in a future paper.

The solutions found here can be viewed as a *mathematically exact subclass* of the full class of multipolar equilibria of the 2D Euler equations and, as we have seen, they share all the same gross features and qualitative properties of the more general class. Significantly, we point out that perhaps one of the principal benefits of the new exact solutions of this paper (as with any exact solution) is that they provide closed form solutions that can be numerically, or perturbatively, ''continued'' into parameter regimes where exact results are not available (in this case, into $\Omega\neq 0$).

Naturally, the linear and nonlinear stability of these new equlibrium solutions is a matter of enormous interest. An examination of the stability of multipolar structures is of great importance[13] and it might be hoped that the stability analysis of the mathematically exact subclass of solutions found here might have many of the same stability properties as the more general class of multipolar equilibria of the two-dimensional Euler equations. Moreover, the availability of *exact* base-state equilibrium solutions (e.g., for a linear stability analysis) might be expected to simplify any such stability analysis. This important problem is left for the future.

Generally speaking, the solutions reveal that, for certain special shapes of uniform vortex patch, it is possible to exactly nullify the irrotational flow induced outside a patch of uniform vorticity by superposing an appropriate finite distribution of line vortices such that the *combined* structure is in overall equilibrium. Note also that the coherent structures found here are, in a sense, completely ''invisible'' in that their presence is undetectable from an examination of the induced far-field flow. Indeed, because the vortices are surrounded by completely quiescent fluid, it is impossible to locally detect the presence of the vortex anywhere outside the support of the vorticity. These vortex patches therefore only interact when they overlap. Such vortices cannot be detected by any remote measurements, and an understanding of such vortical structures is very important in, for example, the context of oceanic data assimilation. We also point out that Leith[18] computed a class of minimum enstrophy (MEM) vortices which possessed exactly this property while Polvani and Carton[15] have numerically calculated a class of nonrotating ($\Omega=0$) tripolar $V$-state solutions which they describe as ''invisible'' in that the irrotational flow field exterior to the patch decays quickly at large distances from the patch.

Finally, when considering the potential flow induced outside a patch of nonzero vorticity from the point of view of the Biot–Savart integral, mathematical analogies between vorticity dynamics and the classical problems of gravitational and electrodynamic potential theory arise. A vorticity distribution can be considered analogous to a mass or charge distribution. It is thus pertinent to point out that the mathematical approach adopted in this paper is intimately related to the *Herglotz Principle* (see, for example, the discussion in Shapiro[19]). Herglotz studied laminae of uniform density bounded by irreducible algebraic curves and showed that the potential of such a laminar is harmonically extendable into the laminar except where the Schwarz function of the bounding curve has singularities. This is sometimes referred to as ''balayage inwards.'' In the realm of vortex dynamics, general considerations of this kind are generally referred to as the study of ''image vorticity.''[16] The idea of image vorticity can be roughly summarized as the (nonunique) replacing of one vorticity distribution in some set $D$ (e.g., a uniform vortex patch, in the present case) by a *different* vorticity distribution (e.g., a finite set of line vortices) such that the irrotational flow induced outside $D$ is the same. From this perspective, we have herein demonstrated that it is possible to *superpose* two different (but specially constructed) distributions of image vorticity in such a way as to obtain exact equilibrium solutions of the Euler equations (that happen to share many of the qualitative features of physically-observable multipolar equilibria).

## ACKNOWLEDGMENTS

The author would like to thank Professor L. M. Polvani for some helpful email discussions. The author would also like to thank Dr. David Hill for help in plotting some of the graphs. The author gratefully acknowledges financial support for this research from the National Science Foundation (Grant No. DMS-9803167).

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