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The Benney hierarchy and the Dirichlet boundary problem in two dimensions

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Abstract

A theoretical connection between reductions of the Benney hierarchy and the Dirichlet problem for Laplace's equation in the plane is made. The connection is used to deduce general formulas for the uniformizations of two spectral functions associated with N-parameter reductions of the hierarchy. Two types of reduction are considered: one type has been considered by previous authors using alternative arguments, the second type is new. The formulas are general and are expressed in terms of the modified Green's function (for Laplace's equation) in arbitrary N-connected, reflectionally-symmetric, planar domains. The Benney moments are found to be purely geometrical quantities associated with these domains.

1. Introduction

A connection has recently been made between the universal Whitham hierarchy and the Dirichlet problem for the Laplace equation in two-dimensional planar domains [1,2]. This connection emerged by first making an association between conformal mappings and integrable hierarchies: in particular, a direct theoretical connection was made between the dispersionless Toda hierarchy and the Laplacian growth problem (or Hele–Shaw problem) in the simply-connected case [3,4]. It was then noticed that the problem of how the moments of a domain change under deformation of the domain exhibits an integrable structure—a result most easily understood from the Hadamard variational formula involving the (Laplacian) Green's function of the domain. Takhtajan [5] has also independently noticed this integrable structure of the Dirichlet problem and the significance of the associated Green's function. As a result, the first-type Green's function for the Dirichlet boundary value problem for Laplace's equation in planar domains now has an important role in understanding the mathematics of the dispersionless Toda hierarchy and universal Whitham hierarchy.

It has been known for a long time that the Hele–Shaw problem admits large classes of solutions in which the time evolution depends only on a finite set

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of evolving quantities. Owing to this circumstance, such solutions are commonly called "exact". Before the connection with integrable systems was made, the exact solutions to the Hele–Shaw problem already enjoyed an abstract mathematical interpretation as time-evolving *quadrature domains* [6]. Richardson [7] was the first to associate Hele–Shaw flows with the theory of quadrature domains (it has since been realized that quite a number of physically-distinct problems in fluid dynamics can be usefully interpreted in terms of quadrature domain theory [8]). Richardson considered a set of moment quantities now known as the "Richardson moments". For certain problems, many of these moments are conserved by the dynamics. The exact solutions of the Hele–Shaw problems [7,9–11] have an interpretation as the algebraic orbits of the universal Whitham hierarchy [2]. Conversely, the methods developed for constructing solutions to the Hele–Shaw problem (e.g., [8,9,12]) might now be profitably employed to construct representations of the algebraic orbits of the Whitham hierarchy.

A tantalizing parallel to all this exists with another fluid dynamical problem: the evolution of long-wavelength incompressible gravity waves on shallow water. Benney [13] studied this problem and showed that it has an infinite number of conserved densities which are polynomials in a set of "moment" quantities. In analogy with the "Richardson moments", these are now known as the "Benney moments". They satisfy an infinite set of partial differential equations dubbed the *Benney hierarchy* or *Benney moment equations*. Intriguingly, Gibbons and Tsarev [14] have shown how the finite reductions of this integrable hierarchy correspond to conformal mappings to slit domains.

A less widely known fact is that the Dirichlet problem for Laplace's equation in the plane has deep theoretical connections with the general problem of conformal mapping to slit domains. The seminal work on this is due to Koebe [15]. Given the connection between the planar Dirichlet problem and the universal Whitham hierarchy, and in light of the connection of the Benney hierarchy to conformal slit mappings, it seems natural to ask whether the Benney hierarchy, or its reductions, can be linked directly with the Dirichlet problem of planar domains. It is the purpose of this Letter to elucidate such a connection.

The focus here is to use the connection we make to find explicit general formulas for the solutions to a spectral problem associated with two specific types of N-parameter reduction. There has been much recent work [16–18] on finding such formulas. In the context of the geometrical interpretation of the reductions described by Gibbons and Tsarev [14], the two reduction types considered here correspond to the following: one corresponds to slit mappings from an upper-half plane to another upper half-plane containing N vertical straight-line slits, the other type corresponds to mappings to an upper half-plane containing N concentric circular-arc slits. Our principle result is to show that the solutions to both these spectral problems can be written, for any N, in terms of the modified Green's function of a reflectionally-symmetric N-connected planar domain in a parametric z-plane. The variable z plays the role of uniformization parameter.

2. The Benney hierarchy

The Benney equations [13] are

$$u_t + uu_x - \left(\int_0^y u_x(x, y', t)\, dy'\right) u_y + h_x = 0,$$

$$h_t + uh_x + \left(\int_0^h u_x(x, y', t)\, dy'\right) u_y = 0. \tag{1}$$

Benney showed that if moments $A_n(x,t)$ are defined by

$$A_n(x,t) = \int_0^h u^n\, dy \tag{2}$$

then they satisfy the infinite set of equations

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1}\frac{\partial A_0}{\partial x} = 0, \quad n = 1, 2, \ldots \tag{3}$$

which are the *Benney moment equations*. An identical set of moment equations can be derived from a Vlasov equation

$$\frac{\partial f}{\partial t_2} + p\frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial f}{\partial p} = 0, \tag{4}$$

where $f(x,p,t)$ is some distribution function and the moments are now defined as

$$A_n = \int_{-\infty}^{\infty} p^n f \, dp. \tag{5}$$

Benney [13] showed that the moment equations have an infinite number of conserved densities which are polynomial in the moments $\{A_n \mid n = 0, 1, \ldots\}$.

The best way to see this is to consider the generating function of the moments defined by

$$\lambda_R(x,p,t) = p + \sum_{n=0}^{\infty} \frac{A_n(x,t)}{p^{n+1}} \tag{6}$$

which is the asymptotic series as $p \to \infty$ of

$$p + \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x,p',t)}{p-p'} \, dp'. \tag{7}$$

The notation $\mathcal{P}\int$ denotes the principal-value integral. Gibbons and Tsarev [14] have found a method for constructing a family of solutions to the above equations. Their method is to define

$$\lambda(p) = p + \int_{\Lambda} \frac{f(x,p',t)}{p-p'} \, dp', \tag{8}$$

where Λ is an indented contour passing below the point p on the real p'-axis. This function has the same large-p asymptotics as the function defined in (7) but, importantly, it can be analytically continued into the upper-half p-plane. Gibbons and Tsarev [14] have shown that N-parameter reductions of the integral equation (8) correspond to slit-mappings from an upper-half p-plane to an upper half λ-plane having a collection of N non-intersecting slits emanating from fixed points on the real λ-axis into the upper-half λ-plane. Let $\{c_j \mid j = 1, \ldots, N\}$ be some fixed choice of Jordan arcs into the upper-half λ-plane from some set of fixed points $\{\lambda_0^{(j)} \mid j = 1, \ldots, N\}$ on the real λ-axis. The N-parameter reductions correspond to conformal mappings from the upper-half p-plane to a collection of N slits taken along these arcs and having end-points at some set of points $\{\hat{\lambda}_j \mid j = 1, \ldots, N\}$ on these arcs. These points are the Riemann invariants of the system and they have characteristic speeds $\hat{p}_i = p(\hat{\lambda}_i)$. In this way, construction of analytical forms for $\lambda(p)$ corresponds to being able to construct analytical formulas for such slit maps. It is this construction that will be the focus of the remainder of this Letter.

3. The modified Green's function

The elements of Dirichlet calculus that will be used in the sequel will now be introduced. The key references for the following material are Koebe [15], Nehari [19] and Schiffer [20].

Let D_z be an arbitrary bounded N-connected planar domain. Suppose D_z is bounded by N smooth Jordan curves. C_0 is taken as the outermost boundary while $\{C_k \mid k = 1, \ldots, N-1\}$ denote the $N-1$ enclosed boundaries (or the boundaries of the finite set of "holes" in the domain). Define the *modified Green's function* as the function $G_0(x,y;x_0,y_0)$ satisfying the following properties:

(i) the function

$$g_0(x,y;x_0,y_0) = G_0(x,y;x_0,y_0) - \log r_0 \tag{9}$$

is harmonic with respect to (x,y) throughout D_z including at the point (x_0,y_0). Here r_0 is

$$r_0 = \sqrt{(x-x_0)^2 + (y-y_0)^2}; \tag{10}$$

(ii) if $\partial G_0/\partial n$ is the normal derivative of G_0 on a curve then

$$\begin{aligned} &G_0(x,y;x_0,y_0) = 0, \quad \text{on } C_0, \\ &G_0(x,y;x_0,y_0) = A_k, \quad \text{on } C_k, \quad k = 1, \ldots, N-1, \\ &\oint_{C_k} \frac{\partial G_0}{\partial n} \, ds = 0, \quad k = 1, \ldots, N-1, \end{aligned} \tag{11}$$

where ds denotes an element of arc and $\{A_k\}$ are constants.

It is convenient to introduce complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. Thus, if the complex number $z_\alpha = x_0 + iy_0$ denotes the complex position of the singularity of the Green's function we will henceforth write $G_0(z,z_\alpha)$ instead of $G_0(x,y;x_0,y_0)$. Since $G_0(z,z_\alpha)$ is a harmonic function of z in D_z (except for the single logarithmic singularity at $z = z_\alpha$) then we define $\tilde{G}_0(z,z_\alpha)$ to be its analytic extension, obtained by adding to $G_0(z,z_\alpha)$ its harmonic conjugate

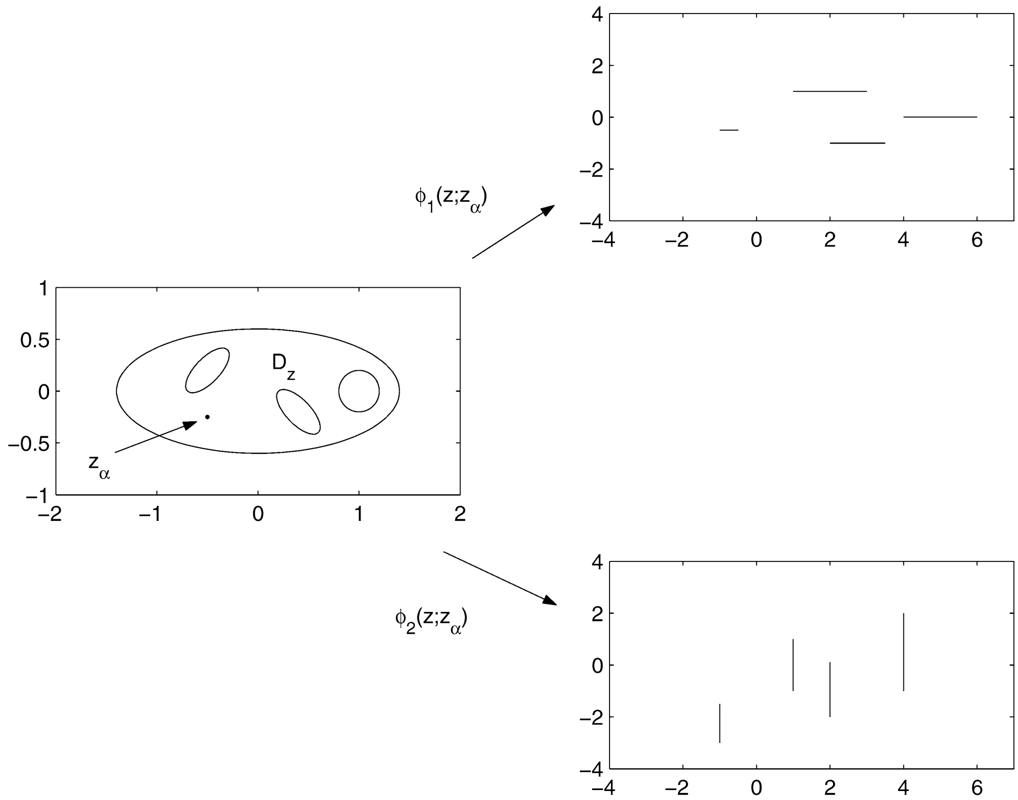

Fig. 1. Schematic illustrating the images of a general quadruply connected domain under the conformal mappings ϕ_1 and ϕ_2 given in (13) and (14).

function $H_0(z, z_\alpha)$, i.e.,

$$\tilde{G}_0(z, z_\alpha) = G_0(z, z_\alpha) + i H_0(z, z_\alpha). \tag{12}$$

It will be in terms of this object that we shall represent the solutions of the reductions of the Benney hierarchy.

4. Slit mappings

It was shown by Koebe [15] (see also Schiffer [20]) that a univalent conformal mapping of an arbitrary multiply connected domain D_z to an N-connected slit mapping consisting of N-slits all of which are parallel to the *real* axis in the image domain is given by

$$\begin{aligned}\phi_1(z; z_\alpha) &\equiv -\frac{1}{i}\frac{\partial}{\partial y_0}\tilde{G}_0(z, z_\alpha) \\ &= \left[\frac{\partial}{\partial \bar{z}_\alpha} - \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z, z_\alpha).\end{aligned} \tag{13}$$

It is also known [15,20] that a univalent conformal mapping taking D_z to a multiply connected domain consisting of N-slits, all of which are parallel to the *imaginary* axis in the image domain, is given by

$$\begin{aligned}\phi_2(z; z_\alpha) &\equiv -\frac{\partial}{\partial x_0}\tilde{G}_0(z, z_\alpha) \\ &= -\left[\frac{\partial}{\partial \bar{z}_\alpha} + \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z, z_\alpha),\end{aligned} \tag{14}$$

where, again, z_α is some point inside D_z. Fig. 1 illustrates the images of a general quadruply connected domain D_z under the two mappings ϕ_1 and ϕ_2.

5. Reflectionally-symmetric domains

The reductions considered by Yu and Gibbons [16] and Baldwin and Gibbons [17,18] treat the case where the slits in the λ-plane are straight vertical lines perpendicular to the real axis: [16] treats the genus-1 case,

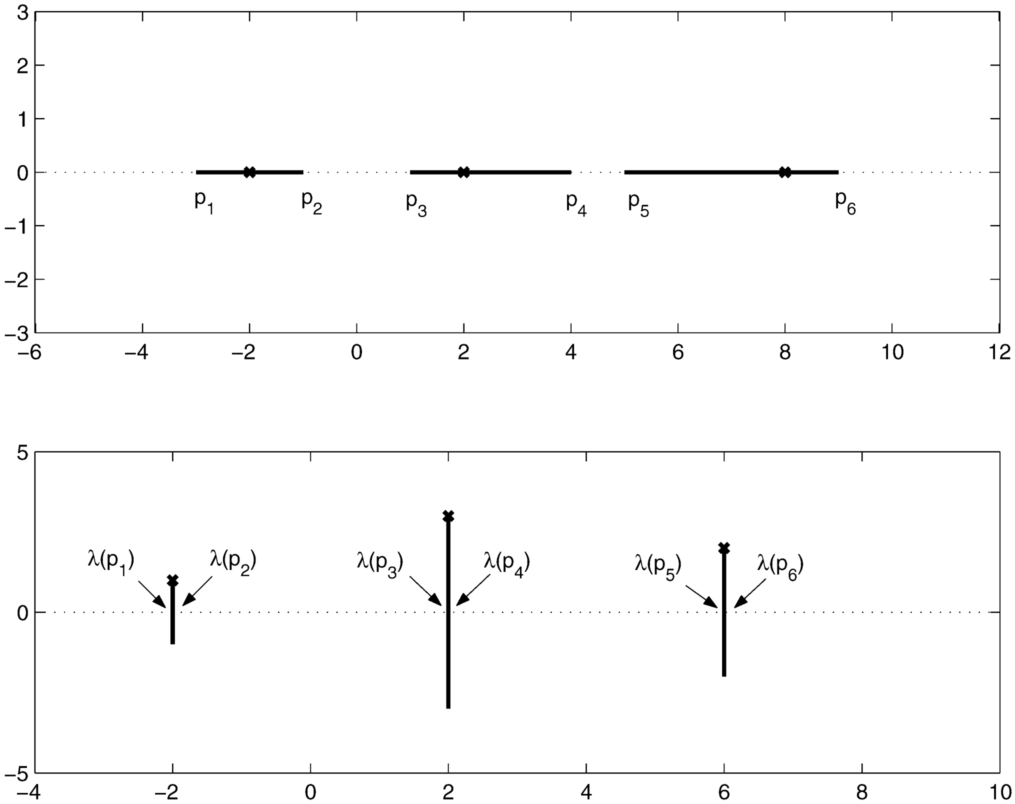

Fig. 2. Schematic illustrating the conformal mapping between p and λ planes for the first-type Benney reductions in the genus-2 case. Crosses, indicating the branch points, show corresponding points in the two planes.

[17] tackles the genus-2 case while higher-genus cases are addressed in [18]. Fig. 2 illustrates the required conformal mappings in the genus-2 case: a conformal mapping from the upper-half p-plane with 3 finite-length slits on the real p-axis to the upper-half λ-plane containing 3 vertical slits is required. Actually, this figure illustrates the case in which the whole p-plane exterior to the three finite-length slits on the real p-axis maps univalently to the whole λ-plane exterior to the three vertical slits with the upper and lower-half p-planes mapping to the upper and lower-half λ-plane, respectively. This will be the class of mappings to be constructed in what follows.

First, it is necessary to restrict the class of domains D_z under consideration. Let D_z now be any bounded N-connected domain in a complex z-plane which is reflectionally-symmetric about the real z-axis and such that all the holes in the domain are centred on the real axis. Let $G_0(z, z_\alpha)$ be the modified Green's function of D_z with logarithmic singularity at some point z_α inside D_z.

6. The generating functions

Equipped with such a reflectionally-symmetric domain D_z, we now pose that the pair of functions $p(z)$ and $\lambda(z)$ is given by

$$\begin{aligned} p(z) &= \phi_1(z; z_\alpha) = \left[\frac{\partial}{\partial \bar{z}_\alpha} - \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z, z_\alpha), \\ \lambda(z) &= \phi_2(z; z_\alpha) + C \\ &= -\left[\frac{\partial}{\partial \bar{z}_\alpha} + \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z, z_\alpha) + C, \end{aligned} \tag{15}$$

where C is some real constant and z_α is some point constrained to be on the real z-axis inside D_z (where z_α is only taken to be real once derivatives have been taken). It is claimed that (15) represents a uniformization, via the parameter z, of the functions $p(z)$ and $\lambda(z)$ solving the spectral problem for the Benney reductions considered in [16–18].

To see this, first note that, being logarithmic derivatives, both $p(z)$ and $\lambda(z)$ have simple poles, with unit residue, at $z = z_\alpha$. It therefore follows that, as $z \to z_\alpha$,

$$\lambda \sim p \sim \frac{1}{z - z_\alpha} + \mathcal{O}(1), \quad \text{as } z \to z_\alpha. \tag{16}$$

The constant C in (15) can be chosen so that $\lambda = p + \mathcal{O}(p^{-1})$ as $z \to z_\alpha$ which is the condition required of $\lambda(p)$ as $p \to \infty$ [18].

Next, it follows from the assumed reflectional symmetry of D_z and the choice of z_α to be on the real z-axis that the images of the boundaries $\{C_j \mid j = 0, 1, \ldots, N-1\}$ under the mapping $p(z)$ will each be finite-length horizontal slits along the real p-axis. Let $\{(a_j, b_j) \mid j = 0, 1, \ldots, N-1\}$ be the two points at which each of the boundary circles $\{C_j \mid j = 0, 1, \ldots, N-1\}$ intersects the real z-axis. These points must map to the two end-points of each slit, that is,

$$p(a_j) = p_{2j+1},$$
$$p(b_j) = p_{2j+2}, \quad j = 0, 1, \ldots, N-1, \tag{17}$$

where all the values $\{p_{2j+1}, p_{2j+2} \mid j = 0, 1, \ldots, N-1\}$ are on the real p-axis. It also follows, on use of the Schwarz reflection principle, that the upper/lower halves of D_z map to the upper/lower halves of the p-plane, respectively.

In a similar way, the reflectional symmetry of D_z and the choice of z_α real also implies that the images of the boundaries $\{C_j \mid j = 0, 1, \ldots, N-1\}$ under the mapping $\lambda(z)$ will each be finite-length vertical slits, symmetric about the real λ-axis. This time, the pair of points (a_j, b_j) will both map to the point where the centre-point of each vertical slit intersects the real λ-axis (strictly speaking, of course, a_j and b_j will map to points on different "sides" of the vertical slit corresponding to the image of C_j). This means that

$$\lambda(a_j) = \lambda_j^0, \qquad \lambda(b_j) = \lambda_j^0, \quad j = 0, 1, \ldots, N-1, \tag{18}$$

where, in the context of the Benney reductions, the set $\{\lambda_j^0 \mid j = 0, 1, \ldots, N-1\}$ are a set of fixed constants [14] determined from the initial conditions. Considering λ now as a function of p, i.e., $\lambda = \lambda(p)$, it follows that

$$\lambda_j^0 = \lambda(p_{2j+1}),$$
$$\lambda_j^0 = \lambda(p_{2j+2}), \quad j = 0, 1, \ldots, N-1 \tag{19}$$

which are exactly the conditions required of $\lambda(p)$ [18]. Again, it follows from the Schwarz reflection principle that the upper/lower halves of D_z map, respectively, to the upper/lower halves of the λ-plane. Thus, we have established that (15) are the required expressions for $p(z)$ and $\lambda(z)$. Further, it should be clear that we can also multiply (15) by arbitrary real constants and still obtain reductions of the Benney system.

Note that it can also be shown that the distribution function can be written in terms of the modified Green's function in the form

$$f(x, p, t) = \frac{2}{\pi} \operatorname{Im}\left[\frac{\partial}{\partial \bar{z}_\alpha} \tilde{G}_0(z, z_\alpha)\right]. \tag{20}$$

Inspection of (15) immediately shows that the Benney moments are, in fact, purely geometrical quantities associated with the domain D_z and some given point z_α inside it. In this respect, the Benney moments are strongly reminiscent of the Richardson moments in the Hele–Shaw problem. The Richardson moments are usually written as integral quantities over the support of some multiply connected domain. Equivalently, on use of Green's theorem, they become line integral quantities around the boundary of the domain. It is possible to write the Benney moments in the same way. On use of the Plemelj formula in (8) as p tends to the real axis and taking the imaginary part, it follows that

$$f = -\frac{1}{\pi} \operatorname{Im}[\lambda]. \tag{21}$$

(21) highlights two properties of f: first, on the real p-axis between the slits $\{[p_{2j+1}, p_{2j+2}] \mid j = 0, \ldots, N-1\}$, $f = 0$; second, since the upper side of each slit in the p-plane corresponds to the upper vertical segment of the image in the λ-plane with the lower side of each slit in the p-plane corresponding to the lower vertical segment in the λ-plane, and owing to the reflectional symmetry of the arrangement, it also follows from (21) that the values of f at any point on the lower side of each slit in the p-plane is the negative of the value at the corresponding point on the upper side. These two properties of f, together with

the expression (5) for A_n, can be used to deduce that

$$A_n = \frac{1}{2}\oint_{\partial D_z} p^n f \frac{dp}{dz}\, dz \tag{22}$$

or, on use of (21),

$$A_n = \frac{1}{2\pi}\oint_{\partial D_z} p^n \,\mathrm{Im}[\bar{\lambda}] \frac{dp}{dz}\, dz \tag{23}$$

which shows that the Benney moments are contour integrals around the boundary of D_z of an integrand purely expressible in terms of the modified Green's function of D_z. Indeed, on use of (15) and (20), it follows that

$$A_n = \frac{1}{2\pi i}\oint_{\partial D_z} \left(\frac{\partial \tilde{G}_0}{\partial \bar{z}_\alpha} - \frac{\partial \tilde{G}_0}{\partial z_\alpha}\right)^n \left(\frac{\partial \tilde{G}_0}{\partial \bar{z}_\alpha} - \overline{\frac{\partial \tilde{G}_0}{\partial \bar{z}_\alpha}}\right) \times \left(\frac{\partial^2 \tilde{G}_0}{\partial z \partial \bar{z}_\alpha} - \frac{\partial^2 \tilde{G}_0}{\partial z \partial z_\alpha}\right) dz. \tag{24}$$

6.1. The elliptic reduction

As a check on our formulation, it is appropriate to check that the new expressions (15) are equivalent to those already derived, using alternative arguments, by previous authors. We therefore now consider the special case of the elliptic reduction previously analyzed by Yu and Gibbons [16].

Consider a doubly connected, reflectionally-symmetric domain. By the Riemann mapping theorem, every such domain is conformally equivalent to a concentric annulus in a parametric plane [19]. Since the boundary value problem satisfied by the modified Green's function is conformally invariant then picking different choices of conformally equivalent domains D_z simply corresponds to inconsequential reparametrizations of the uniformizing variable. It is therefore enough to construct the modified Green's function associated with this annulus. Let it be $\rho < |\zeta| < 1$. Since we have now specified a definite domain, we denote it by D_ζ (reserving the notation D_z for discussion of a general domain). ρ is the conformal modulus [19] of the doubly connected domain. The point $z = z_\alpha$ corresponds to $\zeta = \alpha$.

First, introduce the function

$$P(\zeta) \equiv (1-\zeta)P'(\zeta), \tag{25}$$

where

$$P'(\zeta) \equiv \prod_{k=1}^{\infty}\left(1-\rho^{2k}\zeta\right)\left(1-\rho^{2k}\zeta^{-1}\right). \tag{26}$$

For brevity, we have suppressed the dependence of $P(\zeta)$ on ρ. It follows directly from (25) and (26) that

$$P\left(\rho^2\zeta\right) = -\zeta^{-1}P(\zeta),$$
$$P\left(\zeta^{-1}\right) = -\zeta^{-1}P(\zeta). \tag{27}$$

The modified Green's function $G_0(\zeta,\alpha)$ for this domain is

$$G_0(\zeta,\alpha) = \log\left||\alpha|\frac{P(\zeta\alpha^{-1})}{P(\zeta\bar{\alpha})}\right|, \tag{28}$$

where α is a point inside D_ζ. It can be verified that (28) satisfies all the requirements of the modified Green's function described in Section 3. It follows that the analytic extension of this function is

$$\tilde{G}_0(\zeta,\alpha) = \log\left(|\alpha|\frac{P(\zeta\alpha^{-1})}{P(\zeta\bar{\alpha})}\right). \tag{29}$$

Given (28), expressions for $p(\zeta)$ and $\lambda(\zeta)$ follow from the general results (15). Since

$$\frac{\partial \tilde{G}_0}{\partial \alpha} = \frac{1}{2\alpha} - \frac{1}{\alpha^2}\frac{\zeta P_\zeta(\zeta\alpha^{-1})}{P(\zeta\alpha^{-1})},$$
$$\frac{\partial \tilde{G}_0}{\partial \bar{\alpha}} = \frac{1}{2\bar{\alpha}} - \frac{\zeta P_\zeta(\zeta\bar{\alpha})}{P(\zeta\bar{\alpha})}, \tag{30}$$

where $P_\zeta(\zeta)$ denotes the derivative of $P(\zeta)$ with respect to ζ, it follows that

$$p(\zeta) = \frac{1}{\alpha}\left(K\left(\zeta\alpha^{-1}\right) - K(\zeta\alpha)\right),$$
$$\lambda(\zeta) = \frac{1}{\alpha}\left(K\left(\zeta\alpha^{-1}\right) + K(\zeta\alpha) - 1\right) + C, \tag{31}$$

where $K(\zeta)$ is defined as

$$K(\zeta) \equiv \frac{\zeta P_\zeta(\zeta)}{P(\zeta)} \tag{32}$$

and where we have now taken $\bar{\alpha} = \alpha$. Formulas (31) give an explicit expressions of the functions p and λ in terms of the uniformizing parameter ζ. The constant C is chosen to ensure that $\lambda \sim p + \mathcal{O}(p^{-1})$ as $\zeta \to \alpha$. Straightforward algebra produces

$$C = \frac{1}{\alpha}\left(1 - 2K\left(\alpha^2\right)\right). \tag{33}$$

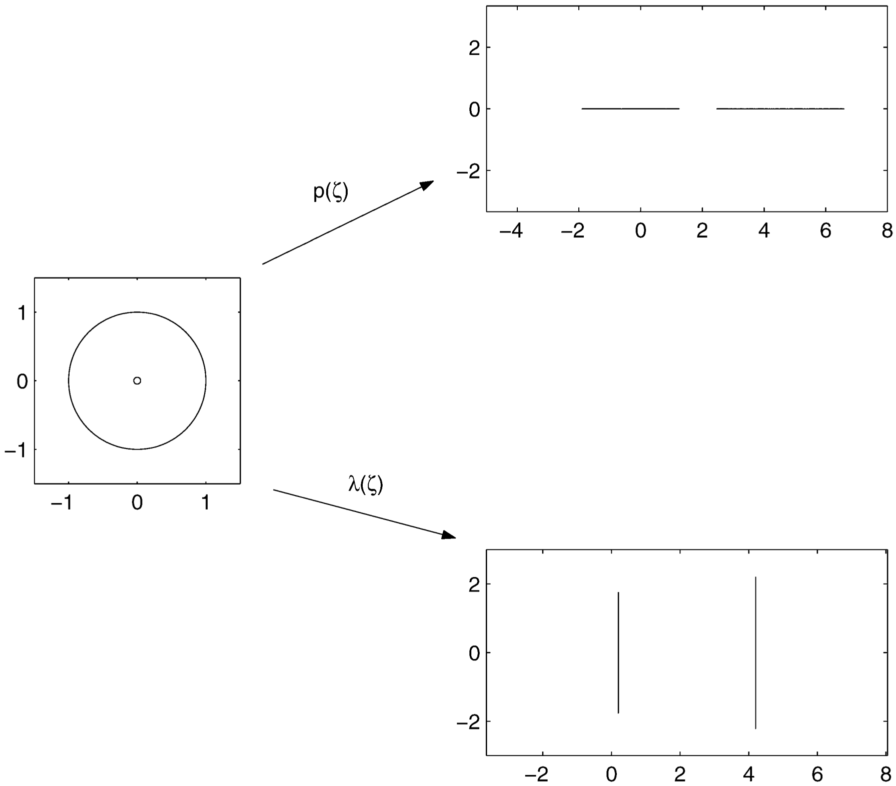

Fig. 3. An illustration of the first type of reduction in the elliptic case. The figures shows the images of the annulus under the conformal mappings (31) with parameter values $\rho = 0.05$ and $\alpha = 0.25$.

Yu and Gibbons [16] use a different uniformizing parameter χ and report their results as

$$p(\chi) = p_4 - \frac{1}{\mathcal{P}(\chi) - \mathcal{P}(\chi_0)},$$
$$\lambda(\chi) = \frac{1}{k}\big(\gamma(\chi + \chi_0) + \gamma(\chi - \chi_0)\big) + C, \tag{34}$$

where p_4, χ_0 and C are constants, $\mathcal{P}(\chi)$ is the Weierstrass elliptic $\mathcal{P}$-function [21] with periods ω_1 and ω_2 and

$$\gamma(\chi) = -\tilde{\zeta}(\chi) + \frac{\tilde{\zeta}(\omega_1)}{\omega_1}\chi, \tag{35}$$

where $\tilde{\zeta}(\chi)$ is the Weierstrass zeta-function [21]. It is possible to verify that (34) are equivalent to (31) if the following identifications are made:

$$\chi = \log\zeta, \qquad \chi_0 = \log\alpha,$$
$$\omega_1 = \pi i, \qquad \omega_2 = 2\log\rho. \tag{36}$$

This confirms the correctness of the newly-derived expressions (15). To illustrate the conformal mappings, Fig. 3 shows how the annulus in the ζ-plane maps to the two types of slit domains in the p and λ-planes. The parameter values $\rho = 0.05$ and $\alpha = 0.25$ are chosen.

7. New reductions

In [16–18], the N-parameter vertical-slit reductions just considered were viewed as Schwarz–Christoffel mappings between the p and λ planes. A natural extension of Schwarz–Christoffel maps is to consider mappings to circular-arc regions. Therefore, consider now a mapping from the upper-half p-plane to an upper-half λ-plane with N slits emanating from the real λ-axis that are all circular arcs. The “centres” of the N circular arcs must be specified. Here we choose all the circular arcs to be concentric, with an arbitrarily chosen centre.

Another result of [15] (also described by Schiffer [20]) is that conformal mappings from any N-connected domain D_z to a conformally equivalent unbounded domain consisting of N concentric arcs of circles can also be written in terms of the modified Green's function of D_z. To do so, choose two points z_α and z_β inside D_z and consider the analytic function of z given by

$$\phi_3(z; z_\alpha, z_\beta) \equiv \exp\left[\tilde{G}_0(z, z_\beta) - \tilde{G}_0(z, z_\alpha)\right]. \tag{37}$$

ϕ_3 maps D_z, in a one-to-one fashion, to an unbounded domain consisting of N concentric arcs of circles. We now pose that

$$\begin{aligned} p(z) &= \phi_1(z, z_\alpha) = \left[\frac{\partial}{\partial \bar{z}_\alpha} - \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z, z_\alpha), \\ \lambda(z) &= A\phi_3(z; z_\alpha, z_\beta) + C \\ &= A\exp\left[\tilde{G}_0(z, z_\beta) - \tilde{G}_0(z, z_\alpha)\right] + C, \end{aligned} \tag{38}$$

where both z_α and z_β are now restricted to be distinct points on the real z-axis inside D_z. It follows from (38) that, as $z \to z_\alpha$, $p \to \infty$ and $\lambda \to \infty$. Constants A and C should therefore be chosen so that $\lambda \sim p + \mathcal{O}(p^{-1})$ as $z \to z_\alpha$.

As before, the image of D_z under the mapping given by $p(z)$ is a set of N real intervals on the real axis in the image plane. Further, by the choice of taking z_β real, it also follows that the common "centre" of the circular arcs will be on the real λ-axis and that the image of D_z under $\lambda(z)$ consists of N concentric circular arcs, each of finite length, which are reflectionally-symmetric about the real λ-axis. Indeed, picking the parameter z_β can be thought of as specifying the common centre of the circular arcs. By the Schwarz reflection principle, it also follows that the upper/lower halves of D_z maps to the upper/lower halves of the p and λ-planes. Therefore, all the conditions required of $p(z)$ and $\lambda(z)$ are satisfied and they are given by (38) with z as the natural uniformizing variable.

It is possible to show, using arguments analogous to those concerning the first-type reduction, that formula (23) expressing the Benney moments in terms of p and λ holds in this case as well. Again this gives an expression for the $\{A_n\}$ as contour integrals around ∂D_z of an integrand dependent only on the modified Green's function of D_z.

7.1. The elliptic case

Let D_ζ be the annular domain $\rho < |\zeta| < 1$ as in Section 6.1. On use of (29) in (38) it follows that

$$\begin{aligned} p(\zeta) &= \frac{1}{\alpha}\left(K\left(\zeta\alpha^{-1}\right) - K(\zeta\bar{\alpha})\right), \\ \lambda(\zeta) &= \tilde{A}\frac{P(\zeta\beta^{-1})P(\zeta\bar{\alpha})}{P(\zeta\bar{\beta})P(\zeta\alpha^{-1})} + \tilde{C} \end{aligned} \tag{39}$$

for some real constants $\tilde{A}$ and $\tilde{C}$ which must be chosen so that $\lambda \sim p + \mathcal{O}(p^{-1})$ as $p \to \infty$. A local expansion of (39) yields

$$\begin{aligned} \tilde{A} &= -\frac{P(\alpha\beta)P'(1)}{\alpha P(\alpha\beta^{-1})P(\alpha^2)}, \\ \tilde{C} &= \frac{1}{\alpha}\left(1 - K\left(\alpha^2\right)\right) \\ &\quad + \alpha\tilde{A}\frac{d}{d\zeta}\left(\frac{P(\zeta\alpha)P(\zeta\beta^{-1})}{P(\zeta\beta)P'(\zeta\alpha^{-1})}\right)\bigg|_{\zeta=\alpha}. \end{aligned} \tag{40}$$

where we have now taken $\bar{\alpha} = \alpha$ and $\bar{\beta} = \beta$. To illustrate the conformal mappings, Fig. 4 shows how the annulus in the ζ-plane maps to the two types of slit domains in the p and λ-planes. The parameter values $\rho = 0.05$, $\alpha = 0.25$ and $\beta = -0.25$ are chosen.

8. Discussion

The key new formulas of this Letter for the two different reduction types are (15) and (38) where $\tilde{G}_0$ is the analytic extension of the modified Green's function of an N-connected planar domain D_z which is reflectionally-symmetric about the real axis and with all its holes centred on this axis. The specific manifestations of (15) and (38) in the elliptic case $N = 2$ have been explicitly constructed. One of these elliptic reductions corresponds to that derived by Yu and Gibbons [14], the other is new. Given these explicit expressions one could now, in principle, make use of the hodograph method of Tsarev [22] to solve the initial value problem. This was done by Yu and Gibbons [16] in the case of the first-type elliptic reduction.

We believe the general results (15) and (38) to be significant for the following reason. Associated

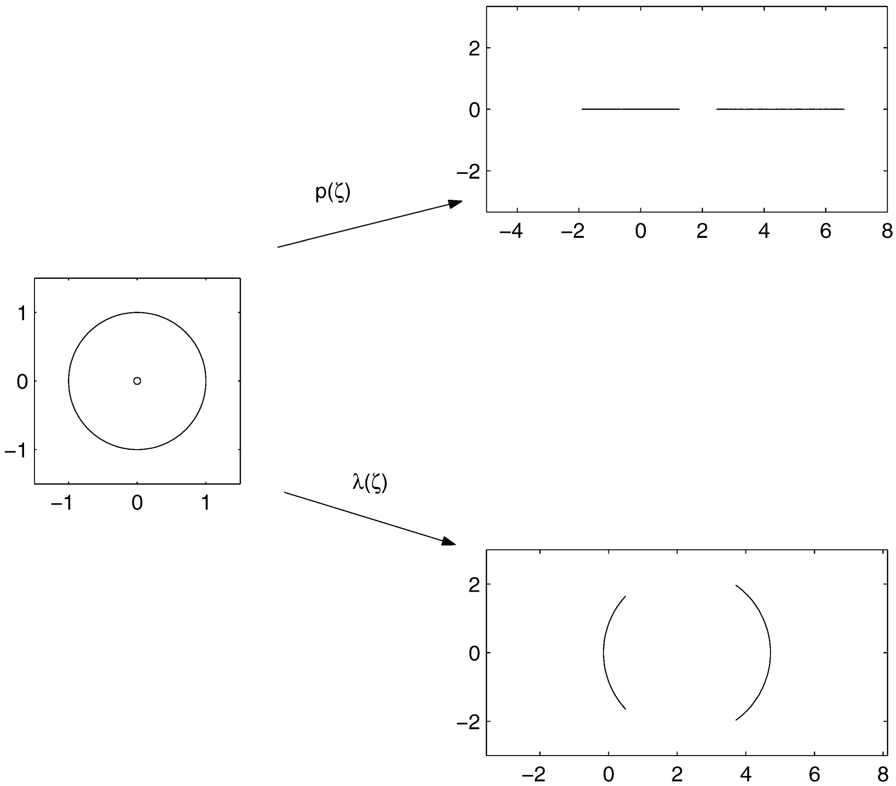

Fig. 4. An illustration of the second type of reduction in the elliptic case. The figures shows the images of the annulus under the conformal mappings (38) with parameter values $\rho = 0.05$ and $\alpha = 0.25$, $\beta = -0.25$.

with any multiply connected planar domain D_z (let us assume, having boundary components that are all smooth Jordan curves) is a compact, symmetric Riemann surface known as the *Schottky double*. It is a Riemann surface consisting of two identical "halves" and endowed with an anti-holomorphic involution providing a mapping from one "half" to the other. It is possible to write an expression for the modified Green's function associated with a given planar domain in terms of the prime form on the Schottky double. Formulas for the Green's function are given, for example, in Krichever et al. [23] where they are represented in terms of the Riemann theta function. Analogous formulas exist for the modified Green's function (since the Green's function and the modified Green's functions are solutions to "dual" problems, as discussed in [23]) which, together with (15) and (38) for the two types of reduction considered here, give formulas for the required uniformizations of p and λ.

Indeed, based on the results of the present Letter, the present author [24] has produced explicit formulas for the general N-parameter case of the two reduction types considered here. The method is based on finding explicit formulas for the modified Green's functions of the canonical class of reflectionally-symmetric multiply connected *circular domains* (i.e., domains whose boundaries are all circles) and then making use of (15) and (38).

The formulation here also suggests that the reductions of the Benney hierarchy can be interpreted as a special class of flows in the extended moduli space of analytic curves. This mirrors recent work in [1,2] where an identification between exact solutions of the equations governing Laplacian growth are reinterpreted as special reductions, known as "algebraic orbits", of the universal Whitham hierarchy. There, the flows are generated by certain meromorphic differentials defined on the Schottky double of the multiply connected planar domain. Here, the generating differentials dp and $d\lambda$ of the two reduction types are both second-kind Abelian differentials with two second-order poles (of vanishing residue), one on each half of the Schottky double of the planar domain D_z. Crowdy and Marshall [12] have already used the Schottky model to represent time-evolving quadrature domains (or, as we now understand [2], the algebraic orbits of the universal Whitham hierarchy). The work in Crowdy [24] extends this mode of representation to N-parameter reductions of the Benney hierarchy.

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