Circulation-induced shape deformations of drops and bubbles: Exact two-dimensional models

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In this paper simple two-dimensional mathematical models for understanding the fluid dynamical problem of how circulation affects the free surface shapes of inviscid drops and bubbles with surface tension are presented. This theoretical paradigm is of interest in many areas of science including large-scale transport processes in chemical engineering. Exact solutions for the finite-amplitude steady-state equilibria of the mathematical models are found. Equilibrium states are shown to exist right up to steady capillary pinch-off in the case of a bubble, the bubbles just before pinch-off having large perimeter-to-area ratios.
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I. INTRODUCTION

An understanding of the free surface dynamics of single free liquid drops represents an important theoretical paradigm in diverse areas of science. Applications arise in cloud physics[1] and in containerless processing in low gravity.[2] Another important area of application is in many large-scale transport processes in chemical engineering[3] involving multi-phase/multi-component dispersions (e.g., liquid–liquid extraction, distillation, direct contact heat transfer). To devise accurate models of such large-scale processes, a thorough understanding of the nonlinear free surface dynamics of *single* drops and bubbles is an important first step.

It is known that mass and heat transport processes between a liquid and a host fluid can be significantly enhanced by ''atomizing'' the liquid, i.e., by stretching and deforming the core liquid until a drop detaches and oscillates before reaching equilibrium. The observed increase in transport rates is believed to be due to the fact that, because of large shear effects at the interface between a detaching liquid ligament and the host fluid, or because of relative translational velocities between the liquid and host fluid (as well as for many other physical reasons), the detached drops can often have significant internal circulations.[4] For this reason, it is desirable to gain a fundamental understanding of the free surface dynamics of single bubbles/drops in the presence of circulations around and inside them. In this vein, Mashayek and Ashgriz[4] recently carried out a numerical investigation of the dynamics of free drops (in zero gravity) with internal circulations generated by constant surface velocities.

In this paper, in an attempt to gain some *analytical* insight into how circulations inside and around blobs and bubbles (at large Reynolds number) can affect their free surface shapes, we consider simple two-dimensional mathematical models. For example, the model of a free drop with circulation considered here consists of a two-dimensional, inviscid, simply-connected droplet of incompressible fluid surrounded by a constant pressure ambient, in zero gravity, held together by surface tension. This is exactly the physical scenario considered (in the 3D static case) by Rayleigh.[5] The important extra ingredient of introducing some circulation inside/around the blobs and bubbles is effected using a line vortex inside the blob. These idealized mathematical models thus represent paradigmatic problems in which a free capillary surface (in a radial geometry and at high Reynolds numbers) interacts nonlinearly with a circulatory inviscid flow field around it.

The first step in studying any dynamical system is to determine its steady-state equilbria. This is the subject of the present paper. Having formulated the models, we demonstrate that the model problems admit exact, finite-amplitude solutions for their steady-state equilibria. These exact solutions were derived by extending a new theoretical approach to the general problem of free surface Euler flows with capillarity recently developed by the present author.[6] Interestingly, the new solutions have intimate mathematical connections with the classic exact solutions for pure capillary water waves as found by Crapper.[7]

The exact solutions presented here are important not least because exact solutions for free surface Euler flows with surface tension are rare—although a few are known.[7–11] The new solutions provide useful analytical insights and, on a practical level, are expected to be important in providing checks on numerical codes designed to compute solutions to more complicated free boundary problems in a radial geometry (i.e., blobs and bubbles) where more physical effects are included (e.g., viscosity, gravity, thermocapillary/electrostatic effects) and exact results are not available.

II. MATHEMATICAL FORMULATION

In order to model the effects of circulation on shape deformations of bubbles and blobs with surface tension we consider two idealized 2D models. To model the effects of circulation outside a constant pressure bubble we consider a single bubble, with surface tension on its boundary, in the presence of a line vortex at physical infinity. Except for the line vortex at infinity, the flow outside the bubble is otherwise assumed to be irrotational. Similarly, to model a blob with internal circulation, we consider a 2D simply-connected blob (with surface tension acting on its boundary) with a single line vortex placed inside it, the flow inside the blob being otherwise irrotational. In this paper, we study the steady-state equilibria of these two mathematical models. The problem in each case is to identify any circulation-induced steady-state equilibrium shapes of the free surface.

Both fluid domains are simply-connected, and by Riemann's theorem, these free boundary problems can be recast as the problem of finding the functional form of a conformal map taking the unit circle in a parametric ζ-plane to the physical fluid region in each case. This conformal map will be denoted $z(\zeta)$.

Since the flow is assumed to be irrotational, it is appropriate to define the complex potential to be

$$w(z)=\phi(x,y)+i\psi(x,y), \tag{1}$$

where $\phi(x,y)$ and $\psi(x,y)$ are the velocity potential and streamfunction, respectively. There will be two boundary conditions on the free surface—a kinematic condition that the free surface be a streamline and a dynamic Bernoulli condition associated with that streamline. The kinematic condition is equivalent to

$$\psi(x,y)=0, \tag{2}$$

on the free surface, while the Bernoulli condition on the free surface can be written as

$$\Gamma+\kappa=\tfrac{1}{2}q^2, \tag{3}$$

where q denotes the speed of the fluid,

$$q^2=\left|\frac{dw}{dz}\right|^2, \tag{4}$$

κ denotes the surface curvature. Γ is the Bernoulli constant.

We define the composed function,

$$W(\zeta)\equiv w(z(\zeta)). \tag{5}$$

The functional form of $W(\zeta)$ for both problems under consideration here is taken to be

$$W(\zeta)=i\gamma\log\zeta, \tag{6}$$

where γ (real) represents a measure of the point vortex strength. Note that on the free surface $|\zeta|=1$,

$$\psi(x,y)\equiv \mathrm{Im}[W]=0, \tag{7}$$

as required.

After some manipulation, the Bernoulli condition on $|\zeta|=1$ can be written in terms of the conformal mapping function. It is convenient to write it in the following form:

$$-\frac{d}{d\zeta}\left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2}+\Gamma z_\zeta(\zeta)=\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})}. \tag{8}$$

Note that it will be assumed that the parameters Γ and γ are specified and that we seek solutions $z(\zeta)$ corresponding to these values of Γ and γ. Specifying Γ corresponds to specifying the pressure inside the bubble or outside the blob, while specifying γ corresponds to specifying the circulation of the line vortex.

The above is a general statement of the problem of free surface potential flow with surface tension in terms of conformal maps and complex potentials. We now consider the details of the two model problems separately. The geometrical differences between the two problems require appropriate modification of the analysis in each case. In the next section the problem of a single bubble in the field of a simple vortex (swirling flow) is considered, while in Sec. IV we consider a finite droplet (or blob) of fluid containing a circulatory flow induced by an isolated line vortex.

III. CIRCULATION-INDUCED BUBBLE DEFORMATIONS

In the case of a single bubble of constant pressure in an infinite fluid, the conformal map $z(\zeta)$ must have a simple pole inside the unit circle. Without loss of generality this is assumed to be at $\zeta=0$. The general form of the map is given by

$$z(\zeta)=\frac{a}{\zeta}+f(\zeta), \tag{9}$$

where a is some constant. There remains a rotational degree of freedom in the Riemann mapping theorem which will be specified in a convenient way later in the analysis. For a physically meaningful solution, $z(\zeta)$ must be a univalent map from the unit ζ-circle to the physical fluid region exterior to the bubble. Necessarily, z_ζ has no zeros in $|\zeta|<1$. Under the assumption that we seek solutions with smooth bubble shapes (i.e., with no corners or cusps), we also assume that z_ζ does not vanish anywhere on $|\zeta|=1$.

The nontrivial part of the problem is to find the functional form for $z(\zeta)$ that satisfies the Bernoulli condition (8) on $|\zeta|=1$. Note that the Bernoulli condition is highly nonlinear in the conformal mapping function, making this a difficult problem in general. In the closely related problem of deep water pure capillary waves, Crapper[7] employed a method of solution involving a hodograph transformation and a very special separation of variables technique to identify exact solutions. The problem of pure capillary water waves has recently been reappraised by the present author[6] using quite different methods. We now show how to use this new approach[6] to find the required steady-state equilibria in the present case.

We now define an important function $S(\zeta)$.

Definition: Define the function $S(\zeta)$ as follows:

$$S(\zeta)\equiv-\frac{d}{d\zeta}\left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2}+\Gamma z_\zeta(\zeta). \tag{10}$$

Note also that the function on the right hand side of (8) is known to be analytic everywhere outside the unit circle. We therefore denote its Laurent expansion (convergent in $|\zeta|\geqslant 1$) as follows:

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})}=\sum_{j=2}^{\infty}\frac{c_j}{\zeta^j}. \tag{11}$$

A finite number of the coefficients $\{c_j\}$ will be important in the development.

By combining the above, it can be shown by generalizing the results of Ref. 6 in a straightforward way that the problem of finding steady solutions for the free surface of a bubble in the presence of a point vortex at infinity (for given Γ and γ) is equivalent to finding $z(\zeta)$ satisfying the following conditions:

(i) $z(\zeta)$ is a univalent conformal map from $|\zeta|\le 1$ to the fluid region having the general form (9)

(ii) $[z_\zeta(\zeta)]^{1/2}$ has only simple pole singularities in $|\zeta|>1$

(iii) $S(\zeta)$ is analytic everywhere outside the unit circle with

$$S(\zeta)\sim\frac{c_2}{\zeta^2}\ \text{as}\ \ \zeta\to\infty, \tag{12}$$

where c_2 is defined in (11).

The reader is referred to Crowdy[6] for more detailed information on this particular mathematical approach to the problem of free surface Euler flows with surface tension. In an Appendix, it is shown how the above analyticity conditions on $z(\zeta)$ and $S(\zeta)$ are equivalent to (8).

A. Calculation of the exact solution

The above information [i.e., conditions (i)-(iii)], in fact, provides a *constructive* means of finding solutions. In particular, it allows us to investigate whether *rational functions* (for $[z_\zeta]^{1/2}$) with a *finite* number of poles are admitted as solutions by providing a set of *necessary and sufficient* conditions that any such function must satisfy. Such solutions are referred to herein as ''exact solutions'' in the sense that they can be written down in terms of a finite set of known parameters. As shown in Crowdy,[6] Crapper's solution[7] for capillary water waves corresponds exactly to a rational function solution of this kind.

Since it is known that $z(\zeta)$ is analytic in the unit circle except for a simple pole at $\zeta=0$, it follows that z_ζ is also analytic inside the unit circle except for a second order pole at the origin. Furthermore, outside the unit circle, $[z_\zeta]^{1/2}$ must have only simple pole singularities. By hypothesizing solutions $[z_\zeta]^{1/2}$ which are rational functions having a specified number of simple poles outside the unit circle and then trying to ensure the required analyticity properties of $S(\zeta)$ outside the unit circle [note that $S(\zeta)$ depends only on $[z_\zeta]^{1/2}$ and its conjugate function] we can examine directly whether (exact) rational function solutions to the problem are admitted. If such a function exists, it is then necessarily an *exact solution* to the free boundary problem provided only that it satisfies the additional requirement that it is a univalent map from the unit circle.

More specifically, at any pole ζ_j of $z_\zeta^{1/2}$ in $|\zeta|>1$, it is clear that, in general, $S(\zeta)$ will have a second order pole at the same point. However, for a solution to the problem, $S(\zeta)$ must be analytic at ζ_j. In general, the requirement that the principal part of $S(\zeta)$ vanishes at ζ_j imposes 2 conditions on the parameters of the mapping $[z_\zeta]^{1/2}$. Further, under the assumption that $z_\zeta^{1/2}$ is a rational function, it can have, at worst, a polar singularity at infinity. Application of the well-known *test-power test*[12] reveals that $S(\zeta)$ can only possibly have the required asymptotic behavior as $\zeta\to\infty$ if

$$z_\zeta^{1/2}\sim p\zeta^{-1},\quad \text{as}\ \ \zeta\to\infty, \tag{13}$$

for some constant p.

By combining this information, the admissible functional structure of $z_\zeta^{1/2}$ can be deduced under the assumption that it is rational. If there are enough free parameters in the mapping function to satisfy the required analyticity properties of $S(\zeta)$ outside the unit circle (i.e., the counting is consistent) then the problem is reduced to solving a consistent finite nonlinear system. Solutions of the nonlinear system, *if they exist*, will constitute steady solutions of the original free boundary problem (subject to the additional conditions of univalency).

Using this constructive approach, the natural first step is to attempt to find a solution with just *one* simple pole of $[z_\zeta(\zeta)]^{1/2}$ in $|\zeta|>1$. As shown in detail in an analogous reformulation of the deep water capillary wave problem in Crowdy,[6] Crapper's exact solution[7] corresponds to a mapping with just a single simple pole of $[z_\zeta(\zeta)]^{1/2}$. However, in the present case, a direct calculation immediately reveals that such a solution is impossible.

Nevertheless, a rational function solution with $z_\zeta^{1/2}$ having *two* simple poles outside the unit circle can be found, i.e.,

$$[z_\zeta(\zeta)]^{1/2}=\frac{R}{\zeta}\left(\frac{(\zeta-\eta_1)(\zeta-\eta_2)}{(\zeta-\zeta_1)(\zeta-\zeta_2)}\right), \tag{14}$$

where $|\eta_1|,|\eta_2|,|\zeta_1|,|\zeta_2|>1$. By the rotational degree of freedom, R is assumed real. With $[z_\zeta]^{1/2}$ given by (14), the corresponding $S(\zeta)$ is given by

$$S(\zeta)=-\frac{d}{d\zeta}\left[\frac{(\zeta-\eta_1)(\zeta-\eta_2)(1-\zeta\overline{\zeta_1})(1-\zeta\overline{\zeta_2})}{\zeta(\zeta-\zeta_1)(\zeta-\zeta_2)(1-\zeta\overline{\eta_1})(1-\zeta\overline{\eta_2})}\right]+\frac{\Gamma R^2}{\zeta^2}\left(\frac{(\zeta-\eta_1)(\zeta-\eta_2)}{(\zeta-\zeta_1)(\zeta-\zeta_2)}\right)^2. \tag{15}$$

The two simple poles of $z_\zeta^{1/2}$ at ζ_1 and ζ_2 impose *four* conditions on the parameters in (14). These result from the conditions of the vanishing principal part of $S(\zeta)$ at ζ_1 and ζ_2. There is but a single further condition on $S(\zeta)$, namely, that it has the behavior given in (12). This imposes a single additional requirement on the mapping function. In total, with $z_\zeta^{1/2}$ of the form (14), there will be *five* conditions to be satisfied for a solution. However, note that there are precisely *five* as yet undetermined parameters in (14)—ζ_1, ζ_2, η_1, η_2 and R.

The resulting *consistent* system of five coupled nonlinear equations is found, after some simple manipulations, to be satisfied by the following parameters:

$$\eta_2=-\eta_1,\qquad \zeta_2=-\zeta_1,\qquad \eta_1=i\frac{1}{\sqrt{3}}\zeta_1,$$

$$\Gamma R^2=\frac{9}{2}\left(\frac{|\zeta_1|^4-1}{|\zeta_1|^4+3}\right),\quad \gamma^2=R^2\left(\left(\frac{|\zeta_1|^4-1}{|\zeta_1|^4+3}\right)-\frac{2}{3}\right). \tag{16}$$

B. The question of uniqueness

We have not so far succeeded in identifying any additional solutions having more than 2 simple poles of $[z_\zeta(\zeta)]^{1/2}$ outside the unit circle. While the counting arguments for the case of, say, *three* simple poles is again consistent, the resulting consistent system of seven nonlinear equations in seven unknowns does not appear to be solvable, at least, not after extensive searches by the present author. The same appears to be true of the case of four simple poles. We do not, however, make any definite statement here about whether or not solutions with $N>2$ poles exist. The intriguing mathematical question of the uniqueness of the above ''two-pole solutions'' therefore remains open at this time.

C. Summary of the exact solution (bubble)

We now provide a concise summary of the exact solutions. It is found that z_ζ can be integrated to the polar (rational) form,

$$z(\zeta)=-A\left(\frac{1}{\zeta}+\frac{8\zeta}{(\zeta^2-\zeta_1^2)}\right). \tag{17}$$

This mapping is a function of the two parameters A and ζ_1. ζ_1 is a parameter of the solutions which we now choose to be real. The derivative of this function is given by

$$z_\zeta(\zeta)=\frac{A}{\zeta^2}\left(\frac{3\zeta^2+\zeta_1^2}{\zeta^2-\zeta_1^2}\right)^2. \tag{18}$$

We choose to fix the area of the bubble to be π. Therefore, we insist that

$$\pi=-\frac{1}{2}\,\mathrm{Im}\oint_{|\zeta|=1}\bar{z}z_\zeta\,d\zeta. \tag{19}$$

This provides the following equation relating A and ζ_1:

$$A=\frac{1}{\sqrt{9-8\left(\frac{3+\zeta_1^4}{1-\zeta_1^4}\right)^2}}. \tag{20}$$

With the area of the bubble specified in this way, the above represents a one-parameter family of exact solutions (parametrized by ζ_1), with the corresponding Γ and γ given by

$$\Gamma=\frac{1}{2A}\left(\frac{\zeta_1^4-1}{\zeta_1^4+3}\right), \tag{21}$$

$$\gamma^2=9A\left(\left(\frac{\zeta_1^4-1}{\zeta_1^4+3}\right)-\frac{2}{3}\right). \tag{22}$$

Note that with A related to ζ_1 via (20), then (21) and (22) provide Γ and γ as functions of the single parameter ζ_1.

Note that it is straightforward to make use of a symbolic manipulator (e.g., MATHEMATICA) to verify that the above map and corresponding choice of parameters does indeed satisfy the Bernoulli condition (8) on $|\zeta|=1$. This was done as an explicit check on the solutions.

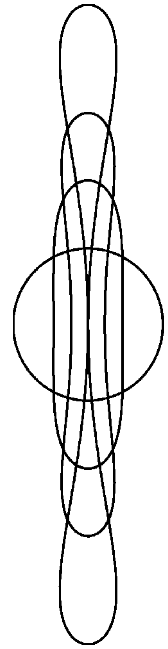

FIG. 1. (Superposed) equilibrium bubble shapes: $\zeta_1=3.0,\ 3.2,\ 3.7,\ \infty$.

We observe that in the limit as the pole position $\zeta_1\to\infty$ the shape of the free surface tends to a circular configuration and $z(\zeta)$ has the following trivial form:

$$z(\zeta)=-\frac{A}{\zeta}. \tag{23}$$

This represents a circular bubble and corresponds to an exact solution of the problem noted in Longuet-Higgins.[13] The new exact solutions (17) represent an analytic continuation of the circular solution of Longuet-Higgins[13] for *finite* ζ_1.

The ratio r of the perimeter to the area can easily be calculated analytically and is given by the formula

$$r=2A\left[-3-4\left(\frac{3+\zeta_1^4}{1-\zeta_1^4}\right)\right]. \tag{24}$$

A plot of r as a function of the pole position ζ_1 is given in Fig. 1. It is clear that the perimeter-to-area ratio gets large as the bubble gets closer to steady pinch-off.

We point out that in the derivation of the solutions it is assumed that Γ and γ are pre-specified parameters. In principle, given Γ and γ, the corresponding values of A and ζ_1 can be found. However, as seen above, now that the exact form of the solutions is known, it is more natural (and convenient) to study them by instead specifying a fixed area of the bubble and examining the solutions as the pole position ζ_1 changes.

For the sake of comparison, we record here the exact solution for deep water pure capillary waves as found originally by Crapper.[7] We present the solutions in the form as rederived (using the same general approach as in the present paper) in Crowdy:[6]

$$z(\zeta)=iA\left(\log\zeta-\frac{4\zeta_1}{(\zeta-\zeta_1)}\right), \tag{25}$$

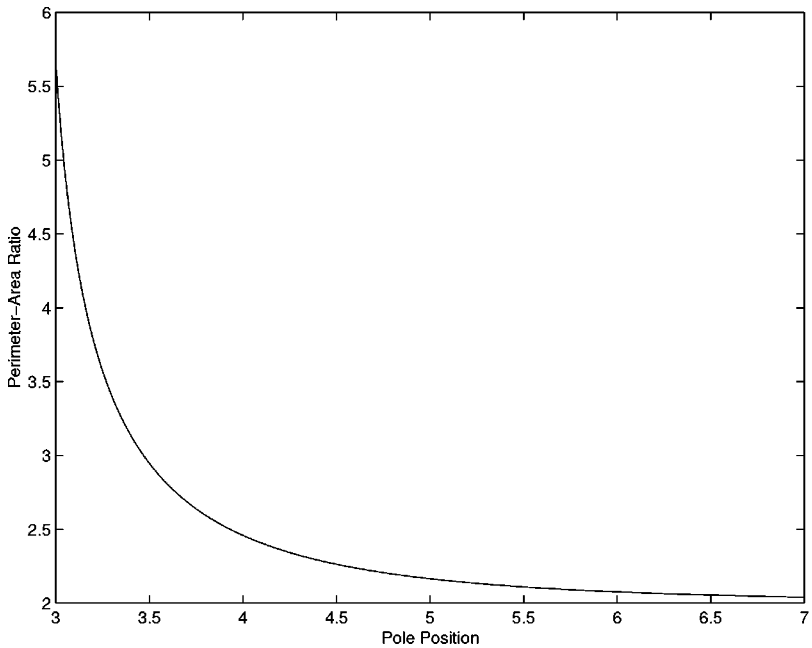

FIG. 2. Perimeter–area ratio (bubble) plotted against pole position ζ_1 (x-axis).

corresponding to

$$z_\zeta(\zeta)=\frac{iA}{\zeta}\frac{(\zeta+\zeta_1)^2}{(\zeta-\zeta_1)^2}, \tag{26}$$

with

$$\Gamma=\frac{1}{2A}\frac{|\zeta_1|^2-1}{|\zeta_1|^2+1}, \tag{27}$$

$$\frac{|W_\zeta|^2}{2}=\frac{c^2}{2k^2}=\Gamma A^2. \tag{28}$$

Solutions are again parametrized by ζ_1—the parameter A being set once the wavelength of the water wave is specified. ζ_1 is a parameter controlling the amplitude of the wave. Note that Crapper's solutions are comparatively simpler than the new solutions found here ($[z_\zeta]^{1/2}$ having only one simple pole in $|\zeta|>1$). The mathematical relationship between the solutions (18) and (26) to the two physical problems is clear.

In Fig. 2, the solutions for various different values of ζ_1 are superposed for comparison. It is found that as $\zeta_1\rightarrow 3.0$, the bubble becomes vertically elongated, eventually begins to pinch at which point the conformal map $z(\zeta)$ ceases to be univalent. This is not surprising—the analogous phenomenon in the case of capillary water waves is well-known[7,10] and the pinching of the bubble in this case can be thought of as the radial analogue to the pinching together of two different sections of the free surface capillary wave as found by Crapper.[7] It is interesting to observe that the same phenomenon also occurs in the present case of a bubble in a swirling flow. As $\zeta_1\rightarrow\infty$, the bubble tends to a circular shape as expected. Figure 3 shows some typical streamlines of a typical solution. In Figs. 4 and 5 the values of Γ and γ plotted as a function of ζ_1 (for a fixed area of the bubble) are given for the range of ζ_1 for which solutions exist. Note that there is a maximum value of γ^2 for which exact solutions exist (i.e., $3.0<\zeta_1<\infty$).

With a view to gaining insight into the fission events of bubbles in a liquid host, Vanden-Broeck and Keller[14] have studied a related two-dimensional model problem of bubble break-up by considering a single constant pressure bubble (with surface tension) and modeling the mean-field effects of the surrounding liquid host (and any other bubbles) using an ambient straining flow. They calculated the steady-state equilibrium shapes of the bubble numerically and observed that, at sufficiently large strain rates, the bubble pinches simultaneously at four different points on its bounding surface. Our results clearly show that steady capillary pinch-off (in fact, as two different parts of the interface come together at a *single* point) can still occur for a bubble in significantly milder ambient flow conditions—i.e., in an ambient circulatory flow rather than the much more singular straining flow considered by Vanden-Broeck and Keller.[14] Moreover, near (steady) pinch-off, the bubble is observed to become very elongated and possesses a large perimeter-to-area ratio. As shown in Fig 2, the perimeter of the bubble at pinch-off is almost three times the perimeter of the circular bubble solution of the same area and it is certainly conceivable, for example, that this much greater surface area of contact might lead to enhanced transport properties between the bubble and

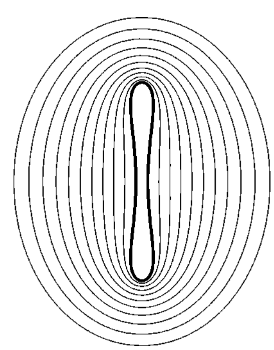

FIG. 3. Typical streamlines around a bubble for $\zeta_1=3.15$.

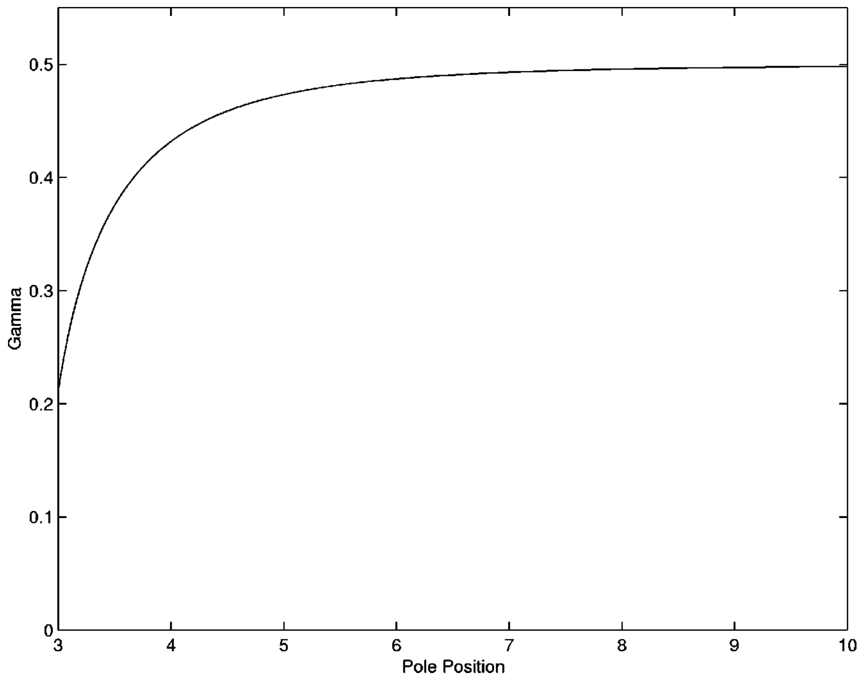

FIG. 4. A plot of Γ (vertical axis) versus pole position ζ_1 (bubble).

the host fluid. In addition, the circulation itself is seen to induce (at least, steady) pinch-off of the bubble into two smaller bubbles and thus it could be argued on the evidence of the above results that the presence of circulation leads to greater ''atomizing'' (or splitting) events and therefore more detached drops in a liquid dispersion. This would similarly provide a mechanism for enhanced transport rates. It is clear, however, that the above results are steady and that a full analysis of the unsteady problem is needed for greater understanding of the circulation-induced free surface dynamics.

IV. CIRCULATION-INDUCED BLOB DEFORMATIONS

We now consider a different but related problem: finding the steady-state shapes of a finite droplet/blob of fluid containing a single point vortex at some point inside it. This problem represents a simple paradigmatic example of the nonlinear interaction between a nontrivial circulatory flow inside a blob and its free (capillary) surface.

In this case, there is no longer a simple pole singularity of $z(\zeta)$ in the unit circle. Without loss of generality, it is

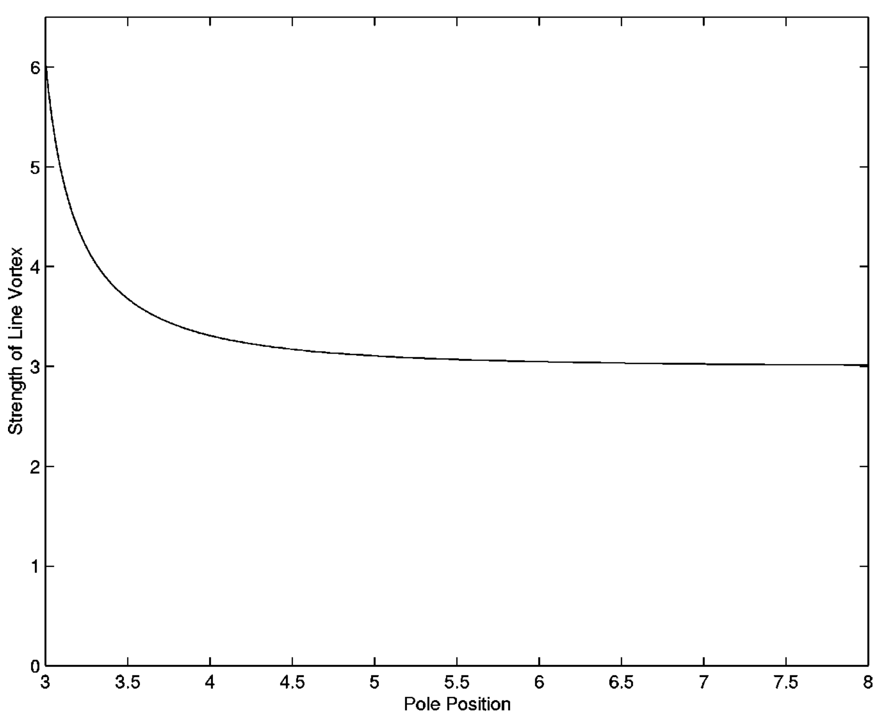

FIG. 5. A plot of γ^2 (vertical axis) versus pole position ζ_1 (bubble).

assumed that the point vortex is placed at the physical origin $z=0$ and that the position of the point vortex corresponds to $\zeta=0$ in a ζ-plane, i.e.,

$$z(0)=0. \tag{29}$$

In this case, the follow Laurent expansion (convergent in $|\zeta|\geqslant 1$) will be important:

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\overline{z}_\zeta(\zeta^{-1})}=\sum_{j=0}^{\infty}\frac{c_j}{\zeta^j}. \tag{30}$$

As in the previous section, it can similarly be shown that the problem of finding steady-state shapes for a droplet containing a single line vortex (for given Γ and γ) is equivalent to finding $z(\zeta)$ satisfying the following conditions:

(i) $z(\zeta)$ is a univalent conformal map from the unit circle to the fluid blob with

$$z(0)=0; \tag{31}$$

(ii) $[z_\zeta(\zeta)]^{1/2}$ has only simple pole singularities outside the unit circle;

(iii) $S(\zeta)$ is analytic everywhere outside the unit circle with

$$S(\zeta)\sim c_0, \quad \text{as} \quad \zeta\to\infty, \tag{32}$$

where c_0 is defined in (30).

A. Calculation of the exact solution

The above information on the analyticity structure of solutions again provides an explicit method of constructing them. In exactly the same way as for the case of a bubble, it can be shown that the system admits exact solutions with *two* poles of the mapping function outside the unit circle. In this case, the map has the functional form

$$[z_\zeta(\zeta)]^{1/2}=R\frac{(\zeta-\eta_1)(\zeta-\eta_2)}{(\zeta-\zeta_1)(\zeta-\zeta_2)}; \tag{33}$$

R is again assumed to be real. The corresponding $S(\zeta)$ is given by

$$S(\zeta)=-\frac{d}{d\zeta}\left[\frac{\zeta(\zeta-\eta_1)(\zeta-\eta_2)(1-\zeta\overline{\zeta_1})(1-\zeta\overline{\zeta_2})}{(\zeta-\zeta_1)(\zeta-\zeta_2)(1-\zeta\overline{\eta_1})(1-\zeta\overline{\eta_2})}\right] +\Gamma R^2\left(\frac{(\zeta-\eta_1)(\zeta-\eta_2)}{(\zeta-\zeta_1)(\zeta-\zeta_2)}\right)^2. \tag{34}$$

Five conditions result from the requirement that the principal part of $S(\zeta)$ vanishes at both ζ_1 and ζ_2 and from the requirement that $S(\zeta)$ satisfies (32). Parameters satisfying the following system are found to provide all the requirements:

$$\eta_2=-\eta_1, \quad \zeta_2=-\zeta_1, \quad \eta_1=i\sqrt{3}\zeta_1,$$

$$\Gamma R^2=\frac{1}{2}\left(\frac{|\zeta_1|^4-1}{3|\zeta_1|^4+1}\right), \tag{35}$$

$$\gamma^2=18R^2\left(\frac{1}{2}\left(\frac{|\zeta_1|^4-1}{3|\zeta_1|^4+1}\right)+\frac{1}{3}\right).$$

Again, we have been unable to identify any solutions corresponding to maps with more than 2 polar singularities in $|\zeta|>1$, however, we make no statement about whether or not additional solutions exist. This question of uniqueness remains open.

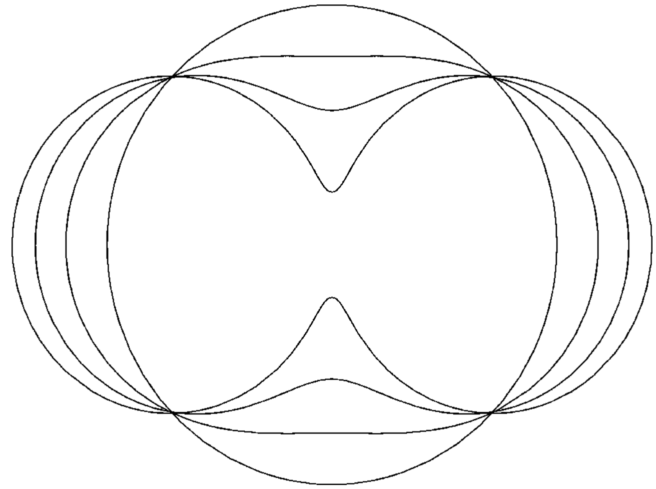

FIG. 6. (Superposed) equilibrium blob shapes: $\zeta_1=1.14,\ 1.5,\ 2.1,\ \infty$.

B. Summary of the exact solutions (blob)

Solving the system of 5 coupled nonlinear equations resulting from the required analyticity properties of $S(\zeta)$, and fixing the area of the blob to be π, we again have a one-parameter family of solutions (parametrized by the pole position ζ_1). Choosing ζ_1 to be real, the exact solution can be represented as follows:

$$z(\zeta)=A\left(\zeta-\frac{8\zeta_1^2\zeta}{\zeta^2-\zeta_1^2}\right). \tag{36}$$

This corresponds to the derivative

$$z_\zeta(\zeta)=A\left(\frac{\zeta^2+3\zeta_1^2}{\zeta^2-\zeta_1^2}\right)^2. \tag{37}$$

The area of the blob is fixed to be π so that

$$\pi=\frac{1}{2}\oint_{|\zeta|=1}\overline{z}z_\zeta d\zeta, \tag{38}$$

which gives A as a function of the parameter ζ_1, i.e.,

$$A=\sqrt{\frac{1}{9+8\left(\frac{1+3\zeta_1^4}{1-\zeta_1^4}\right)^2}}, \tag{39}$$

and the corresponding Γ and γ given as functions of ζ_1 by

$$\Gamma=\frac{1}{2A}\left(\frac{\zeta_1^4-1}{3\zeta_1^4+1}\right), \tag{40}$$

$$\gamma^2=18A\left(\frac{1}{3}+\frac{1}{2}\left(\frac{\zeta_1^4-1}{3\zeta_1^4+1}\right)\right). \tag{41}$$

The result is a one-parameter family of exact solutions (parametrized by ζ_1). In Fig. 6, the shapes of the blob for various different values of ζ_1 are superposed for comparison. It

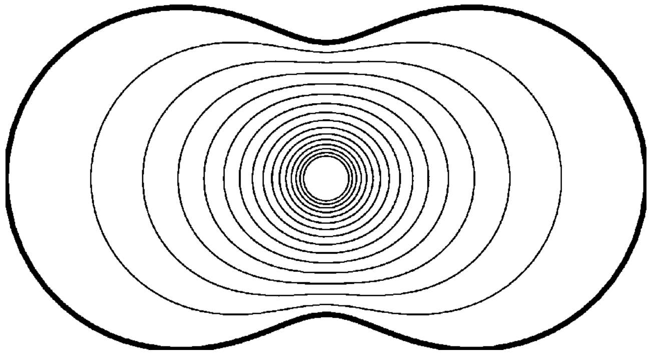

FIG. 7. Typical streamlines inside a blob ($\zeta_1 = 1.5$).

is found that solutions exist for all values of $\zeta_1 > 1$. As $\zeta_1 \to \infty$, the shape of the blob tends to a circular shape, as expected. As ζ_1 moves closer to the unit circle, the shape of the blob resembles two separate (but symmetrical) blobs joined by a necking region that grows thinner and pinches in towards the point vortex at the origin—the curvature in the pinching region increases markedly as $\zeta_1 \to 1$. Some typical streamlines are shown in Fig. 7. This plot is also a confirmation that the mapping function $z(\zeta)$ is a univalent function throughout the unit circle. Finally, Figs. 8 and 9 show the corresponding values of Γ and γ^2 as functions of the pole position ζ_1.

Note that in the preceding analysis we have implicitly made use of the *non-self-induction hypothesis* that is usual in the consideration of steady solutions involving point vortices (see Saffman[15]). However, for the self-consistency of the steady solutions, it is important to check that the velocity of the point vortex is *zero* under the assumption of non-self-induction. In this case, the symmetry of the solutions force this to be true. Alternatively, it can be seen from the exact solutions that

$$z(\zeta) = \zeta + O(\zeta^3), \quad \text{for small} \quad \zeta, \tag{42}$$

and therefore that

$$z_\zeta(\zeta) = 1 + O(\zeta^2). \tag{43}$$

Thus, the fluid velocity at the point vortex (i.e., at $\zeta = 0$ or $z = 0$) is

$$\frac{dw}{dz} = \frac{W_\zeta}{z_\zeta} = \frac{i\gamma}{\zeta}(1 + O(\zeta^2)). \tag{44}$$

Using (42) then implies that

$$\frac{dw}{dz} = \frac{i\gamma}{z}(1 + O(z^2)), \quad \text{as} \quad z \to 0. \tag{45}$$

Thus it is clear that, under the non-self-induction hypothesis, the velocity is $O(z)$ as $z \to 0$. The point vortex is therefore steady as required for consistency.

V. DISCUSSION

In this paper the effects of circulations inside and around single blobs and bubbles with surface tension have been modeled using line vortices. Clearly, this idealization is not physical—real (viscous) blobs with internal circulations generally contain smooth distributions of vorticity. Nevertheless, a line vortex provides a convenient model that is amenable to mathematical analysis. The class of mathematical models used in this paper has been shown to admit exact finite-amplitude solutions for the steady circulation-induced shape deformations of the bubbles and blobs where surface tension forces are in exact balance with the hydrodynamic pressure forces.

The model problems considered here have assumed that some nonzero circulatory flows have been induced, by *some* physical mechanism, inside or around a blob or bubble. Some examples of physical mechanisms possibly giving rise

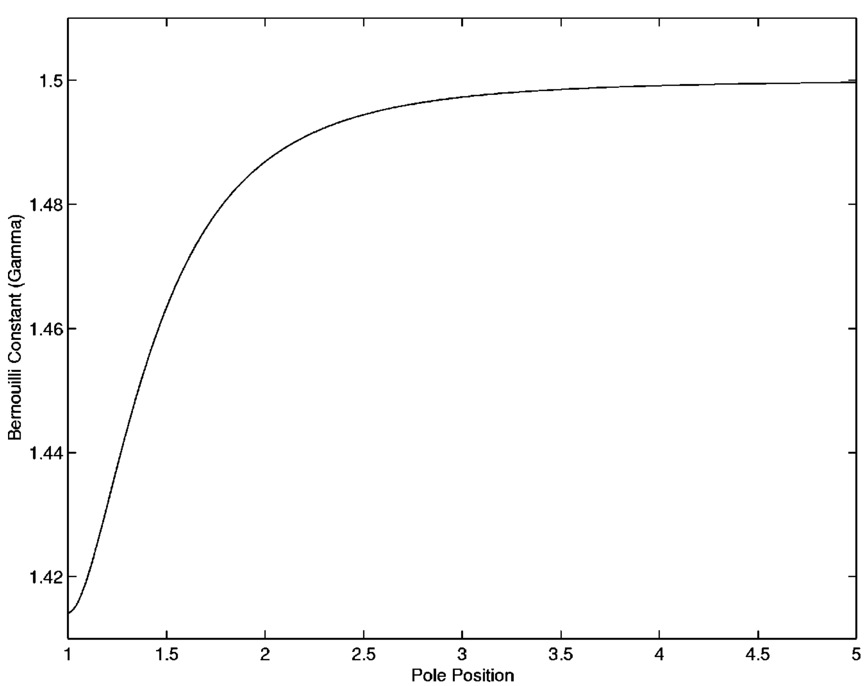

FIG. 8. A plot of Γ (vertical axis) versus pole position ζ_1 (blob).

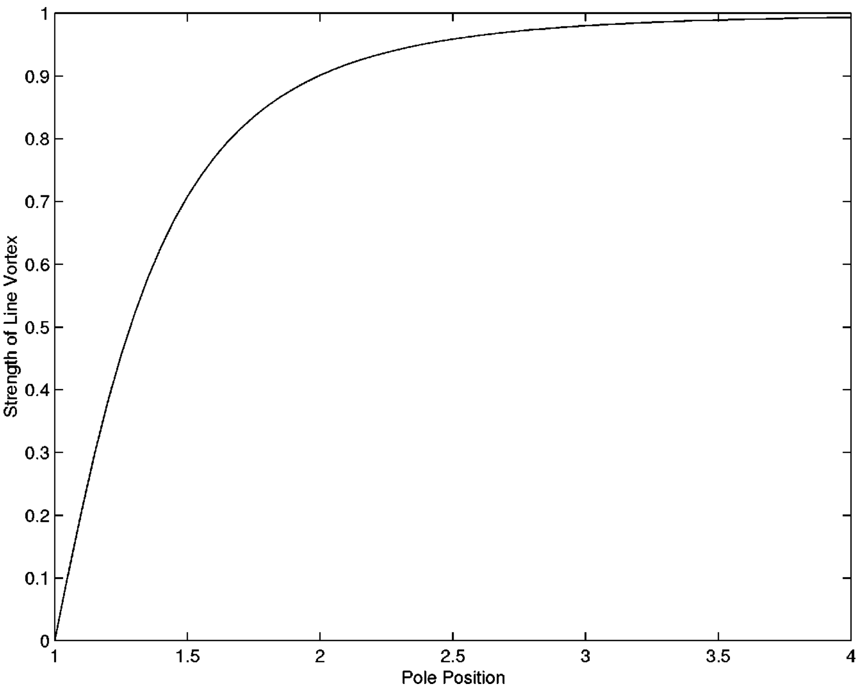

FIG. 9. A plot of γ^2 (vertical axis) versus pole position ζ_1 (blob).

to such circulations were briefly mentioned in the Introduction: a circulation in a blob can be induced by various mixing and atomizing events (e.g., blobs formed by tearing and stretching of liquid ligaments from a core liquid), by shear effects between the blob and host fluid, by thermo-capillary effects and various thermodynamics effects such as the presence of a nonisothermal host fluid. The precise details of how a blob has gained an internal circulation have not been modeled here, only the subsequent interaction of the gross circulatory flow with the free capillary surface.

A natural extension of the present work is to study the time-dependent evolution of the simple mathematical models. This might involve a study of complex singularity dynamics and might throw some analytical light on the nature of shape oscillations of bubbles and drops in the presence of internal/external circulations. In addition, the stability of the new equilibria found here is also clearly of interest.

Finally, it is noted that the new mathematical approach implicit in the presentation herein has wide utility. Indeed, the equilibrium solutions found in this paper can be generalized to doubly-connected fluid domains and the present author has recently identified exact solutions for the problem of steady capillary waves on a fluid annulus.[11] Perhaps not unexpectedly in light of the present results, these new annular solutions are generalizations of the classic exact solutions for finite amplitude waves on finite sheets of fluid as identified by Kinnersley[10] (which are themselves generalizations of Crapper's[7] exact solution). Moreover, the methods herein can also be generalized to solve quite different physical problems, for example, the problem of steady singularity-driven Hele-Shaw flows in the presence of surface tension. In this problem too, a wide class of new exact solutions[16] have been identified using methods similar to those employed here.

APPENDIX: ANALYTICITY OF $S(\zeta)$

The following theorem shows that satisfying the Bernoulli condition on the bubble boundary is equivalent to ensuring certain global analyticity properties of the function $S(\zeta)$ in the extended complex plane *outside* the unit circle.

Theorem: The Bernoulli condition on the free surface of the bubble is equivalent to $S(\zeta)$ being analytic everywhere in $|\zeta|\geqslant 1$ with

$$S(\zeta)\sim\frac{c_2}{\zeta^2},\quad \text{as } \zeta\to\infty, \tag{A1}$$

where c_2 is defined in (11).

Proof: First, assume that the Bernoulli condition holds on $|\zeta|=1$. This implies

$$S(\zeta)=\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\overline{z}_\zeta(\zeta^{-1})}. \tag{A2}$$

By analytic continuation, this also holds off the unit circle. It is clear that this implies immediately that $S(\zeta)$ is analytic in $|\zeta|\geqslant 1$ and that $S(\zeta)\sim(c_2/\zeta^2)$.

Conversely, assume that $S(\zeta)$ is analytic outside the unit circle, including at infinity where $S(\zeta)\sim(c_2/\zeta^2)$. Given these conditions on $S(\zeta)$ it is clear that it can be written in the form

$$S(\zeta)=\frac{H(\zeta)}{\overline{z}_\zeta(\zeta^{-1})}, \tag{A3}$$

for some $H(\zeta)$ (to be determined) which is analytic in $|\zeta|\geqslant 1$ and tends to a constant as $\zeta\to\infty$. This is because $\overline{z}_\zeta^{-1}(\zeta^{-1})$ is analytic outside the unit circle and $\sim\zeta^2$ as $\zeta\to\infty$. Note that

$$\overline{S(\zeta)\overline{z_\zeta}(\zeta^{-1})} = S(\zeta)\overline{z_\zeta}(\zeta^{-1}), \tag{A4}$$

on $|\zeta| = 1$. This can be seen after some manipulation using the definition of $S(\zeta)$. Equations (A3) and (A4) imply that $H(\zeta)$ is real on the unit circle, i.e.,

$$\overline{H}(\zeta^{-1}) = H(\zeta), \tag{A5}$$

on $|\zeta| = 1$. Equation (A5) furnishes the analytic continuation of $H(\zeta)$ into $|\zeta| \leqslant 1$ and, in particular, reveals that it is analytic everywhere in $|\zeta| \leqslant 1$. Thus, $H(\zeta)$ has been shown to be analytic everywhere in the finite plane, bounded as $\zeta \to \infty$ and real on the unit circle. By Liouville's theorem, $H(\zeta)$ is necessarily a real constant function. The additional condition that $S(\zeta) \sim c_2/\zeta^2$ sets this constant and finally implies that

$$H(\zeta) = \gamma^2 = W_\zeta(\zeta)\overline{W_\zeta}(\zeta^{-1}). \tag{A6}$$

Equation (A3) is then equivalent to the Bernoulli condition and the theorem is proved.

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