PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2005) **461**, 2653–2678
doi:10.1098/rspa.2005.1480
Published online 23 June 2005

The Schwarz–Christoffel mapping to bounded multiply connected polygonal domains

By Darren Crowdy

Department of Mathematics, Imperial College of Science, Technology and Medicine, 180 Queen's Gate, London SW7 2AZ, UK
(d.crowdy@imperial.ac.uk)

A formula for the generalized Schwarz–Christoffel mapping from a bounded multiply connected circular domain to a bounded multiply connected polygonal domain is derived. The theory of classical Schottky groups is employed. The formula for the derivative of the mapping function contains a product of powers of Schottky–Klein prime functions associated with a Schottky group relevant to the circular pre-image domain. The formula generalizes, in a natural way, the known mapping formulae for simply and doubly connected polygonal domains.

Keywords: conformal mapping; Schwarz–Christoffel; multiply connected

1. Introduction

The construction of the conformal mapping from the upper-half plane or unit disc in a pre-image plane to a given simply connected polygonal region is a well-known classical result of complex analysis (Nehari 1952; Ablowitz & Fokas 1997). The conformal mappings are known as Schwarz–Christoffel maps and the subject, being of very general applicability, already commands an extensive literature. The monograph by Driscoll & Trefethen (2002) provides a recent review of the study of such mappings and gives a comprehensive list of references. It also surveys some of the many applications of the formula in various branches of science.

A natural and long-standing question is how to generalize the classical Schwarz–Christoffel mapping formula to the case of multiply connected polygonal regions. Only a few results addressing this question currently exist and almost all of these pertain to the case of mapping to doubly connected polygonal regions. Given the general applicability of Schwarz–Christoffel mappings (henceforth abbreviated to S–C mappings), it is clear that the derivation of generalized formulae mapping to polygonal regions of arbitrary finite connectivity is of some interest.

Embree & Trefethen (1999) have used ideas involving S–C mappings of multiply connected domains to construct the first-type Green's function in such domains. However, the domains considered have reflectional symmetry about the real axis, thereby allowing use of the Schwarz reflection principle to reduce the problem to one of constructing a simply connected S–C mapping to 'half'

Received 9 October 2004
Accepted 22 March 2005

the domain. Another study that addresses the more general question of mapping to polygonal domains of connectivity greater than two is the recent paper by DeLillo *et al.* (2005). In the latter paper, the authors derive a formula for mapping from a finitely connected unbounded circular pre-image region to an unbounded conformally equivalent polygonal region. The derivation relies on an extension of an idea originally presented in DeLillo *et al.* (2001) involving consideration of an infinite sequence of reflections in circles needed to satisfy the relevant argument conditions (on the derivative of the mapping function) on the segments of each pre-image circle mapping to the sides of the polygonal region.

If one is interested in an S–C mapping from a *bounded* multiply connected circular pre-image region to a *bounded* multiply connected polygonal region, no such formula currently exists in the literature. The derivation of such a formula is the subject of this paper. Further, the theoretical approach used here is conceptually different to that used by DeLillo *et al.* (2005) for the unbounded case. Here, results from classical function theory are used to construct the mapping. In principle, it should also be possible to extend the approach of DeLillo *et al.* (2005) to the case of mapping to bounded polygonal domains.

Although our mathematical approach is different, the reader will nevertheless discern certain common mathematical threads in the two constructions. In particular, although this is not mentioned in their paper, the infinite sequences of reflections in circles appearing in the construction presented by DeLillo *et al.* (2005) is naturally associated with the theory of classical Schottky groups of Möbius mappings (Beardon 1984). In turn, associated with any such Schottky group is a fundamental function known as the *Schottky–Klein prime function* (Baker 1995). The key result of this paper is to show that an S–C mapping to bounded polygonal domains can be written, in a natural way, as a product of powers of this prime function.

The Schottky groups associated with the simply and doubly connected cases are, respectively, the trivial group and the loxodromic group of Möbius mappings (the latter is defined more precisely later). In these cases, the product of powers of Schottky–Klein prime functions associated with the two groups naturally reduce to the formulae for the simply and doubly connected cases that have already appeared in the literature (Driscoll & Trefethen 2002). The final form of the mapping formula derived in this paper is

$$z(\zeta) = A + B\int^{\zeta} S(\zeta')\prod_{k=1}^{n_0}[\omega(\zeta';a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta';a_k^{(j)})]^{\beta_k^{(j)}}\,\mathrm{d}\zeta', \tag{1.1}$$

with

$$S(\zeta) \equiv \left(\frac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})}\right), \tag{1.2}$$

and where $\omega(\zeta,\gamma)$ is the relevant Schottky–Klein prime function. $M+1$ is the connectivity of the domain. All other parameters appearing in (1.1) and (1.2) are explained in the main body of the paper. What we wish to emphasize here is that the single formula (1.1) encapsulates all S–C mappings to finitely connected bounded polygonal regions. All that differs from one topology to the next is the relevant Schottky group and hence the definition of the Schottky–Klein prime function $\omega(\zeta,\gamma)$. This generality in the formula (1.1) would seem attractive from both a conceptual and an implementational viewpoint.

For example, in the simply connected case when the Schottky group is just the trivial group, the associated prime function is

$$\omega(\zeta,\gamma) = (\zeta-\gamma), \tag{1.3}$$

while the function $S(\zeta)$ is just a constant when $M=0$. With these identifications, it should be clear that (1.1) then has the same functional form as the well-known S–C mapping formula from a unit disc as given in eqn (2.4) of Driscoll & Trefethen (2002). In a similar way, it will be shown in §8*b* how (1.1) reduces to the formula for mappings from an annulus to doubly connected polygonal domains as recorded in eqn (4.21) of Driscoll & Trefethen (2002).

2. Mathematical formulation

Let the target region D_z in a complex z-plane be a bounded $(M+1)$-connected polygonal region. $M=0$ is the simply connected case. Let P_0 denote the outer boundary polygon and let the M smaller enclosed polygons be $\{P_j | j=1,\ldots,M\}$. Let P_j have n_j edges where $n_j \geq 2$ are integers. The set of interior angles at each vertex of polygon P_j are

$$\pi(\beta_k^{(j)}+1), \quad k=1,\ldots,n_j, \tag{2.1}$$

where

$$\sum_{k=1}^{n_0} \beta_k^{(0)} = -2, \quad \sum_{k=1}^{n_j} \beta_k^{(j)} = 2, \quad j=1,\ldots,M. \tag{2.2}$$

The parameters $\{\pi\beta_k^{(j)} | j=0,1,\ldots,M\}$ are the *turning angles* (Driscoll & Trefethen 2002). Let the straight-line edges of polygon P_j be given by the following linear equations

$$\bar{z} = \epsilon_k^{(j)} z + \kappa_k^{(j)}, \quad k=1,\ldots,n_j, \tag{2.3}$$

where $\epsilon_k^{(j)}$ and $\kappa_k^{(j)}$ are a set of complex constants and $|\epsilon_k^{(j)}|=1$. For a given target polygon, the parameters $\{\epsilon_k^{(j)},\kappa_k^{(j)}\}$ are specified.

We seek a conformal mapping to D_z from a conformally equivalent multiply connected circular domain D_ζ. Let D_ζ be the unit ζ-disc with M smaller circular discs excised. Let the boundaries of these smaller circular discs be denoted $\{C_j | j=1,\ldots,M\}$ and let $|\zeta|=1$ be denoted C_0. The complex numbers $\{\delta_j | j=1,\ldots,M\}$ will be taken to denote the centres of the enclosed circular discs, while the real numbers $\{q_j | j=1,\ldots,M\}$ will denote their radii. Figure 1 shows a schematic.

To proceed with the construction, an intermediate η-plane will be introduced. Consider a conformal mapping $\eta(\zeta)$ taking the multiply connected circular domain D_ζ to a conformally equivalent circular-slit domain D_η. Figure 2 shows a schematic. Let the image of C_0 under this mapping be the unit circle in the η-plane which will be called L_0. The M circles $\{C_j | j=1,\ldots,M\}$ will be taken to have finite-length circular-slit images, centred on $\eta=0$, and labelled

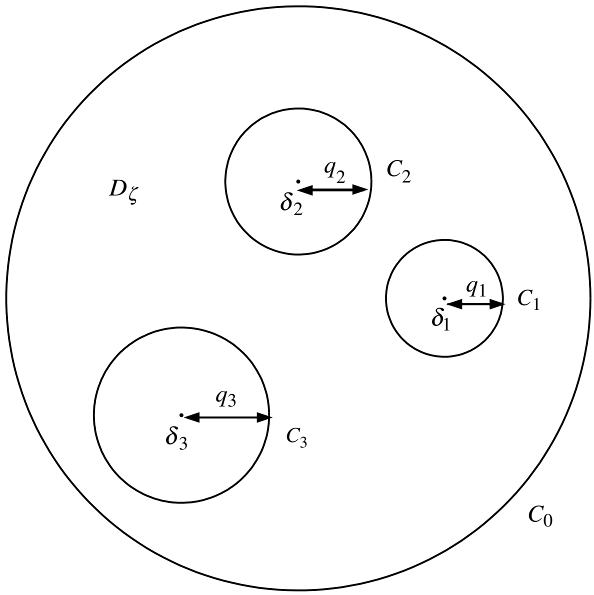

Figure 1. A multiply connected circular region D_ζ. The case shown, with three enclosed circles, is quadruply connected. C_0 denotes the unit circle. There are M interior circles (the case $M=3$ is shown here), each labelled $\{C_j|j=1,\ldots,M\}$. The centre of circle C_j is δ_j and its radius is q_j.

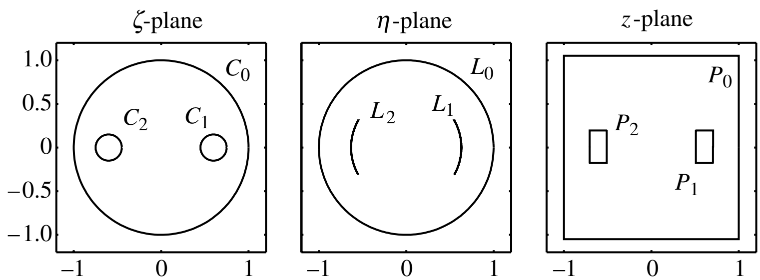

Figure 2. Schematic of the three complex planes for the case of a triply connected domain. The circular domain D_ζ in the ζ-plane, the circular-slit domain D_η in the η-plane and the polygonal domain D_z in the z-plane are shown. The schematic shows the circles C_0, C_1 and C_2, the slits L_0, L_1 and L_2 and the corresponding polygons P_0, P_1 and P_2.

$\{L_j|j=1,\ldots,M\}$. Let the arc L_j be specified by the conditions

$$|\eta| = r_j, \quad \arg[\eta] \in [\phi_1^{(j)}, \phi_2^{(j)}]. \tag{2.4}$$

It is clear that, for $j=1,\ldots,M$, there will be two pre-image points on the circle C_j in the ζ-plane corresponding to the two endpoints of the circular-slit L_j. These two pre-image points, labelled $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$, satisfy the conditions

$$\left.\begin{aligned} \eta(\gamma_1^{(j)}) &= r_j e^{i\phi_1^{(j)}}, \quad \eta_\zeta(\gamma_1^{(j)}) = 0, \\ \eta(\gamma_2^{(j)}) &= r_j e^{i\phi_2^{(j)}}, \quad \eta_\zeta(\gamma_2^{(j)}) = 0, \end{aligned} \quad j = 1,\ldots,M. \right\} \tag{2.5}$$

These $2M$ zeros of η_ζ are all simple zeros since the points $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$ map to the ends of a slit and the arguments of $\eta(\zeta)-\eta(\gamma_1^{(j)})$ and $\eta(\zeta)-\eta(\gamma_2^{(j)})$ change by 2π as ζ passes through these points. This fact will be important later.

The idea of the construction of the S–C mapping is to consider conditions on the derivative of the mapping function in the intermediate η-plane. These conditions turn out to be easier to handle than those in the original ζ-plane, basically because the conditions take the same functional form on all boundaries (which is not the case in the original ζ-plane). Once these conditions on $z(\eta)$ are satisfied in the η-plane, the functional form of the required mapping function $z(\zeta)=z(\eta(\zeta))$ can be deduced.

3. Schottky groups

To proceed with the construction, first define M Möbius maps $\{\phi_j|j=1,\ldots,M\}$ corresponding, respectively, to the conjugation maps on the circles $\{C_j|j=1,\ldots,M\}$. That is, if C_j has equation

$$|\zeta-\delta_j|^2=(\zeta-\delta_j)(\bar{\zeta}-\bar{\delta}_j)=q_j^2, \tag{3.1}$$

then

$$\bar{\zeta}=\bar{\delta}_j+\frac{q_j^2}{\zeta-\delta_j}, \tag{3.2}$$

and so

$$\phi_j(\zeta)\equiv\bar{\delta}_j+\frac{q_j^2}{\zeta-\delta_j}. \tag{3.3}$$

If ζ is a point on C_j, then its complex conjugate

$$\bar{\zeta}=\phi_j(\zeta). \tag{3.4}$$

Next, introduce the Möbius maps

$$\theta_j(\zeta)\equiv\bar{\phi}_j(\zeta^{-1})=\delta_j+\frac{q_j^2\zeta}{1-\bar{\delta}_j\zeta}, \tag{3.5}$$

where the conjugate function $\bar{\phi}_j$ is defined by

$$\bar{\phi}_j(\zeta)=\overline{\phi_j(\bar{\zeta})}. \tag{3.6}$$

For $j=1,\ldots,M$, let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta|=1$ (i.e. the circle obtained by the transformation $\zeta\mapsto 1/\bar{\zeta}$). It is easily verified that the image of the circle C_j' under the transformation θ_j is the circle C_j. Since the M circles $\{C_j\}$ are non-overlapping, so are the M circles $\{C_j'\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of Möbius mappings generated by compositions of the M basic Möbius maps $\{\theta_j|j=1,\ldots,M\}$ and their inverses $\{\theta_j^{-1}|j=1,\ldots,M\}$ and including the identity map. Beardon (1984) gives a general discussion of such groups. A very accessible

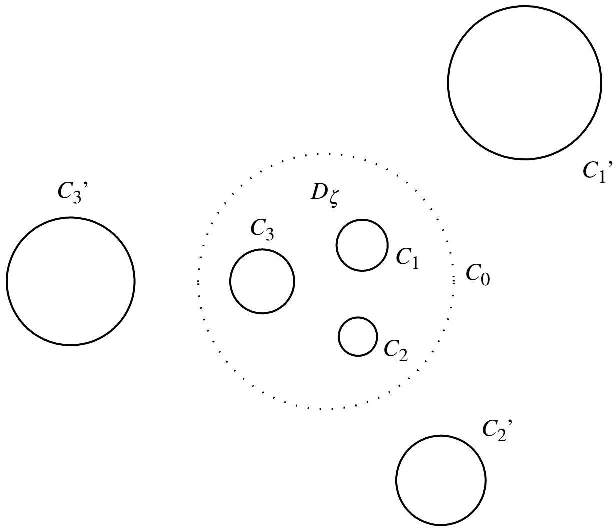

Figure 3. The set of Schottky circles associated with a quadruply connected domain. The region exterior of all six circles $\{C_j, C_j' | j=1,2,3\}$ is the fundamental region. The part of the fundamental region interior to the unit circle C_0 is D_ζ.

discussion of Schottky groups and their mathematical properties can also be found in a recent monograph by Mumford *et al.* (2002).

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j\}$ and $\{C_j'\}$. A schematic is shown in figure 3. This region is known as the *fundamental region* associated with the Schottky group. This fundamental region can be understood as having two 'halves'—the half that is inside the unit circle but exterior to the circles C_j is D_ζ, and the region that is outside the unit circle and exterior to the circles C_j' is the other half. The region is called fundamental because the whole complex plane is tessellated by an infinite sequence of 'copies' of this region obtainable by conformally mapping the fundamental region by elements of the Schottky group. Any point in the plane that can be reached by the action of a finite composition of the basic generating maps on a point in the fundamental region is called an *ordinary point* of the group. Any point not obtainable in this way is a *singular point* of the group. The set of $2M$ circles $\{C_j, C_j' | j=1, \ldots, M\}$ are known as the set of *Schottky circles*.

There are two important properties of the Möbius maps introduced above. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}, \quad \forall\, \zeta. \tag{3.7}$$

This can be verified using the definitions (3.3) and (3.5) (or, alternatively, by considering the geometrical effect of each map). The second property, which follows from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)}, \quad \forall\, \zeta. \tag{3.8}$$

4. The Schottky–Klein prime function

Following the discussion in ch. 12 of Baker (1995), the *Schottky–Klein prime function* is defined as

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma), \tag{4.1}$$

where the function $\omega'(\zeta,\gamma)$ is given by

$$\omega'(\zeta,\gamma) = \prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)}, \tag{4.2}$$

and where the product is over all mappings in Θ''—the set of all mappings in Θ excluding the identity map and all inverses. It is emphasized that the prime notation is not used here to denote differentiation. The function $\omega(\zeta,\gamma)$ is single-valued on the whole ζ-plane and has a zero at γ and all points equivalent to γ under the mappings of the group Θ. Again following Baker (1995), we proceed under the assumption that the infinite product defining the prime function is convergent. Whether this is true will depend, in general, on the distribution of Schottky circles in the ζ-plane. A basic rule of thumb is that convergence is good provided the Schottky circles are well-separated in the ζ-plane. Of course, the final formula derived here will only be valid provided such convergence criteria are satisfied. Similar convergence criteria arise in the construction of DeLillo *et al.* (2005).

The Schottky–Klein prime function has some important transformation properties that will be needed in the construction of the S–C mapping. One such property is that it is antisymmetric in its arguments, that is,

$$\omega(\zeta,\gamma) = -\omega(\gamma,\zeta). \tag{4.3}$$

This is clear from inspection of (4.1) and (4.2). A second important property is given by

$$\frac{\omega(\theta_j(\zeta),\gamma_1)}{\omega(\theta_j(\zeta),\gamma_2)} = \beta_j(\gamma_1,\gamma_2)\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}, \tag{4.4}$$

where θ_j is one of the generating maps of the group. A detailed derivation of this result is given in ch. 12 of Baker (1995). A formula for $\beta_j(\gamma_1,\gamma_2)$ is

$$\beta_j(\gamma_1,\gamma_2) = \prod_{\theta_k \in \Theta_j} \frac{(\gamma_1-\theta_k(B_j))(\gamma_2-\theta_k(A_j))}{(\gamma_1-\theta_k(A_j))(\gamma_2-\theta_k(B_j))}, \tag{4.5}$$

where the set Θ_j denotes all compositions of the basic mappings of the group Θ which do not have a positive or negative power of θ_j at the right-hand end. A_j and B_j are the two fixed points of the mapping θ_j satisfying

$$\theta_j(A_j) = A_j, \quad \theta_j(B_j) = B_j. \tag{4.6}$$

A_j and B_j are the two solutions of a quadratic equation and satisfy an equation of the form

$$\frac{\theta_j(\zeta)-B_j}{\theta_j(\zeta)-A_j} = \mu_j \mathrm{e}^{\mathrm{i}\kappa_j}\frac{\zeta-B_j}{\zeta-A_j}, \tag{4.7}$$

for some real constants μ_j, κ_j. A_j and B_j are distinguished by the condition that $|\mu_j|<1$ in (4.7). A third property of $\omega(\zeta,\gamma)$ which will also be useful later is

$$\bar{\omega}(\zeta^{-1},\gamma^{-1}) = -\frac{1}{\zeta\gamma}\omega(\zeta,\gamma), \tag{4.8}$$

where the conjugate function $\bar{\omega}(\zeta,\gamma)$ is defined by

$$\bar{\omega}(\zeta,\gamma) = \overline{\omega(\bar{\zeta},\bar{\gamma})}. \tag{4.9}$$

A detailed derivation of (4.8) is given in an appendix to Crowdy & Marshall (2005).

It is convenient to categorize all possible compositions of the basic maps according to their *level*. As an illustration, consider the case in which there are four basic maps $\{\theta_j|j=1,2,3,4\}$. The identity map is considered to be the *level-zero map*. The four basic maps, together with their inverses, $\{\theta_j^{-1}|j=1,2,3,4\}$ constitute the eight *level-one maps*. All possible combinations of any *two* of these eight level-one maps that do not reduce to the identity, e.g.

$$\theta_1(\theta_1(\zeta)),\ \theta_1(\theta_2(\zeta)),\ \theta_1(\theta_3(\zeta)),\ \theta_1(\theta_4(\zeta)),\ \theta_2(\theta_1(\zeta)),\ \theta_2(\theta_2(\zeta)),\ldots, \tag{4.10}$$

will be called the *level-two maps*, all possible combinations of any *three* of the eight level-one maps that do not reduce to a lower level map will be called the *level-three maps*, and so on.

On a practical note, to write a function routine to numerically calculate $\omega(\zeta,\gamma)$, it is necessary to truncate the infinite product in (4.2). This is done in a natural way by including all Möbius maps up to some chosen level and truncating the contribution to the product from all higher-level maps. The truncation that includes all maps up to level three have been used to compute the examples in this paper. The software program MATLAB is particularly suited to construction of the Schottky–Klein prime function, since the action of an element of the Schottky group on the point ζ can be written as multiplication by a 2×2 matrix on the vector $(\zeta,\ 1)^{\mathrm{T}}$—a linear algebra operation that is performed very efficiently in MATLAB.

5. Three special functions

In this section, a series of propositions will outline the properties of three special functions, which will be called $\{F_j(\zeta;\zeta_1,\zeta_2)|j=1,2,3\}$, needed in the construction of the S–C mapping formula. These special functions are all constructed as ratios of Schottky–Klein prime functions as defined in the previous section.

Proposition 5.1. *If ζ_1 and ζ_2 are any two distinct points on C_0, then the function*

$$F_1(\zeta;\zeta_1,\zeta_2)\equiv\frac{\omega(\zeta,\zeta_1)}{\omega(\zeta,\zeta_2)}, \tag{5.1}$$

has constant argument on each of the circles $\{C_j|j=0,1,\ldots,M\}$.

Proof. First consider points ζ on C_0. There,

$$\left.\begin{aligned}\overline{F_1(\zeta;\zeta_1,\zeta_2)} &= \frac{\overline{\omega(\zeta,\zeta_1)}}{\overline{\omega(\zeta,\zeta_2)}} = \frac{\bar{\omega}(\zeta^{-1},\zeta_1^{-1})}{\bar{\omega}(\zeta^{-1},\zeta_2^{-1})},\\ &= \frac{\zeta_2\,\omega(\zeta,\zeta_1)}{\zeta_1\,\omega(\zeta,\zeta_2)}, \qquad \text{by (4.8)},\\ &= \frac{\zeta_2}{\zeta_1}F_1(\zeta;\zeta_1,\zeta_2),\end{aligned}\right\} \tag{5.2}$$

which implies that the argument of $F_1(\zeta;\zeta_1,\zeta_2)$ is constant on C_0.

Now consider points ζ on any of the enclosed circles $\{C_j|j=1,\dots,M\}$. On C_j,

$$\left.\begin{aligned}\overline{F_1(\zeta;\zeta_1,\zeta_2)} &= \frac{\overline{\omega(\zeta,\zeta_1)}}{\overline{\omega(\zeta,\zeta_2)}} = \frac{\bar{\omega}(\phi_j(\zeta),\zeta_1^{-1})}{\bar{\omega}(\phi_j(\zeta),\zeta_2^{-1})}, \qquad \text{by (3.4)},\\ &= \frac{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\zeta_1^{-1})}{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\zeta_2^{-1})}, \quad \text{by (3.5)}.\end{aligned}\right\} \tag{5.3}$$

But now (4.4) can be used to deduce that

$$\left.\begin{aligned}\overline{F_1(\zeta;\zeta_1,\zeta_2)} &= \overline{\beta_j(\bar{\zeta}_1^{-1},\bar{\zeta}_2^{-1})}\frac{\bar{\omega}(\zeta^{-1},\zeta_1^{-1})}{\bar{\omega}(\zeta^{-1},\zeta_2^{-1})}, \qquad \text{by (4.4)},\\ &= \overline{\beta_j(\bar{\zeta}_1^{-1},\bar{\zeta}_2^{-1})}\frac{\zeta_2}{\zeta_1}F_1(\zeta;\zeta_1,\zeta_2), \quad \text{by (4.8)},\end{aligned}\right\} \tag{5.4}$$

so the argument of $F_1(\zeta;\zeta_1,\zeta_2)$ is therefore constant on each of the circles $\{C_j|j=0,1,\dots,M\}$. ■

Proposition 5.2. *If ζ_1 and ζ_2 are two distinct points on a particular choice of enclosed circle C_j (for some $j=1,\dots,M$), then the function*

$$F_2(\zeta;\zeta_1,\zeta_2) \equiv \frac{\omega(\zeta,\zeta_1)}{\omega(\zeta,\zeta_2)}, \tag{5.5}$$

has constant argument on each of the circles $\{C_k|k=0,1,\dots,M\}$.

Proof. First consider points ζ on C_0. There,

$$\left.\begin{aligned}\overline{F_2(\zeta;\zeta_1,\zeta_2)} &= \frac{\overline{\omega(\zeta,\zeta_1)}}{\overline{\omega(\zeta,\zeta_2)}} = \frac{\bar{\omega}(\zeta^{-1},\phi_j(\zeta_1))}{\bar{\omega}(\zeta^{-1},\phi_j(\zeta_2))}, \qquad \text{by (3.4)},\\ &= \frac{\bar{\omega}(\phi_j(\zeta_1),\zeta^{-1})}{\bar{\omega}(\phi_j(\zeta_2),\zeta^{-1})}, \qquad \text{by (4.3)},\\ &= \frac{\bar{\omega}(\bar{\theta}_j(\zeta_1^{-1}),\zeta^{-1})}{\bar{\omega}(\bar{\theta}_j(\zeta_2^{-1}),\zeta^{-1})}, \quad \text{by (3.5)},\end{aligned}\right\} \tag{5.6}$$

but we can now use the transformation property (4.4) to deduce

$$\left.\begin{aligned} \overline{F_2(\zeta;\zeta_1,\zeta_2)} &= \overline{\beta_j(\bar{\zeta}^{-1},\bar{\zeta}^{-1})}\frac{\bar{\omega}(\zeta_1^{-1},\zeta^{-1})}{\bar{\omega}(\zeta_2^{-1},\zeta^{-1})} \\ &= \frac{\bar{\omega}(\zeta^{-1},\zeta_1^{-1})}{\bar{\omega}(\zeta^{-1},\zeta_2^{-1})}, && \text{by (4.3)}, \\ &= \frac{\zeta_2}{\zeta_1}F_2(\zeta;\zeta_1,\zeta_2), && \text{by (4.8)}. \end{aligned}\right\} \tag{5.7}$$

Now consider points ζ on any of the enclosed circles $\{C_k|k=1,\ldots,M\}$. On C_k,

$$\left.\begin{aligned} \overline{F_2(\zeta;\zeta_1,\zeta_2)} = \frac{\overline{\omega(\zeta,\zeta_1)}}{\overline{\omega(\zeta,\zeta_2)}} &= \frac{\bar{\omega}(\phi_k(\zeta),\phi_j(\zeta_1))}{\bar{\omega}(\phi_k(\zeta),\phi_j(\zeta_2))}, && \text{by (3.4)}, \\ &= \frac{\bar{\omega}(\bar{\theta}_k(\zeta^{-1}),\bar{\theta}_j(\zeta_1^{-1}))}{\bar{\omega}(\bar{\theta}_k(\zeta^{-1}),\bar{\theta}_j(\zeta_2^{-1}))}, && \text{by (3.5)}. \end{aligned}\right\} \tag{5.8}$$

Now, on use of the transformation property (4.4),

$$\left.\begin{aligned} \overline{F_2(\zeta;\zeta_1,\zeta_2)} &= \overline{\beta_k(\theta_j(\bar{\zeta}_1^{-1}),\theta_j(\bar{\zeta}_2^{-1}))}\frac{\bar{\omega}(\zeta^{-1},\bar{\theta}_j(\zeta_1^{-1}))}{\bar{\omega}(\zeta^{-1},\bar{\theta}_j(\zeta_2^{-1}))}, \\ &= \overline{\beta_k(\theta_j(\bar{\zeta}_1^{-1}),\theta_j(\bar{\zeta}_2^{-1}))}\frac{\bar{\omega}(\bar{\theta}_j(\zeta_1^{-1}),\zeta^{-1})}{\bar{\omega}(\bar{\theta}_j(\zeta_2^{-1}),\zeta^{-1})}, && \text{by (4.3)}. \\ &= \overline{\beta_k(\theta_j(\bar{\zeta}_1^{-1}),\theta_j(\bar{\zeta}_2^{-1}))\beta_j(\bar{\zeta}^{-1},\bar{\zeta}^{-1})}\frac{\bar{\omega}(\zeta_1^{-1},\zeta^{-1})}{\bar{\omega}(\zeta_2^{-1},\zeta^{-1})}, && \text{by (4.4)}, \\ &= \overline{\beta_k(\theta_j(\bar{\zeta}_1^{-1}),\theta_j(\bar{\zeta}_2^{-1}))}\frac{\bar{\omega}(\zeta^{-1},\zeta_1^{-1})}{\bar{\omega}(\zeta^{-1},\zeta_2^{-1})}, && \text{by (4.3) and (4.5)}, \\ &= \overline{\beta_k(\theta_j(\bar{\zeta}_1^{-1}),\theta_j(\bar{\zeta}_2^{-1}))}\frac{\zeta_2}{\zeta_1}F_2(\zeta;\zeta_1,\zeta_2), && \text{by (4.8)}. \end{aligned}\right\} \tag{5.9}$$

Thus, $F_2(\zeta;\zeta_1,\zeta_2)$ has constant argument on each of the circles $\{C_k|k=0,1,\ldots,M\}$. ■

Proposition 5.3. *Let ζ_1 and ζ_2 be any two distinct ordinary points of a given Schottky group. Then the function*

$$F_3(\zeta;\zeta_1,\zeta_2) \equiv \frac{\omega(\zeta,\zeta_1)\omega(\zeta,\bar{\zeta}_1^{-1})}{\omega(\zeta,\zeta_2)\omega(\zeta,\bar{\zeta}_2^{-1})}, \tag{5.10}$$

has constant argument on each of the circles $\{C_k|k=0,1,\ldots,M\}$.

Proof. The proof is analogous to those of the previous two propositions. ■

6. Conformal mapping to circular slit domain

Koebe (1914) established the existence of a conformal mapping from a given multiply connected region to a circular disc contained concentric circular arc slits (see also Schiffer 1950; Nehari 1952) but gives no explicit formulae. It turns out that the conformal mapping from the circular domain D_ζ in the ζ-plane to the circular-slit domain D_η in the η-plane can be constructed using the Schottky–Klein prime function. The author has not found an explicit construction of this particular mapping (from a circular domain to a circular-slit domain) anywhere in the literature so what follows appears to be a subsidiary new result of the present paper.

Suppose that the point α in the domain D_ζ is to map to $\eta=0$. Provided that the correspondence between no other points between the ζ and z-planes has been specified, one is free to arbitrarily specify the value of α. The conformal map $\eta(\zeta)$ taking the circular domain D_ζ to the circular-slit domain D_η is given by

$$\eta(\zeta) = \frac{\omega(\zeta, \alpha)}{|\alpha|\omega(\zeta, \bar{\alpha}^{-1})}. \tag{6.1}$$

First, let us verify that $\eta(\zeta)$ has constant modulus on all the circles $\{C_j | j=0,1,\ldots,M\}$. For points ζ on C_0,

$$\left.\begin{aligned} \overline{\eta(\zeta)} &= \frac{1}{|\alpha|}\frac{\bar{\omega}(\zeta^{-1}, \bar{\alpha})}{\bar{\omega}(\zeta^{-1}, \alpha^{-1})}, \\ &= |\alpha|\frac{\omega(\zeta, \bar{\alpha}^{-1})}{\omega(\zeta, \alpha)}, \qquad \text{by (4.8)}, \\ &= \frac{1}{\eta(\zeta)}, \end{aligned}\right\} \tag{6.2}$$

where we have used the fact that $\bar{\zeta}=\zeta^{-1}$ on C_0. Thus the image of C_0 lies on the unit circle L_0 in the η-plane.

On the other hand, for points ζ on any one of the interior circles $\{C_j | j=1,\ldots,M\}$,

$$\left.\begin{aligned} \overline{\eta(\zeta)} &= \frac{1}{|\alpha|}\frac{\bar{\omega}(\phi_j(\zeta), \bar{\alpha})}{\bar{\omega}(\phi_j(\zeta), \alpha^{-1})}, \qquad \text{by (3.4)}, \\ &= \frac{1}{|\alpha|}\frac{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}), \bar{\alpha})}{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}), \alpha^{-1})}, \quad \text{by (3.5)}. \end{aligned}\right\} \tag{6.3}$$

But now the transformation property (4.4) can be used to deduce that

$$\left.\begin{aligned} \overline{\eta(\zeta)} &= \frac{1}{|\alpha|}\overline{\beta_j(\alpha, \bar{\alpha}^{-1})}\frac{\bar{\omega}(\zeta^{-1}, \bar{\alpha})}{\bar{\omega}(\zeta^{-1}, \alpha^{-1})}, \quad \text{by (4.4)}, \\ &= \overline{\beta_j(\alpha, \bar{\alpha}^{-1})}|\alpha|\frac{\omega(\zeta, \bar{\alpha}^{-1})}{\omega(\zeta, \alpha)}, \qquad \text{by (4.8)}, \\ &= \overline{\beta_j(\alpha, \bar{\alpha}^{-1})}\frac{1}{\eta(\zeta)}. \end{aligned}\right\} \tag{6.4}$$

(6.4) immediately implies that, on C_j,

$$|\eta(\zeta)|^2 = \overline{\beta_j(\alpha, \bar{\alpha}^{-1})}. \tag{6.5}$$

On use of (4.5), a formula for $\beta_j(\alpha, \bar{\alpha}^{-1})$ is

$$\beta_j(\alpha, \bar{\alpha}^{-1}) = \prod_{\theta_k \in \Theta_j} \frac{(\alpha - \theta_k(B_j))(\bar{\alpha}^{-1} - \theta_k(A_j))}{(\alpha - \theta_k(A_j))(\bar{\alpha}^{-1} - \theta_k(B_j))}. \tag{6.6}$$

It is clear from (6.5) that the quantities $\beta_j(\alpha, \bar{\alpha}^{-1})$ must be real and positive, but this is not immediately apparent from the formula (6.6). This is shown explicitly in an appendix of Crowdy & Marshall (2005).

We now briefly outline why C_0 maps to the whole of L_0 while the circles $\{C_j | j = 1, \ldots, M\}$ map to finite-length circular slits. Since $\eta(\zeta)$ is meromorphic in D_ζ, by the argument principle, since $\eta(\zeta)$ has just one simple zero in D_ζ then

$$1 = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_\zeta} \frac{\eta_\zeta}{\eta} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} [\log \eta(\zeta)]_{\partial D_\zeta}, \tag{6.7}$$

where ∂D_ζ denotes the directed boundary of D_ζ (i.e. C_0 traversed in an anticlockwise direction and all the $\{C_j | j = 1, \ldots, M\}$ traversed in a clockwise direction) and the square brackets denote the change in value of the function enclosed in the brackets on making a single traversal of this boundary. But a natural way to pick a branch of the function $\log \eta(\zeta)$ is to join the logarithmic singularities at α and $\bar{\alpha}^{-1}$ with a branch cut in the fundamental region and to similarly join all pairs of equivalent points in all equivalent regions. Then $\log \eta(\zeta)$ does not change value on traversing any of the C_j for $j = 1, \ldots, M$ since C_j does not cross any of these branch cuts. This means that the circles $\{C_j | j = 1, \ldots, M\}$ map to circular slits in the η-plane because the change in argument of $\eta(\zeta)$ is zero on traversing $\{C_j | j = 1, \ldots, M\}$ (so the image in the η-plane does not encircle the origin). The same is not true of C_0 owing to the presence of the branch cut joining α and $\bar{\alpha}^{-1}$. Since the integral around all the enclosed circles is zero, it can be deduced from (6.7) that the change in argument of $\eta(\zeta)$ around C_0 is precisely 2π. This means that the image of C_0 under the map $\eta(\zeta)$ encircles $\eta = 0$ exactly once so that the image is the entire unit circle L_0.

The function $\eta(\zeta)$ in (6.1) is an explicit construction of a function $f(\zeta; \alpha)$ introduced in a more abstract setting in eqn (A1.21) of Schiffer (1950). There, it is constructed from the first-type Green's function of some given domain and it is proven that this function conformally maps that given domain univalently on to the interior of a unit circle slit along M circular arcs around the origin. Here, we have essentially chosen the given domain to be D_ζ and explicitly constructed the relevant $f(\zeta; \alpha)$ (here, the function $\eta(\zeta)$).

Finally, the zeros, $\{\gamma_1^{(j)}, \gamma_2^{(j)} | j = 1, \ldots, M\}$, of η_ζ are purely functions of the chosen α and the conformal moduli $\{q_j, \delta_j | j = 1, \ldots, M\}$.

7. Properties of the S–C mapping function

Let the conformal map $z(\eta)$ map the circular-slit domain D_η to the bounded polygonal region D_z. In this section, the properties required of this function will be outlined.

First, by definition, $z(\eta)$ must be an analytic function everywhere inside D_η. Furthermore, it must have branch point singularities on the circular arcs $\{L_j | j=0,1,\ldots,M\}$. Define the *prevertices* (Driscoll & Trefethen 2002) in the ζ-plane to be the points

$$\{a_k^{(0)} | k = 1, \ldots, n_0\} \tag{7.1}$$

on C_0, and the points

$$\{a_k^{(j)} | k = 1, \ldots, n_j\} \tag{7.2}$$

on each of the enclosed circles $\{C_j | j=1,\ldots,M\}$.

Now let the image of the point $a_k^{(j)}$ under the mapping $\eta(\zeta)$ be $\tilde{a}_k^{(j)}$ so that

$$\tilde{a}_k^{(j)} = \eta(a_k^{(j)}). \tag{7.3}$$

Then, locally, we must have

$$z_\eta(\tilde{a}_k^{(j)}) = (\eta - \tilde{a}_k^{(j)})^{\beta_k^{(j)}} f(\eta), \tag{7.4}$$

where $f(\eta)$ is some function that is analytic at $\eta = \tilde{a}_k^{(j)}$. Equivalently, if considering the composed function $z(\zeta) = z(\eta(\zeta))$, we must have

$$z_\zeta(a_k^{(j)}) = (\zeta - a_k^{(j)})^{\beta_k^{(j)}} g(\zeta), \tag{7.5}$$

where $g(\zeta)$ is some function that is analytic at $\zeta = a_k^{(j)}$. Except for these branch point singularities, the mapping must be analytic at all other points on the circular arcs.

Next, in order that the segments of the circular arcs $\{L_j | j=0,1,\ldots,M\}$ between these branch point singularities map to straight-line segments in the z-plane, the mapping function must satisfy the property that the quantity

$$\eta z_\eta(\eta) \tag{7.6}$$

has piecewise-constant argument on all the circular arcs $\{L_j | j=0,1,\ldots,M\}$. To see this, consider the kth line segment of the polygon P_j in the z-plane. On this line, it is known from (2.3) that

$$\bar{z} = \epsilon_k^{(j)} z + \kappa_k^{(j)}, \tag{7.7}$$

for some constants $\epsilon_k^{(j)}$ and $\kappa_k^{(j)}$. This means that, on the portion of the circular arc L_j mapping to this line segment, differentiation with respect to z means that we must have

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}z}{\mathrm{d}\eta}\right)^{-1} = \epsilon_k^{(j)}. \tag{7.8}$$

But, on this portion of L_j, we also have

$$\bar{z} = \overline{z(\eta)} = \bar{z}(\bar{\eta}) = \bar{z}(r_j^2 \eta^{-1}). \tag{7.9}$$

Therefore, (7.8) becomes

$$\frac{r_j^2\eta^{-1}\bar{z}_\eta(r_j^2\eta^{-1})}{\eta z_\eta(\eta)} = -\epsilon_k^{(j)}. \tag{7.10}$$

Since $\bar{\eta} = r_j^2\eta^{-1}$ on L_j, (7.10) is equivalent to

$$\overline{\eta z_\eta(\eta)} = -\eta z_\eta(\eta)\epsilon_k^{(j)} \tag{7.11}$$

on C_j. Thus, on this portion of L_j, $\eta z_\eta(\eta)$ has a constant argument. Clearly, ηz_η must have piecewise constant argument on all segments of each of the arcs $\{L_j|j=0,1,\ldots,M\}$.

Similar arguments reveal that the equivalent conditions in the original ζ-plane are that, on C_j, the quantity

$$(\zeta - \delta_j)z_\zeta(\zeta) \tag{7.12}$$

must have piecewise constant argument. A difficulty then arises because it is *different* functions of ζ which must have piecewise constant argument on the various circles in the ζ-plane. The transformation to the η-plane ensures at least that it is the same function of η, i.e. the function ηz_η, which must have piecewise constant argument on *all* the circular arcs in the η-plane.

8. Construction of the S–C mapping function

The conformal mapping $z(\zeta)$ from D_ζ to D_z will now be constructed. First, pick an arbitrary point γ_j on each of the circles $\{C_j|j=0,\ldots,M\}$. It is required to construct a mapping from D_η to D_z satisfying the condition that $\eta z_\eta(\eta)$ has piecewise constant argument on the segments of the circular arcs $\{L_j|j=0,\ldots,M\}$ between the prevertices $\{\tilde{a}_k^{(j)}\}$. But this is equivalent to the condition that ηz_η has piecewise constant argument on the segments of the original circles $\{C_j|j=0,\ldots,M\}$ between the prevertices $\{a_k^{(j)}\}$. We must also ensure that ηz_η has the requisite branch point singularities on these circles in the ζ-plane.

Consider the function

$$\prod_{k=1}^{n_0}(F_1(\zeta;a_k^{(0)},\gamma_0)^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}(F_2(\zeta;a_k^{(j)},\gamma_j))^{\beta_k^{(j)}}. \tag{8.1}$$

Since this function is a product of various powers of the special functions F_1 and F_2 considered in propositions 5.1 and 5.2 then it will have piecewise constant argument on all the circles $\{C_j|j=0,\ldots,M\}$. It also has the correct branch point singularities at the points $\{a_k^{(j)}|k=1,\ldots,n_j\}$. However, in addition to the required branch points, on use of the two relations (2.2) it is easy to check that this function also has a second-order zero at γ_0 and M second order poles at the points $\{\gamma_j|j=1,\ldots,M\}$.

Now multiply (8.1) by the quantity

$$\prod_{j=1}^{M}F_2(\zeta;\gamma_j,\gamma_1^{(j)})F_2(\zeta;\gamma_j,\gamma_2^{(j)}), \tag{8.2}$$

which can be seen to have a second-order zero at the M points $\{\gamma_j | j=1, \ldots, M\}$ and simple poles at the $2M$ points $\{\gamma_1^{(j)}, \gamma_2^{(j)} | j=1, \ldots, M\}$. Multiplying (8.1) by this function therefore has the effect of shifting the M second order poles at the arbitrarily chosen points $\{\gamma_j | j=1, \ldots, M\}$ to produce instead $2M$ simple poles of the function at the points $\{\gamma_1^{(j)}, \gamma_2^{(j)} | j=1, \ldots, M\}$. Recall from (2.5) that the latter set of points are precisely the positions of the zeros of the conformal mapping $\eta_\zeta(\zeta)$—a fact that will be useful in what follows. Note crucially that since (8.2) is a product of F_2-functions, we have effected this shift in the poles of the function without affecting the important property that it has piecewise constant argument on the circles $\{C_j | j=0, \ldots, M\}$.

The new modified function, which can be written

$$\prod_{k=1}^{n_0} (F_1(\zeta; a_k^{(0)}, \gamma_0))^{\beta_k^{(0)}} \prod_{j=1}^{M} F_2(\zeta; \gamma_j, \gamma_1^{(j)}) F_2(\zeta; \gamma_j, \gamma_2^{(j)}) \prod_{k=1}^{n_j} (F_2(\zeta; a_k^{(j)}, \gamma_j))^{\beta_k^{(j)}}, \quad (8.3)$$

is now multiplied by a second function given by

$$F_3(\zeta; \alpha, \gamma_0). \quad (8.4)$$

This removes the second-order zero at the arbitrarily chosen point γ_0 and replaces it with two first-order zeros at the points α and $\bar{\alpha}^{-1}$. Recall that α is the point in the ζ-plane which maps to $\eta=0$ in D_η. Again, because (8.4) is one of the F_3-functions introduced in proposition 5.3, this shift in the zeros of the function has been effected without altering the property that it has piecewise constant argument on the circles $\{C_j | j=0, \ldots, M\}$. The new function can now be written as

$$F_3(\zeta; \alpha, \gamma_0) \prod_{k=1}^{n_0} (F_1(\zeta; a_k^{(0)}, \gamma_0)^{\beta_k^{(0)}} \prod_{j=1}^{M} F_2(\zeta; \gamma_j, \gamma_1^{(j)}) F_2(\zeta; \gamma_j, \gamma_2^{(j)}) \prod_{k=1}^{n_j} (F_2(\zeta; a_k^{(j)}, \gamma_j))^{\beta_k^{(j)}}. \quad (8.5)$$

This representation as a product of the functions F_1, F_2 and F_3 highlights the fact that it has piecewise constant argument on the circles $\{C_j | j=0, \ldots, M\}$. However, it can be rewritten, after cancellations, as

$$U(\eta) \equiv \frac{\omega(\zeta, \alpha)\omega(\zeta, \bar{\alpha}^{-1})}{\prod_{j=1}^{M} \omega(\zeta, \gamma_1^{(j)})\omega(\zeta, \gamma_2^{(j)})} \prod_{k=1}^{n_0} [\omega(\zeta, a_k^{(0)})]^{\beta_k^{(0)}} \prod_{j=1}^{M} \prod_{k=1}^{n_j} [\omega(\zeta, a_k^{(j)})]^{\beta_k^{(j)}}, \quad (8.6)$$

an equation which defines the function $U(\eta)$. Consider now the function

$$V(\eta) = \frac{U(\eta)}{\eta z_\eta}, \quad (8.7)$$

where z_η is the derivative of the S–C mapping we are seeking. First, note that $V(\eta)$ is analytic everywhere inside and on the unit η-circle L_0. This is because both $U(\eta)$ and ηz_η have the same branch point singularities at the prevertices $\{\tilde{a}_j^{(k)}\}$ on $\{L_j | j=0, 1, \ldots, M\}$ and so these cancel in the quotient. Note also that the zero of the denominator at $\eta=0$ is removed by the zero of $U(\zeta)$ at $\zeta=\alpha$. Further, by the construction of $U(\eta)$, both $U(\eta)$ and ηz_η have piecewise constant argument on each segment between the branch points on L_0 and have the same changes in argument on passing through the branch points $\{a_j^{(0)}|j=1,\dots,n_0\}$. From this we can deduce that, everywhere on L_0, the argument of $V(\eta)$ is a constant. Equivalently,

$$\overline{V(\eta)} = \epsilon V(\eta), \quad \text{on } L_0, \tag{8.8}$$

for some complex constant ϵ. But (8.8) can be written

$$\bar{V}(\eta^{-1}) = \epsilon V(\eta), \tag{8.9}$$

which furnishes the analytic continuation of $V(\eta)$ to the exterior of L_0. In particular, $V(\eta)$ is seen to be analytic everywhere outside the unit η-circle L_0 and is bounded at infinity. $V(\eta)$ is therefore an entire function bounded at infinity and, by Liouville's theorem, is necessarily a constant. It can be concluded that

$$\eta z_\eta(\eta) = \tilde{B}\frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})}\prod_{k=1}^{n_0}[\omega(\zeta,a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta,a_k^{(j)})]^{\beta_k^{(j)}}, \tag{8.10}$$

where $\tilde{B}$ is some complex constant. But, by the chain rule,

$$\eta z_\eta(\eta) = \eta(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\eta}, \tag{8.11}$$

which implies the following expression for $\mathrm{d}z/\mathrm{d}\zeta$:

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{\tilde{B}}{\eta(\zeta)}\frac{\mathrm{d}\eta(\zeta)}{\mathrm{d}\zeta}\frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})}\prod_{k=1}^{n_0}[\omega(\zeta,a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta,a_k^{(j)})]^{\beta_k^{(j)}}. \tag{8.12}$$

To check the consistency of the formula, first note that the pole of the right-hand side at $\zeta=\alpha$ (arising because $\eta(\zeta)$ vanishes there) is a removable pole owing to the presence of $\omega(\zeta,\alpha)$ in the numerator. Second, the simple zeros of $\mathrm{d}\eta/\mathrm{d}\zeta$ at the points $\{\gamma_1^{(j)}\gamma_2^{(j)}|j=1,\dots,M\}$ (cf. §2) do not produce unwanted zeros of $\mathrm{d}z/\mathrm{d}\zeta$ at these points since they are exactly cancelled by the simple zeros appearing in the denominator. By direct calculation based on the formula (6.1), we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}\zeta} = \frac{1}{|\alpha|}\left(\frac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})^2}\right). \tag{8.13}$$

On substitution into (8.12),

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = BS(\zeta)\prod_{k=1}^{n_0}[\omega(\zeta,a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta,a_k^{(j)})]^{\beta_k^{(j)}}, \tag{8.14}$$

where

$$S(\zeta) \equiv \left(\frac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})}\right), \tag{8.15}$$

and B is some constant. Equation (8.14) is the required formula for $\mathrm{d}z/\mathrm{d}\zeta$. On integration of (8.14) with respect to ζ, the final formula (1.1) is obtained, with A being a constant of integration.

(*a*) *The simply connected case*

In the case of a simply connected domain there are no enclosed circles and hence no non-trivial generating Möbius maps. The Schottky group is therefore the trivial group and the associated Schottky–Klein prime function is just

$$\omega(\zeta,\gamma) = (\zeta-\gamma). \tag{8.16}$$

Moreover, $S(\zeta)$ reduces to a constant in this case. In turn, (1.1) reduces to the well-known S–C mapping from the unit disc (Driscoll & Trefethen 2002).

(*b*) *The doubly connected case*

Without loss of generality, any doubly connected domain can be obtained by a conformal mapping from some annulus $q<|\zeta|<1$ in a parametric ζ-plane where the value of the parameter q is determined by the image domain. In this case, $\delta_1=0$ and $q_1=q$, so that the single Möbius map given by (3.5) is

$$\theta_1(\zeta) = q^2\zeta. \tag{8.17}$$

The Schottky group in this case has just one generator and is sometimes referred to as the *loxodromic group*. Clearly, its elements are $\{\theta_1^j | j\in\mathbb{Z}\}$. The associated Schottky–Klein prime function is

$$\omega(\zeta,\gamma) = -\frac{\gamma}{C^2}P(\zeta/\gamma,q), \tag{8.18}$$

where

$$P(\zeta,q) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)(1-q^{2k}\zeta^{-1}) \tag{8.19}$$

and

$$C \equiv \prod_{k=1}^{\infty}(1-q^{2k}). \tag{8.20}$$

The transformation properties of $P(\zeta,q)$ corresponding to (4.4) and (4.8), respectively, are

$$\left.\begin{aligned} \frac{P(q^2\zeta\gamma_1^{-1},q)}{P(q^2\zeta\gamma_2^{-1},q)} &= \frac{\gamma_1}{\gamma_2}\frac{P(\zeta\gamma_1^{-1},q)}{P(\zeta\gamma_2^{-1},q)}, \\ P(\zeta^{-1},q) &= -\zeta^{-1}P(\zeta,q). \end{aligned}\right\} \tag{8.21}$$

It can also be shown directly from the infinite product definition that

$$P(q^2\zeta,q) = -\zeta^{-1}P(\zeta,q). \tag{8.22}$$

By using a rotational degree of freedom in the mapping function we can assume, without loss of generality, that the point α mapping to $\eta=0$ is real. Then, on use of the relations

$$\left.\begin{aligned}\omega(\zeta,\alpha) &= -\alpha C^{-2}P(\zeta\alpha^{-1},q), \quad \omega(\zeta,\alpha^{-1}) = -\alpha^{-1}C^{-2}P(\zeta\alpha,q),\\ \omega_\zeta(\zeta,\alpha) &= -C^{-2}P_\zeta(\zeta\alpha^{-1},q), \quad \omega_\zeta(\zeta,\alpha^{-1}) = -C^{-2}P_\zeta(\zeta\alpha,q),\end{aligned}\right\} \tag{8.23}$$

$S(\zeta)$ then takes the form

$$S(\zeta) = \frac{\alpha^{-1}P_\zeta(\zeta\alpha^{-1},q)P(\zeta\alpha,q) - \alpha P_\zeta(\zeta\alpha,q)P(\zeta\alpha^{-1},q)}{\gamma_1\gamma_2 P(\zeta\gamma_1^{-1},q)P(\zeta\gamma_2^{-1},q)}. \tag{8.24}$$

Consider now the function $T(\zeta,q)\equiv\zeta^2 S(\zeta)$, which, after some rearrangement, can be written

$$T(\zeta,q) = \zeta\left(\frac{K(\zeta\alpha^{-1},q) - K(\zeta\alpha,q)}{L(\zeta,q)}\right), \tag{8.25}$$

where

$$K(\zeta,q) = \zeta\frac{P_\zeta(\zeta,q)}{P(\zeta,q)}, \quad L(\zeta,q) = \frac{P(\zeta\gamma_1^{-1},q)P(\zeta\gamma_2^{-1},q)}{P(\zeta\alpha^{-1},q)P(\zeta\alpha,q)}. \tag{8.26}$$

Note first that $T(\zeta,q)$ is a meromorphic function everywhere except at $\zeta=0$ or ∞, which are singular points of the loxodromic group. It is easy to make use of (8.21) and (8.22) to verify that

$$K(q^2\zeta,q) = K(\zeta,q) - 1, \quad L(q^2\zeta,q) = \gamma_1\gamma_2 L(\zeta,q). \tag{8.27}$$

On use of (8.27), it follows that

$$T(q^2\zeta,q) = \frac{q^2}{\gamma_1\gamma_2}T(\zeta,q). \tag{8.28}$$

But, by the choice of taking α to be real, the mapping $\eta(\zeta)$ satisfies

$$\bar{\eta}(\zeta) = \eta(\zeta), \quad \bar{\eta}_\zeta(\zeta) = \eta_\zeta(\zeta). \tag{8.29}$$

Therefore, if γ_1 is a zero of η_ζ so that $\eta_\zeta(\gamma_1)=0$, then

$$\overline{\eta_\zeta(\gamma_1)} = \bar{\eta}_\zeta(\bar{\gamma}_1) = \eta_\zeta(\bar{\gamma}_1) = 0. \tag{8.30}$$

Thus, $\bar{\gamma}_1$ is also a zero of η_ζ so that $\gamma_2=\bar{\gamma}_1$. Also, since γ_1 is on the circle $|\zeta|=q$, it is clear that $\gamma_1\gamma_2=\gamma_1\bar{\gamma}_1=q^2$. It follows from (8.28) that

$$T(q^2\zeta,q) = T(\zeta,q). \tag{8.31}$$

Meromorphic functions satisfying the functional equation (8.31) are known as *loxodromic functions* (Valiron 1947). (Alternatively, using a simple logarithmic transformation, the analysis here can be rephrased in terms of the more familiar elliptic functions.) The fundamental region for this group is the annulus $q<|\zeta|<q^{-1}$. Note that $T(\zeta,q)$ is analytic everywhere in this fundamental region. It follows from Liouville's theorem for loxodromic functions (Valiron 1947) that

$T(\zeta, q)$ must be a constant function. Thus, (8.14) produces the result

$$z_\zeta = \frac{\tilde{B}}{\zeta^2} \prod_{k=1}^{n_0} [P(\zeta/a_k^{(0)}, q)]^{\beta_k^{(0)}} \prod_{k=1}^{n_1} [P(\zeta/a_k^{(1)}, q)]^{\beta_k^{(1)}}, \tag{8.32}$$

for some constant $\tilde{B}$. On further use of the transformation property (8.22), as well as (2.2), this can be rewritten as

$$z_\zeta = B \prod_{k=1}^{n_0} [P(\zeta/a_k^{(0)}, q)]^{\beta_k^{(0)}} \prod_{k=1}^{n_1} [P(q^2\zeta/a_k^{(1)}, q)]^{\beta_k^{(1)}}, \tag{8.33}$$

for some constant B. The formula given in DeLillo *et al.* (2001) and Driscoll & Trefethen (2002) is

$$z(\zeta) = A + B \int^{\zeta} \prod_{k=1}^{m} \left[\Theta\left(\frac{\zeta'}{q z_{0,k}} \right) \right]^{-\beta_{0,k}} \prod_{k=1}^{n} \left[\Theta\left(\frac{q\zeta'}{z_{1,k}} \right) \right]^{\beta_{1,k}} \mathrm{d}\zeta', \tag{8.34}$$

where

$$\Theta(\zeta) \equiv \prod_{k=0}^{\infty} (1 - q^{2k+1}\zeta)(1 - q^{2k+1}\zeta^{-1}), \tag{8.35}$$

and where A and B are constants. It is easy to check from (8.19) and (8.35) that

$$\Theta(\zeta q^{-1}) = P(\zeta, q). \tag{8.36}$$

Finally, on use of (8.36), the integral of (8.33) is seen to be identical to (8.34) with appropriate identification of the respective notations. This provides an important check on the validity of the new formula (1.1). It also provides a new derivation of the S–C mapping to doubly connected polygonal domains.

9. Analysis of a triply connected example

Consider the construction of S–C mappings to simply connected polygonal regions. While the functional form of the mapping function is known up to the specification of a finite set of parameters, the actual construction of the map to any given target polygonal region requires the solution of a *parameter problem* to determine the location of the pre-vertices in the ζ-plane. Chapter 1 of the monograph by Driscoll & Trefethen (2002) provides an instructive general discussion of this parameter problem. For multiply connected domains, there is still a parameter problem to solve, but now there are additional parameters—the conformal moduli $\{q_j, \delta_j | j = 1, \ldots, M\}$ in the pre-image ζ-plane—which must also be determined. The parameter problem has been well-studied in the case of simply connected domains, and even for doubly connected domains (Dappen 1987, 1988; Hu 1998), and much work remains to be done to find effective numerical methods to solve the parameter problem in the multiply connected case now that general formulae for the mapping have been found.

To check the validity and viability of formula (1.1) we now present a triply connected example which negotiates the parameter problem in the simplest way. The target domain is chosen to have sufficient geometrical symmetry that the

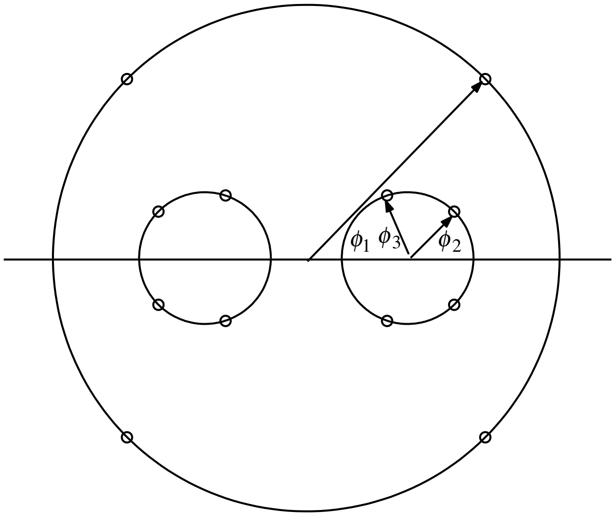

Figure 4. Definition sketch of the parameters ϕ_1, ϕ_2 and ϕ_3 in the ζ-plane for the triply connected example. The circles denote the positions of the prevertices.

complications of solving the parameter problem are significantly reduced. This is not a limitation of the derived formula; more complicated non-symmetric domains can also be constructed on solution of the relevant parameter problem.

Consider a triply connected target region consisting of an outer rectangle, centred at the origin $z=0$, with two equal enclosed rectangles excised. The domain is taken to be reflectionally symmetric about both the real and imaginary axes in the z-plane. A conformal mapping from a conformally equivalent triply connected circular domain to this target domain will be constructed based on (1.1).

The parameter A can be chosen so that the origin is correctly placed (i.e. $z(0)=0$), while B can be thought of as governing the area of the outer rectangle. Here we chose B so that the outer rectangle extends horizontally between ± 1. By the symmetry of the target domain, we expect the prevertices on the unit ζ-circle to be symmetrically disposed. We therefore take them to be at

$$e^{\pm i\phi_1}, \quad e^{\pm i(\pi-\phi_1)}, \tag{9.1}$$

where ϕ_1 is an adjustable real parameter. Figure 4 shows a schematic illustrating ϕ_1 as the argument of the branch point on C_0. It can be thought of as governing the aspect ratio (or height) of the outer rectangle.

With regard to the enclosed polygons, by the symmetry we expect

$$q_1 = q_2 = q, \quad \delta_1 = -\delta_2 = \delta, \tag{9.2}$$

where δ is taken to be real so that C_1 and C_2 are centred on the real axis. The two real parameters q and δ will be picked arbitrarily. This can be thought of as specifying the centre and area of the two symmetrically enclosed rectangles.

It is easy to deduce from the interior angles of the polygonal region that we must take

$$\beta_k^{(0)} = -\tfrac{1}{2}, \ \ k=1,2,3,4, \qquad \beta_k^{(j)} = \tfrac{1}{2}, \ \ k=1,2,3,4, \quad \text{and} \quad j=1,2. \tag{9.3}$$

We also make the choice $\alpha=0$. In this case, the formula (6.1) is not well defined and must be replaced by the formula

$$\eta(\zeta) = \frac{\omega(\zeta,0)}{\omega(\zeta,\infty)} = \frac{\zeta\omega'(\zeta,0)}{\omega'(\zeta,\infty)}, \tag{9.4}$$

which is the appropriate limit of (6.1) as $\alpha\to 0$. With α, q and δ now specified, the values of $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$ for $j=1$, 2 can now be determined. This is done using a simple one-dimensional Newton iteration on the argument, $\hat{\phi}$ say, of the point $\gamma_1^{(1)}$ relative to the point δ. The equation to be solved is that η_ζ as given by the derivative of (9.4) vanishes at $\zeta=\delta+q\,\mathrm{e}^{\mathrm{i}\hat{\phi}}$. Then it follows from the symmetry that

$$\begin{aligned} \gamma_1^{(1)} &= \delta + q\,\mathrm{e}^{\mathrm{i}\hat{\phi}}, & \gamma_2^{(1)} &= \delta + q\,\mathrm{e}^{-\mathrm{i}\hat{\phi}}, \\ \gamma_1^{(2)} &= -\delta + q\,\mathrm{e}^{\mathrm{i}(\pi-\hat{\phi})}, & \gamma_2^{(2)} &= -\delta + q\,\mathrm{e}^{-\mathrm{i}(\pi-\hat{\phi})}. \end{aligned} \tag{9.5}$$

With q and δ specified so that the area and centre of the enclosed rectangles have essentially been set, one expects there to remain only a single real degree of freedom associated with each of the enclosed rectangles. This degree of freedom can be thought of as governing the aspect ratio of these rectangles. By the symmetry, we therefore take the prevertices on C_1 to be at

$$\delta + q\,\mathrm{e}^{\pm\mathrm{i}\phi_2}, \quad \delta + q\,\mathrm{e}^{\pm\mathrm{i}(\pi-\phi_3)}, \tag{9.6}$$

where ϕ_2 and ϕ_3 are real parameters. Only one of these two parameters, ϕ_2 say, should be freely specifiable. The value of ϕ_3 is then determined by the condition that the lengths of the sides of the image polygon should be such that the polygon closes. By the symmetry, we expect the pre-vertices on C_2 to be at

$$-\delta + q\,\mathrm{e}^{\pm\mathrm{i}\phi_3}, \quad -\delta + q\,\mathrm{e}^{\pm\mathrm{i}(\pi-\phi_2)}. \tag{9.7}$$

Figure 4 shows a schematic. In this way, the parameter problem reduces to that of finding the value of the single parameter ϕ_3.

Figure 5 shows the ζ- and η-planes for $q=0.2$ and $\delta=0.5$ under the mapping (9.4). Figure 6 shows triply connected polygonal regions for different choices of ϕ_2 with $\phi_1=\pi/4$. When $\phi_2=0$, $\phi_3=0$ and the outer rectangle contain two horizontal slits sitting on the real axis. As ϕ_2 increases, these slits turn into rectangles with large aspect ratios. Eventually, it is found that there is a critical value of ϕ_2 at which ϕ_2 and ϕ_3 coalesce and sum to π. This means that the two square root branch points at ϕ_2 and ϕ_3 have coalesced to produce a simple zero on the interior circles C_1 and C_2, which results in the formation of a vertical slit centred on the real axis in the z-plane. As ϕ_2 increases from zero to this critical value, the aspect ratio of the enclosed rectangles decreases. Various configurations between the two limiting cases of horizontal and vertical enclosed slits are shown in figure 6. A graph of the solution of the parameter problem for ϕ_3 as a function of ϕ_2 is shown in figure 7. For comparison, a similar set of polygonal regions for the case of conformal moduli given by $q=0.15$ and $\delta=0.6$ are shown in figure 8.

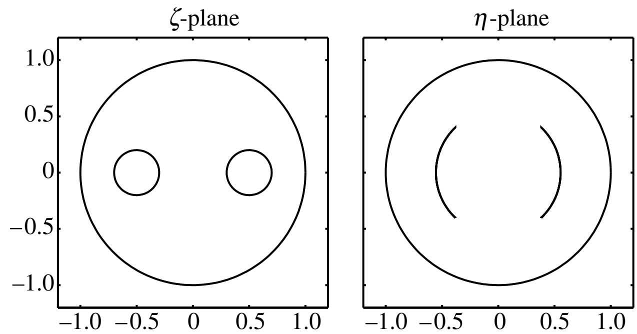

Figure 5. The ζ-plane and η-plane under the map (9.4) for parameter values $q=0.2$, $\delta=0.5$. The circles C_1 and C_2 are shown along with their images, L_1 and L_2, under the conformal mapping (9.4).

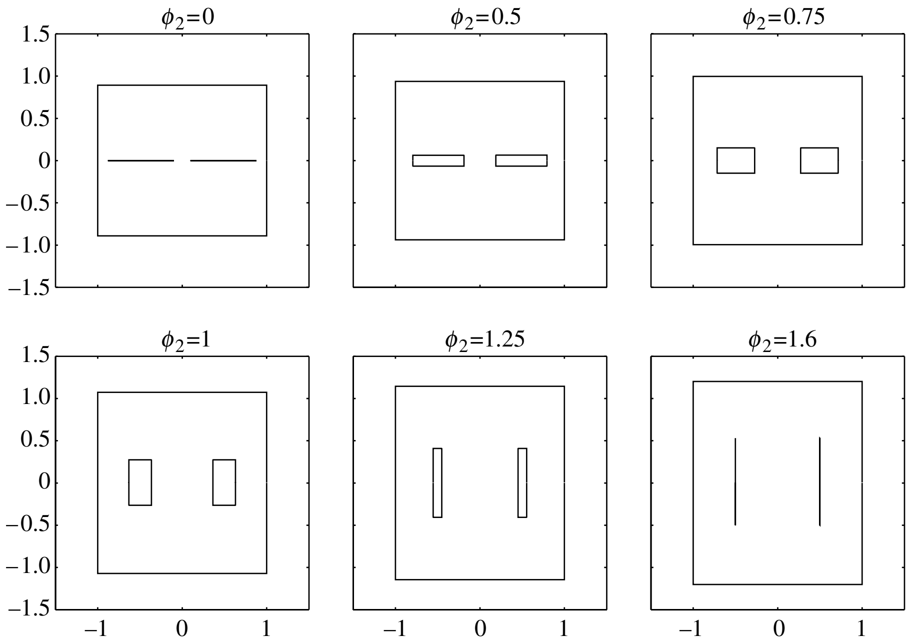

Figure 6. Typical polygons for $q=0.2$, $\delta=0.5$, $\phi_1=\pi/4$ and $\phi_2=0$, 0.5, 0.75, 1, 1.25 and 1.6. The interior polygons exhibit a gradual transition, through a sequence of rectangles of differing aspect ratios, from a horizontal slit to a vertical slit.

10. Discussion

By use of elements of classical function theory, the formula (1.1) for the S–C mapping from a bounded, multiply connected circular domain to a bounded, multiply connected polygonal domain has been constructed. It reduces to well-known formulae in the case of simply and doubly connected domains. Some example triply connected domains have been constructed to demonstrate the efficacy of the formula in practice. The formulation extends naturally to the case of unbounded polygonal domains and the details are presented elsewhere (Crowdy submitted).

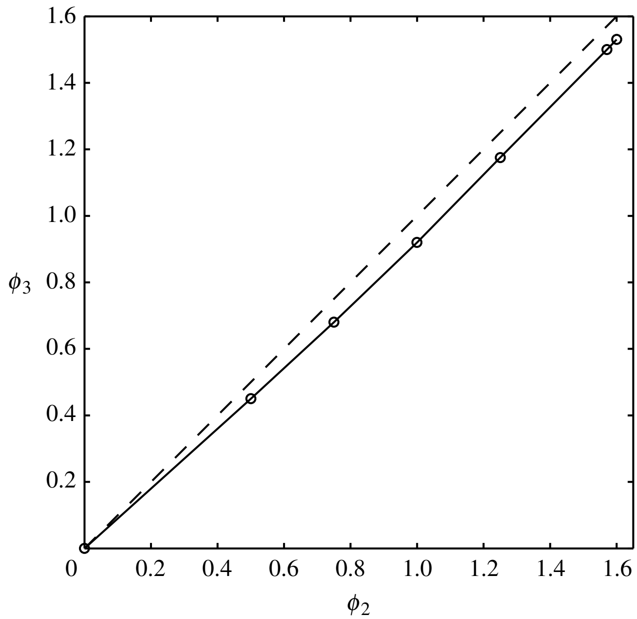

Figure 7. Graph of the solution of parameter problem for ϕ_3 as a function of ϕ_2 for $q=0.2$, $\delta=0.5$, $\phi_1=\pi/4$. The dashed line shows the graph $\phi_3=\phi_2$ and highlights the fact that the solution of the parameter problem for ϕ_3 is always found to be slightly less than ϕ_2.

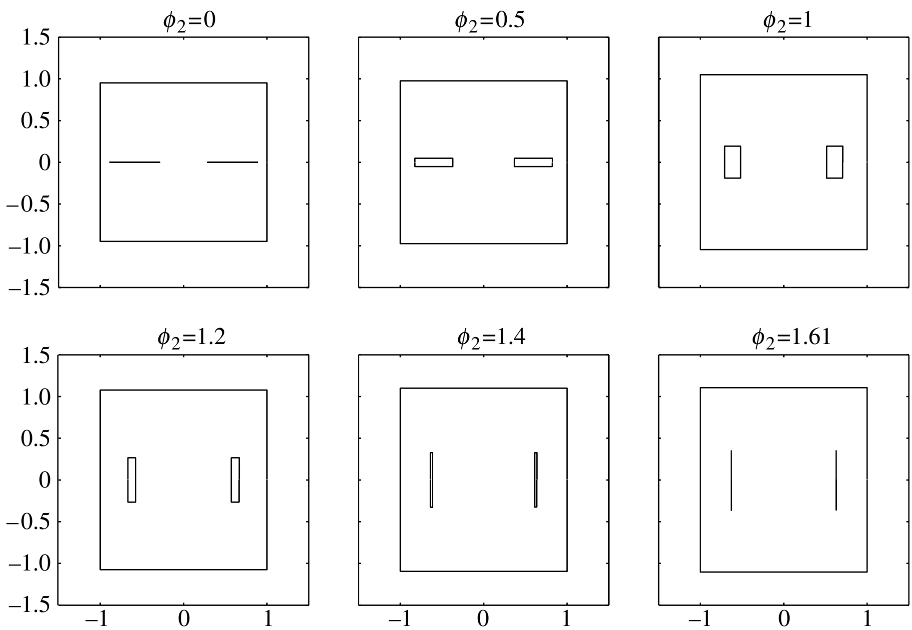

Figure 8. Typical polygons for $q=0.15$, $\delta=0.6$, $\phi_1=\pi/4$ and $\phi_2=0$, 0.5, 1, 1.2, 1.4 and 1.61. The interior polygons exhibit a gradual transition, through a sequence of rectangles of differing aspect ratios, from a horizontal slit to a vertical slit.

There have been a number of different derivations of the formula (8.34) for the S–C mapping of a doubly connected annulus. Akhiezer (1928; see also Akhiezer 1970) appeared to be the first to derive it using elements of elliptic function theory. Komatu (1945) independently derived it much later using the Villat integral representation for functions analytic in the annulus. The monograph by Henrici (1986) contains another derivation based on use of the argument principle. DeLillo *et al.* (2001) give yet another argument based on consideration of a pre-Schwarzian function and use of the maximum principle for harmonic functions. Also, Driscoll & Trefethen (2002) have provided a geometrical interpretation of the construction in DeLillo *et al.* (2001). In this paper, as a special case of a more general formula giving the S–C formula for multiply connected polygons of arbitrary finite connectivity, yet another construction of the doubly connected formula, distinct from all those just listed, has been presented. Here, the approach was motivated by recognition of the function $\Theta(\zeta)$ defined in (8.35) as the particular manifestation of a Schottky–Klein prime function relevant to the annulus. By considering prime functions relevant to more general circular domains, and by making use of an intermediate η-plane, the mapping formula (1.1) has been constructed in a natural way.

The consequence of all this is that it is now possible to write down, in a concise fashion, a general formula for the S–C mapping to a bounded polygonal domain in any finite connectivity. It is

$$z(\zeta) = A + B\int^{\zeta} S_M(\zeta')\prod_{k=1}^{n_0}[\omega(\zeta', a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta', a_k^{(j)})]^{\beta_k^{(j)}}\,\mathrm{d}\zeta', \tag{10.1}$$

where the second double product is not included, of course, if $M=0$. Only the definition of $S_M(\zeta)$ changes with the connectivity, specifically,

$$S_M(\zeta) = \left\{\begin{array}{ll} 1 & M=0, \\ \dfrac{1}{\zeta^2} & M=1, \\ \dfrac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})} & M\geq 2. \end{array}\right\} \tag{10.2}$$

It is expected that the function $S_M(\zeta)$ for $M\geq 2$ can be rewritten in a number of different ways. In particular, it should be possible to rewrite it so that there is ultimately no evidence of the intermediate η-plane used in the construction (note that (10.2) contains vestiges of the slit-mapping $\eta(\zeta)$ in the appearance of parameters α and $\{\gamma_1^{(j)},\gamma_2^{(j)}\}$). One way of rewriting this function has been found in Crowdy (submitted), but the subject requires further investigation. While any such re-expression of $S_M(\zeta)$ is desirable, it is no impediment to the direct implementation of formula (10.1) in practice.

A principal contribution of this paper is the association of the general S–C formula with the Schottky–Klein prime function. We believe this association to be significant, especially when it comes to optimizing the numerical implementation of the S–C mapping formula to multiply connected domains. This is because the Schottky–Klein prime function has intimate connections with the more commonly employed Riemann theta functions, which often have better convergence properties than the former. Baker (1995) cites explicit relations between the Schottky–Klein prime function and the Riemann theta function. Indeed, $\Theta(\zeta, q)$ defined in (8.35) is related to the first Jacobi theta function $\Theta_1(\tau, q)$ by

$$\Theta(\zeta q^{-1}, q) = -\frac{\mathrm{i}e^{-\tau/2}}{Cq^{1/4}}\Theta_1(\mathrm{i}\tau/2, q) \tag{10.3}$$

where $\tau = -\log\zeta$ and C is defined in (8.20) (Whittaker & Watson 1927). Just as Hu (1998) has found that different representations of the first Jacobi theta function can lead to improved convergence properties when performing a numerical implementation of the doubly connected mapping formula (8.34), similar benefits of convergence may be afforded by rewriting formula (10.1) in terms of Riemann theta functions. This is a subject of ongoing investigation. There are many interesting open questions to be answered concerning the numerical issues associated with the construction of multiply connected S–C mappings based on the formula (10.1).

The author acknowledges useful discussions with J. Marshall. The author also thanks Alan Elcrat for sending him a preprint of DeLillo *et al.* (2005).

References

Ablowitz, M. J. & Fokas, A. S. 1997 *Complex variables*. Cambridge: Cambridge University Press.

Akhiezer, N. I. 1928 Aerodynamical investigations. *Ukrain. Akad. Nauk. Trudi. Fiz.-Mat. Viddilu* **7**.

Akhiezer, N. I. 1970 *Elements of the theory of elliptic functions Translations of mathematical monographs*, 79. Providence: American Mathematical Society.

Baker, H. 1995 *Abelian functions*. Cambridge: Cambridge University Press.

Beardon, A. F. 1984 *A primer on Riemann surfaces London mathematical society lecture note series*, 78. Cambridge: Cambridge University Press.

Crowdy D. G. Submitted. Schwarz–Christoffel mappings to unbounded multiply connected polygonal regions. *Math. Proc. Camb. Phil. Soc.*

Crowdy, D. G. & Marshall, J. S. 2005 Analytical formulae for the Kirchhoff–Routh path function in multiply connected domains. *Proc. R. Soc. A* **461**, 2477–2501. (doi:10.1098/rspa.2005.1492.)

Dappen, H. 1987 Wind-tunnel wall corrections on a two-dimensional plate by conformal mapping. *AIAA J.* **25**, 1527–1530.

Dappen, H. 1988 Die Schwarz–Christoffel Abbildung fur zweifach zusammenhangende Gebiete mit Anwendungen. Ph.D thesis, ETH, Zurich.

DeLillo, T. K., Elcrat, A. R. & Pfaltzgraff, J. A. 2001 Schwarz–Christoffel mapping of the annulus. *SIAM Rev.* **43**, 469–477.

DeLillo, T. K., Elcrat, A. R. & Pfaltzgraff, J. A. 2005 Schwarz–Christoffel mapping of multiply-connected domains. *J. d'Analyse* **94**, 17–48.

Driscoll, T. & Trefethen, L. N. 2002 *Schwarz–Christoffel mapping Cambridge mathematical monographs*. Cambridge: Cambridge University Press.

Embree, M. & Trefethen, L. N. 1999 Green's functions for multiply connected domains via conformal mapping. *SIAM Rev.* **41**, 745–761.

Henrici, P. 1986 *Applied and computational complex analysis*, vol. 3. New York: Wiley Interscience.

Hu, C. 1998 Algorithm 785: a software package for computing Schwarz–Christoffel conformal transformations for doubly-connected polygonal regions. *ACM Trans. Math. Software* **24**, 317–333.

Koebe, P. 1914 Abhandlungen zur theorie der konformen abbildung. *Acta Math.* **41**, 305–344.
Komatu, Y. 1945 Darstellung der in einem Kreisringe analytischen Funktionem nebst den Anwendungen auf konforme Abbildung uber Polygonalringebiete. *Jpn. J. Math.* **19**, 203–215.
Mumford, D., Series, C. & Wright, D. 2002 *Indra's pearls.* Cambridge: Cambridge University Press.
Nehari, Z. 1952 *Conformal mapping.* New York: McGraw-Hill.
Schiffer, M. 1950 *Some recent developments in the theory of conformal mapping. Appendix to: R. Courant, Dirichlet's principle, conformal mapping and minimal surfaces.* New York: Interscience.
Valiron, G. 1947 *Cours d'Analyse Mathematique: Theorie des fonctions*, 2nd edn. Paris: Masson et Cie.
Whittaker, E. T. & Watson, G. N. 1927 *A course of modern analysis.* Cambridge: Cambridge University Press.