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Conformal mappings to a doubly connected polycircular arc domain

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The explicit construction of the conformal mapping of a concentric annulus to a doubly connected polygonal domain was first reported by Akhiezer in 1928. The construction of an analogous formula for the case of a polycircular arc domain, i.e. for a doubly connected domain whose boundaries are a union of circular arc segments, has remained an important open problem. In this paper, we present this explicit formula. We first introduce a new method for deriving the classical formula of Akhiezer and then show how to generalize the method to the case of a doubly connected polycircular arc domain. As an analytical check of the formula, a special exact solution for a doubly connected polycircular arc mapping is derived and compared with that obtained from the more general construction. As an illustrative example, a doubly connected polycircular arc domain arising in a classic potential flow problem considered in the last century by Lord Rayleigh is considered in detail.

Keywords: conformal mapping; doubly connected; circular arc

1. Introduction

Schwarz–Christoffel mapping is the name commonly given to a conformal mapping from a simple 'canonical' domain to a polygonal domain having boundaries that are all straight-line segments. Such maps are important in both theory and applications. This is the central topic of a recent monograph by Driscoll & Trefethen (2002). There, the role of such mappings in a wide range of circumstances, from transport problems in heterogeneous media and fluid dynamics to more abstract theoretical applications in approximation theory, is expertly surveyed.

The Schwarz–Christoffel formula (henceforth abbreviated to S–C formula) for a mapping to a simply connected polygonal domain dates back to the 1860s (Driscoll & Trefethen 2002), while the formula relevant to doubly connected domains was first derived by Akhiezer (1928). The natural question of generalizing this formula to polygonal domains of arbitrary finite connectivity has remained open until recently. Apparently, developments in this direction were partly discouraged by knowledge of the difficulty, even with the known simply and doubly connected formulae, of solving

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the so-called 'parameter problem' (as well as various other difficulties such as 'crowding'; Driscoll & Trefethen 2002). These practical impediments have now been overcome and readily transferable software operating on platforms, such as MATLAB, are available. Now, S–C mappings of even very complicated domains can be constructed with the click of a mouse. Driscoll has created a MATLAB package called S–C Toolbox (www.math.udel.edu/-driscoll/SC) based on an earlier FORTRAN program developed by Trefethen.

Influenced by the above developments, the long-standing theoretical problem of finding general formulae for S–C mappings to higher connected polygonal domains has recently been solved. DeLillo *et al.* (2001) were the first to produce a multiply connected S–C mapping from an unbounded circular preimage region to the unbounded region exterior to a finite collection of polygonal objects. Their arguments rely on the use of reflection principles and are an extension of an approach to the doubly connected S–C mapping presented in DeLillo *et al.* (2004). The S–C formula to bounded multiply connected polygonal region was first derived by Crowdy (2005), who introduced the powerful formalism of classical function theory into the analysis of this problem. Indeed, employing the machinery of Schottky groups, Crowdy was able to write the formula, in a natural way, as a product of powers of a special transcendental function known as the *Schottky–Klein prime function* (Baker 1995). The approach can also be readily extended to unbounded polygonal regions (Crowdy 2007).

A natural extension of the theory of S–C mappings is the related theory of mappings to polycircular arc domains. These are domains whose boundaries are a collection of arcs of circles. Since straight-line segments are particular cases of circular arcs, this theory includes the theory of S–C mappings as a special case. Nehari (1952) and Ablowitz & Fokas (1997) discuss conformal mappings to simply connected polycircular arc domains. A numerical construction of such mappings to simply connected regions has been carried out by Bjørstad & Grosse (1987) and Howell (1993). Given the recent development of multiply connected S–C formulae, it is natural to ask about the construction of the analogous formulae for conformal maps to multiply connected polycircular arc domains. This question apparently remains open even for doubly connected domains.

This paper presents the theory for the doubly connected case; the general case of arbitrary connectivity will appear in a subsequent publication (Crowdy & Fokas in preparation). Since the doubly connected case is likely to be the most important for applications, and since it has special features not shared by the case of higher connectivity, we feel that the doubly connected situation deserves a separate treatment.

In the case when a doubly connected polycircular arc domain reduces to a doubly connected polygonal domain, the relevant formula is known. It appears to have been first reported by Akhiezer (1928). It was later rederived, using separate methods, by Komatu (1945) and DeLillo *et al.* (2001). Yet another approach is implicit in the construction of S–C mappings to higher connected domains of Crowdy (2005). All these methods produce, for the derivative of the required mapping, an explicit formula up to a finite set of accessory parameters. In order to unify our approach with the known formula for the S–C mapping to a doubly connected polygonal region, we consider the latter case first. In fact, while there exist many different derivations of the doubly connected S–C formula, the one given here appears to be new.

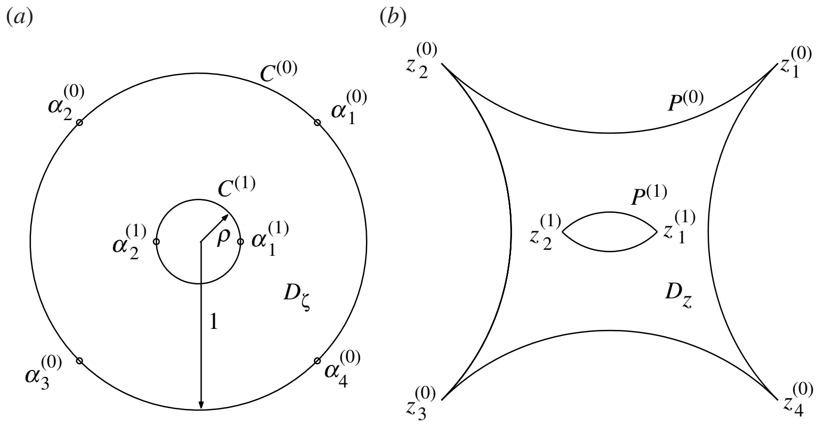

Figure 1. Schematic of (*a*) the preimage annulus $\rho<|\zeta|<1$ and (*b*) a typical target polycircular arc domain D_z.

2. The conformal mapping problem

Let D_ζ denote the annulus $\rho<|\zeta|<1$ in a complex ζ-plane, where $0<\rho<1$ is a real parameter. It is known that any doubly connected target domain D_z in a complex z-plane can be conformally mapped to such an annulus D_ζ for some choice of the conformal modulus ρ (Nehari 1952). ρ is determined by the target domain. Let the boundary $|\zeta|=1$ be denoted by $C^{(0)}$ and let $|\zeta|=\rho$ be denoted by $C^{(1)}$.

Let the target domain D_z, in a complex z-plane, be a bounded doubly connected polycircular arc domain. It is characterized by the fact that it is a doubly connected region having two boundaries each of which is a (continuous) union of circular arc segments. Let $P^{(0)}$ and $P^{(1)}$ denote the two boundaries of D_z. Then, $P^{(0)}$ is a union of n_0 circular arc segments, each defined by the equations

$$\left|z-\Delta_k^{(0)}\right|^2=\left(Q_k^{(0)}\right)^2,\quad k=1,\dots,n_0 \tag{2.1}$$

for some complex parameters $\{\Delta_k^{(0)}|k=1,\dots,n_0\}$ and some real parameters $\{Q_k^{(0)}|k=1,\dots,n_0\}$. Similarly, $P^{(1)}$ is a union of n_1 circular arc segments with associated equations

$$\left|z-\Delta_k^{(1)}\right|^2=\left(Q_k^{(1)}\right)^2,\quad k=1,\dots,n_1. \tag{2.2}$$

The conformal mapping problem is to find the functional form of a conformal mapping $z(\zeta)$ from the annulus D_ζ to the polycircular region D_z. Let $P^{(0)}$ and $P^{(1)}$ correspond to the image under the mapping $z(\zeta)$ of $C^{(0)}$ and $C^{(1)}$, respectively. Let the points $\{z_k^{(j)}|j=0,1;\ k=1,\dots,n_j\}$ denote the vertices of D_z (i.e. the points at which the distinct circular arc segments making up the boundaries intersect). We can adopt the convention that these are ordered in an anticlockwise fashion around each boundary contour (figure 1). The *prevertices* on the circles $C^{(0)}$ and $C^{(1)}$ in the ζ-plane will be denoted $\{\alpha_k^{(j)}|j=0,1;\ k=1,\dots,n_j\}$, where

$$z(\alpha_k^{(j)})=z_k^{(j)}. \tag{2.3}$$

3. Function theory in an annulus

Consider the annulus $\rho<|\zeta|<1$. The following function will play a primary role when performing analysis in this annulus,

$$P(\zeta) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{3.1}$$

This function, to within a constant of proportionality, is the *Schottky–Klein prime function* (ch. 12 of Baker 1995) associated with the annulus $\rho<|\zeta|<1$. It is convergent for all finite $\zeta\neq 0$.

It is easy to verify directly from the definition (3.1) (appendix A) that

$$P(\rho^2\zeta) = -\zeta^{-1}P(\zeta), \quad P(\zeta^{-1}) = -\zeta^{-1}P(\zeta). \tag{3.2}$$

Next, define

$$K(\zeta) \equiv \frac{\zeta P'(\zeta)}{P(\zeta)}, \tag{3.3}$$

which has a simple pole at $\zeta=1$ with residue $+1$. By making use of equation (3.2), it follows that

$$K(\rho^2\zeta) = K(\zeta)-1, \quad K(\zeta^{-1}) = 1-K(\zeta). \tag{3.4}$$

Finally, define

$$L(\zeta) \equiv \zeta\frac{\mathrm{d}K(\zeta)}{\mathrm{d}\zeta}. \tag{3.5}$$

Using equation (3.4), it can be verified that

$$L(\rho^2\zeta) = L(\zeta), \quad L(\zeta^{-1}) = L(\zeta). \tag{3.6}$$

$L(\zeta)$ has a second-order pole at $\zeta=1$ with strength -1, i.e. near $\zeta=1$,

$$L(\zeta) = -\frac{1}{(\zeta-1)^2} + \text{analytic}. \tag{3.7}$$

$L(\zeta)$ is an example of a *loxodromic function* (Valiron 1947): a function $G(\zeta)$ is defined to be a loxodromic function if it is meromorphic everywhere inside (and on the boundary of) the *fundamental annulus* $\rho\leq|\zeta|<\rho^{-1}$ and which additionally satisfies the functional relation

$$G(\rho^2\zeta) = G(\zeta). \tag{3.8}$$

The annulus $\rho\leq|\zeta|<\rho^{-1}$ is called fundamental because, given the singularity structure of $G(\zeta)$ in this annulus, the singularity structure in all other annuli filling out the complex ζ-plane follows from the functional relation (3.8). Since the only singularity of $L(\zeta)$ in the fundamental annulus is a second-order pole at $\zeta=1$, and since it satisfies equation (3.6), it follows that $L(\zeta)$ is a loxodromic function.

4. Doubly connected S–C mappings

In this section, we will rederive the known formula for the S–C mapping to a doubly connected polygonal region (Akhiezer 1928; Komatu 1945; Driscoll & Trefethen 2002). The derivation here appears to be new.

(a) Maps from circles centred at $\zeta = 0$ to straight lines

Let $z(\zeta)$ denote the associated conformal mapping, then if ζ is on a circle $C^{(0)}$ or $C^{(1)}$, it follows that $z(\zeta)$ is on a straight-line segment. Functions which have this property are characterized in the following proposition:

Proposition 4.1. *Let the functions $F(z)$ and $\mathcal{S}(\zeta)$ be defined by the equations*

$$F(z) \equiv \zeta \frac{\mathrm{d}z(\zeta)}{\mathrm{d}\zeta} \tag{4.1}$$

and

$$\mathcal{S}(\zeta) \equiv \zeta \frac{\mathrm{d}^2 z(\zeta)}{\mathrm{d}\zeta^2} \left(\frac{\mathrm{d}z(\zeta)}{\mathrm{d}\zeta} \right)^{-1}. \tag{4.2}$$

Now, suppose that ζ is on a circle centred at the origin of the complex ζ-plane and z is on a straight-line segment of the complex z-plane. Then,

$$\mathrm{Re}\left[\frac{\mathrm{d}F(z)}{\mathrm{d}z} \right] = 0 \tag{4.3}$$

and

$$\mathrm{Re}[\mathcal{S}(\zeta)] = -1. \tag{4.4}$$

Proof. A circle in the complex ζ-plane, denoted by C, which is centred at the origin and has radius r, is specified by the equation $\bar{\zeta} = r^2 \zeta^{-1}$, thus

$$\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta} = -\frac{\bar{\zeta}}{\zeta}, \quad \text{on } C. \tag{4.5}$$

If $\zeta \in C$, then $\overline{z(\zeta)}$ is also a function of $\bar{\zeta}$ so that the chain rule implies

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\zeta} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\zeta}} \frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta} = -\frac{\bar{\zeta}}{\zeta} \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\zeta}} \tag{4.6}$$

or

$$\zeta \frac{\mathrm{d}\bar{z}}{\mathrm{d}\zeta} = -\overline{\left(\zeta \frac{\mathrm{d}z}{\mathrm{d}\zeta} \right)}. \tag{4.7}$$

Now, a straight-line segment in the complex z-plane, denoted by P, is specified by the equation

$$\bar{z} = \epsilon z + \delta, \tag{4.8}$$

where ϵ and δ are some complex constants. Thus

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}z} = \epsilon, \quad z \in P. \tag{4.9}$$

The chain rule implies

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\zeta} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \epsilon\frac{\mathrm{d}z}{\mathrm{d}\zeta}, \quad z \in P. \tag{4.10}$$

Replacing, in equation (4.7), $\mathrm{d}\bar{z}/\mathrm{d}\zeta$ by the above expression, equation (4.7) becomes

$$\epsilon\zeta\frac{\mathrm{d}z}{\mathrm{d}\zeta} = -\overline{\left(\zeta\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)}, \quad \zeta \in C, \quad z \in P. \tag{4.11}$$

Using the definition of $F(z)$, this can be written as

$$\frac{\bar{F}}{F} = -\epsilon. \tag{4.12}$$

Differentiating this equation with respect to z, we find

$$\frac{1}{F}\frac{\mathrm{d}\bar{F}}{\mathrm{d}z} - \frac{\bar{F}}{F^2}\frac{\mathrm{d}F}{\mathrm{d}z} = 0. \tag{4.13}$$

The chain rule implies

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}z} = \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}}\frac{\mathrm{d}\bar{z}}{\mathrm{d}z} = \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}}\epsilon = -\frac{\bar{F}}{F}\frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}}, \tag{4.14}$$

where we have used equations (4.9) and (4.12). Finally, using equations (4.14) in (4.13), we find

$$-\frac{\bar{F}}{F^2}\left(\overline{\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)} + \left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)\right) = 0, \tag{4.15}$$

which implies the result (4.3).

The definitions of $F(z)$ and $\mathcal{S}(\zeta)$ imply

$$\frac{\mathrm{d}F}{\mathrm{d}z} = 1 + \mathcal{S}. \tag{4.16}$$

Indeed,

$$\frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\mathrm{d}F}{\mathrm{d}\zeta}\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-1} = \left[\frac{\mathrm{d}z}{\mathrm{d}\zeta} + \zeta\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}\right]\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-1}. \tag{4.17}$$

Equations (4.3) and (4.16) imply equation (4.4). ■

(*b*) *The construction of* $\mathcal{S}(\zeta)$

The target doubly connected domain D_z in the complex z-plane is specified by two closed polygons $P^{(0)}$ and $P^{(1)}$ which consist, respectively, of n_0 and n_1 straight-line segments. Each of these segments can be specified by

$$\bar{z} = \epsilon_k^{(j)} z + \delta_k^{(j)}, \quad j = 0, 1, \tag{4.18}$$

where, for $j=0$, $1 \le k \le n_0$ and, for $j=1$, $1 \le k \le n_1$. Let $z_k^{(j)}$ denote the vertices of the polygons and let $\alpha_k^{(j)}$ denote the associated prevertices in the ζ-plane. At each vertex of the polygon, the boundary direction shifts by a *turning angle*.

Let the turning angle at prevertex $\alpha_k^{(j)}$ be $\pi\beta_k^{(j)}$. The polygons $P^{(0)}$ and $P^{(1)}$ are closed, thus it is necessary that

$$\sum_{k=1}^{n_0} \beta_j^{(0)} = -2, \quad \sum_{k=1}^{n_1} \beta_j^{(1)} = 2. \tag{4.19}$$

Local arguments imply that near the prevertex $\alpha_k^{(j)}$,

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \left(\zeta - \alpha_k^{(j)}\right)^{\beta_k^{(j)}} g_k^{(j)}(\zeta), \tag{4.20}$$

where the function $g_k^{(j)}(\zeta)$ is analytic at $\alpha_k^{(j)}$.

Equation (4.4) will play a crucial role in the determination of the conformal mapping formula. In this respect, we note the following:

- (i) if z is on the segment specified by equation (4.18), then equation (4.11) is valid with ϵ replaced by $\epsilon_k^{(j)}$. This is a statement of the fact that $\zeta(\mathrm{d}z/\mathrm{d}\zeta)$ has piecewise constant argument on the two circles $C^{(0)}$ and $C^{(1)}$. This condition forms the basis of the approach of Crowdy (2005) for the construction of a general S–C mapping formula to polygonal domains of arbitrary finite connectivity. We note that although equation (4.11) with ϵ replaced by $\epsilon_k^{(j)}$ depends on the values of both j and k, both equations (4.3) and (4.4) are valid for *all* values of j and k,
- (ii) condition (4.4) is precisely the one used by DeLillo *et al.* (2001) in their construction of the doubly connected S–C formula and also in a generalization thereof to multiply connected polygonal domains (DeLillo *et al.* 2004). A geometrical interpretation of condition (4.4) is that the curvature on each straight-line segment of the polygonal boundary is zero.

Proposition 4.2. *Let $z(\zeta)$ be the conformal mapping from the annulus $\rho<|\zeta|<1$ in the complex ζ-plane to the doubly connected domain D_z in the complex z-plane bounded by the two closed polygons $P^{(0)}$ and $P^{(1)}$. These polygons are specified by the turning angles $\{\pi\beta_k^{(j)}|k=1,\dots,n_j;\ j=0,1\}$ and the prevertices $\{\alpha_k^{(j)}|k.=1,\dots,n_j;\ j=0,1\}$. Let $\mathcal{S}(\zeta)$ be defined in terms of $z(\zeta)$ by equation* (4.2). *Then, $\mathcal{S}(\zeta)$ is given by*

$$\mathcal{S}(\zeta) = \sum_{k=1}^{n_0} \beta_k^{(0)} K(\zeta/\alpha_k^{(0)}) + \sum_{k=1}^{n_1} \beta_k^{(1)} K(\zeta/\alpha_k^{(1)}) - 2, \tag{4.21}$$

where $K(\zeta)$ is defined in equation (3.3).

Proof. The function $\mathcal{S}(\zeta)$ has the following properties:

- (i) if ζ is on $C^{(0)}$ or $C^{(1)}$, then $\mathrm{Re}[\mathcal{S}(\zeta)] = -1$,
- (ii) it has a simple pole singularity at each of the points $\{\alpha_k^{(0)}|k=1,\dots,n_0\}$ and $\{\alpha_k^{(1)}|k=1,\dots,n_1\}$ with associated residues $\{\alpha_k^{(0)}\beta_k^{(0)}|k=1,\dots,n_0\}$ and $\{\alpha_k^{(1)}\beta_k^{(1)}|k=1,\dots,n_1\}$, and
- (iii) it is a loxodromic function.

Property (i) is a consequence of equation (4.4), (ii) follows from (4.20) and, in order to establish (iii), we must show that $\mathcal{S}(\zeta)$ satisfies the functional equation (3.8). In this respect, we define the Schwarz conjugate of $f(z)$ by

$$\hat{f}(z) = \overline{f(\bar{z})}. \tag{4.22}$$

Equation (4.4) can be written as

$$\overline{\mathcal{S}(\zeta)} + \mathcal{S}(\zeta) = -2 \tag{4.23}$$

or

$$\hat{\mathcal{S}}(\bar{\zeta}) + \mathcal{S}(\zeta) = -2. \tag{4.24}$$

Hence,

$$\begin{aligned} \hat{\mathcal{S}}(\zeta^{-1}) + \mathcal{S}(\zeta) &= -2, \quad \text{on } C^{(0)}, \\ \hat{\mathcal{S}}(\rho^2\zeta^{-1}) + \mathcal{S}(\zeta) &= -2, \quad \text{on } C^{(1)}. \end{aligned} \tag{4.25}$$

By analytic continuation, these equations imply

$$\hat{\mathcal{S}}(\zeta^{-1}) = \hat{\mathcal{S}}(\rho^2\zeta^{-1}), \quad \text{in } D_\zeta \tag{4.26}$$

or, equivalently,

$$\mathcal{S}(\zeta) = \mathcal{S}(\rho^2\zeta), \quad \text{in } D_\zeta. \tag{4.27}$$

We shall now construct $\mathcal{S}(\zeta)$ explicitly. Consider the candidate function

$$\mathcal{S}_C(\zeta) = \sum_{k=1}^{n_0} \beta_k^{(0)} K\left(\zeta/\alpha_k^{(0)}\right) + \sum_{k=1}^{n_1} \beta_k^{(1)} K\left(\zeta/\alpha_k^{(1)}\right) + c \tag{4.28}$$

for some constant c.

(i) The first equation in equation (3.4) implies that

$$\mathcal{S}_C(\rho^2\zeta) = \mathcal{S}_C(\zeta) - \sum_{k=1}^{n_0} \beta_k^{(0)} - \sum_{k=1}^{n_1} \beta_k^{(1)}, \tag{4.29}$$

therefore $\mathcal{S}_C(\zeta)$ is a loxodromic function in the light of the conditions (4.19).

(ii) The function $K(\zeta)$ has a simple pole at $\zeta = 1$ with unit residue, hence

$$K(\zeta\alpha^{-1}) = \frac{\alpha}{\zeta - \alpha} + \text{analytic}, \tag{4.30}$$

thus $\mathcal{S}_C(\zeta)$ has the required simple pole singularities with the correct residues.

(iii) For ζ on $C^{(0)}$,

$$\overline{\mathcal{S}_C(\zeta)} = \sum_{k=1}^{n_0} \beta_k^{(0)} K\left(\alpha_k^{(0)}/\zeta\right) + \sum_{k=1}^{n_1} \beta_k^{(1)} K\left(\alpha_k^{(1)}/\rho^2\zeta\right) + c. \tag{4.31}$$

Now equation (3.4) implies that

$$\begin{aligned} K\left(\alpha_k^{(0)}/\zeta\right) &= 1 - K\left(\zeta/\alpha_k^{(0)}\right), \\ K\left(\alpha_k^{(1)}/(\rho^2\zeta)\right) &= 1 - K\left(\rho^2\zeta/\alpha_k^{(1)}\right) = 2 - K\left(\zeta/\alpha_k^{(1)}\right). \end{aligned} \tag{4.32}$$

Substituting these relations into equation (4.31), we obtain

$$\overline{\mathcal{S}_C(\zeta)} = 2 + 2c - \mathcal{S}_C(\zeta). \tag{4.33}$$

A similar relation is valid for ζ on $C^{(1)}$. Thus, the condition $\mathrm{Re}[\mathcal{S}_C] = -1$ implies that we must choose $c = -2$.

In summary, the candidate function $\mathcal{S}_C(\zeta)$ with $c = -2$ has all the properties (i)–(iii) required of $\mathcal{S}(\zeta)$. By an application of Liouville's theorem (for loxodromic functions) to the ratio $\mathcal{S}(\zeta)/\mathcal{S}_C(\zeta)$, it can be deduced that this ratio is equal to unity so that $\mathcal{S}(\zeta)$ is given by equation (4.21). This completes the proof of proposition 4.2. ■

(*c*) *S–C mapping to doubly connected polygonal domains*

Using, in equation (4.21), the definitions of $K(\zeta)$ and $\mathcal{S}(\zeta)$ (equations (3.3) and (4.2)) and dividing by ζ, we find

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-1} = \sum_{k=1}^{n_0} \frac{\beta_k^{(0)}}{\alpha_k^{(0)}} \frac{P'(\zeta/\alpha_k^{(0)})}{P(\zeta/\alpha_k^{(0)})} + \sum_{k=1}^{n_1} \frac{\beta_k^{(1)}}{\alpha_k^{(1)}} \frac{P'(\zeta/\alpha_k^{(1)})}{P(\zeta/\alpha_k^{(1)})} - \frac{2}{\zeta}. \tag{4.34}$$

Integration of this equation yields

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{B}{\zeta^2} \prod_{k=1}^{n_0} \left[P\left(\zeta/a_k^{(0)}\right)\right]^{\beta_k^{(0)}} \prod_{k=1}^{n_1} \left[P\left(\zeta/a_k^{(1)}\right)\right]^{\beta_k^{(1)}}, \tag{4.35}$$

for some complex constant B. A further integration yields the S–C formula for the doubly connected case (Akhiezer 1928; Komatu 1945; Driscoll & Trefethen 2002).

5. Doubly connected polycircular arc maps

We now turn our attention to constructing the mapping to doubly connected polycircular arc domains. In particular, we first present the analogue of proposition 4.1.

(*a*) *Maps from circles centred at* $\zeta = 0$ *to circular arcs*

Proposition 5.1. *Let the functions* $F(z)$ *and* $\mathcal{T}(\zeta)$ *be defined respectively by equation* (4.1) *and*

$$\mathcal{T}(\zeta) \equiv \zeta^2 \{z(\zeta), \zeta\}, \tag{5.1}$$

where $\{z(\zeta), \zeta\}$ *is the Schwarzian derivative* (*Nehari* 1952; *Ablowitz & Fokas* 1997) *defined by*

$$\{z(\zeta), \zeta\} \equiv \frac{\mathrm{d}^3 z/\mathrm{d}\zeta^3}{\mathrm{d}z/\mathrm{d}\zeta} - \frac{3}{2}\left(\frac{\mathrm{d}^2 z/\mathrm{d}\zeta^2}{\mathrm{d}z/\mathrm{d}\zeta}\right)^2. \tag{5.2}$$

Suppose that ζ *is on a circle centred at the origin in the complex* ζ*-plane and* z *is on a circular arc in the complex* z*-plane. Then*

$$\overline{\left(F\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} - \frac{1}{2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2\right)} = F\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} - \frac{1}{2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2 \tag{5.3}$$

and

$$\overline{\mathcal{T}(\zeta)} = \mathcal{T}(\zeta). \tag{5.4}$$

Proof. It was shown in the proof of proposition 4.1 that if ζ is on a circle C centred at the origin, then equation (4.7) is valid.

Now, a circular arc, denoted by P, is specified by an equation of the form

$$\bar{z}-\bar{\Delta}=\frac{Q^2}{z-\Delta}, \tag{5.5}$$

for some $\Delta\in\mathbb{C}$ and some real $Q>0$. Differentiating this with respect to z, we find

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}=-\frac{\bar{z}-\bar{\Delta}}{z-\Delta}. \tag{5.6}$$

The chain rule implies

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\zeta}=\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}=-\left(\frac{\bar{z}-\bar{\Delta}}{z-\Delta}\right)\frac{\mathrm{d}z}{\mathrm{d}\zeta}. \tag{5.7}$$

Replacing, in this equation, $\mathrm{d}\bar{z}/\mathrm{d}\zeta$ by the right-hand side of equation (4.6), we find

$$\overline{\left(\frac{\zeta\mathrm{d}z/\mathrm{d}\zeta}{z-\Delta}\right)}=\frac{\zeta\mathrm{d}z/\mathrm{d}\zeta}{z-\Delta},\quad \zeta\in C,\quad z\in P. \tag{5.8}$$

This equation is the analogue of equation (4.11).

Differentiating equation (5.6) with respect to z produces

$$\frac{\mathrm{d}^2\bar{z}}{\mathrm{d}z^2}=-\frac{\mathrm{d}\bar{z}/\mathrm{d}z}{z-\Delta}+\frac{\bar{z}-\bar{\Delta}}{(z-\Delta)^2}=2\frac{\bar{z}-\bar{\Delta}}{(z-\Delta)^2}, \tag{5.9}$$

where we have used equation (5.6). Combining this equation with equation (5.6), we find

$$\frac{\mathrm{d}\bar{z}/\mathrm{d}z}{\mathrm{d}^2\bar{z}/\mathrm{d}z^2}=-\frac{1}{2}(z-\Delta). \tag{5.10}$$

Differentiating equation (5.10) with respect to z, we obtain

$$-\frac{\mathrm{d}^3\bar{z}}{\mathrm{d}z^3}+\frac{3}{2}\frac{(\mathrm{d}^2\bar{z}/\mathrm{d}z^2)^2}{\mathrm{d}\bar{z}/\mathrm{d}z}=0,\quad \zeta\in C,\quad z\in P. \tag{5.11}$$

It should be emphasized that equation (5.11) has no dependence on either Q or Δ.
We will now rewrite equation (5.11) in terms of the function $F(z)$. The chain rule implies

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\zeta}=\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\zeta}}\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta} \tag{5.12}$$

or

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}=\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{\zeta}}\left(-\frac{\bar{\zeta}}{\zeta}\right), \tag{5.13}$$

where we have made use of equation (4.5) to replace $\mathrm{d}\bar{\zeta}/\mathrm{d}\zeta$ by $-\bar{\zeta}/\zeta$. This equation can be rewritten in the form

$$-\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}=\overline{\left(\zeta\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)}\Big/\left(\zeta\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)=\frac{\bar{F}}{F}. \tag{5.14}$$

Direct calculations lead to the following:

$$\begin{aligned} \frac{\mathrm{d}^2\bar{z}}{\mathrm{d}z^2} &= \frac{\bar{F}}{F^2}\left(\frac{\mathrm{d}F}{\mathrm{d}z} + \overline{\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)}\right), \\ \frac{\mathrm{d}^3\bar{z}}{\mathrm{d}z^3} &= -\frac{\bar{F}}{F^3}\left(\bar{F}\overline{\left(\frac{\mathrm{d}^2F}{\mathrm{d}z^2}\right)} - F\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \overline{\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2} + 3\frac{\mathrm{d}F}{\mathrm{d}z}\overline{\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)} + 2\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2\right). \end{aligned} \tag{5.15}$$

Substitution of equation (5.15) into equation (5.11) yields equation (5.3).

On differentiation of equation (4.1), the definition of $F(z)$, with respect to z, it is easy to verify that the following relations are valid:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}z} &= 1 + \zeta\frac{\mathrm{d}\zeta}{\mathrm{d}z}\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2}, \\ \frac{\mathrm{d}^2F}{\mathrm{d}z^2} &= \left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2} + \zeta\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2} + \zeta\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^3z}{\mathrm{d}\zeta^3}. \end{aligned} \tag{5.16}$$

These imply that

$$F\frac{\mathrm{d}^2F}{\mathrm{d}z^2} - \frac{1}{2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2 = -\frac{1}{2} + \zeta^2\frac{\mathrm{d}z}{\mathrm{d}\zeta}\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2} + \zeta^2\frac{\mathrm{d}\zeta}{\mathrm{d}z}\frac{\mathrm{d}^3z}{\mathrm{d}\zeta^3} - \frac{1}{2}\left(\zeta\frac{\mathrm{d}\zeta}{\mathrm{d}z}\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2}\right)^2. \tag{5.17}$$

Using the identity

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2} = -\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^3, \tag{5.18}$$

equation (5.17) simplifies as follows:

$$F\frac{\mathrm{d}^2F}{\mathrm{d}z^2} - \frac{1}{2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2 = -\frac{1}{2} + \zeta^2\frac{\mathrm{d}\zeta}{\mathrm{d}z}\frac{\mathrm{d}^3z}{\mathrm{d}\zeta^3} - \frac{3}{2}\zeta^2\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^2\left(\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}\right)^2. \tag{5.19}$$

Finally, this equation and the definition of $\mathcal{T}(\zeta)$ yield equation (5.4). This completes the proof of proposition 5.1. ∎

(*b*) *The construction of* $\mathcal{T}(\zeta)$

The following proposition is the analogue of proposition 4.2.

Proposition 5.2. *Let $z(\zeta)$ be the conformal mapping of the annulus $\rho<|\zeta|<1$ in the complex ζ-plane to the doubly connected domain D_z in the complex z-plane bounded by the two closed polycircular curves $P^{(0)}$ and $P^{(1)}$. These curves are specified by equations* (2.1) *and* (2.2). *Let $\{\pi\beta_k^{(j)}|k=1,\ldots,n_j;\ j=0,1\}$ be the turning angles at the prevertices $\{\alpha_k^{(j)}|k=1,\ldots,n_j;\ j=0,1\}$, respectively.*

Let $\mathcal{T}(\zeta)$ *be defined in terms of* $z(\zeta)$ *by equation* (5.1). *Then,* $\mathcal{T}(\zeta)$ *is given by*

$$\mathcal{T}(\zeta) = \sum_{k=1}^{n_0} \left(\frac{\left[\beta_k^{(0)}\right]^2 - 1}{2} \right) L\left(\zeta/\alpha_k^{(0)}\right) + \mathrm{i}\gamma_k^{(0)} K\left(\zeta/\alpha_k^{(0)}\right) + \sum_{k=1}^{n_1} \left(\frac{\left[\beta_k^{(1)}\right]^2 - 1}{2} \right) L\left(\zeta/\alpha_k^{(1)}\right) + \mathrm{i}\gamma_k^{(1)} K\left(\zeta/\alpha_k^{(1)}\right) + c, \tag{5.20}$$

where $\{\gamma_k^{(j)} | k=1, \ldots, n_j;\ j=0,1\}$ *are real constants satisfying the conditions*

$$\sum_{k=1}^{n_0} \gamma_k^{(0)} + \sum_{k=1}^{n_1} \gamma_k^{(1)} = 0, \tag{5.21}$$

while c *is a complex constant satisfying*

$$c - \bar{c} = -\mathrm{i} \sum_{k=1}^{n_1} \gamma_k^{(1)}. \tag{5.22}$$

Proof. The function $\mathcal{T}(\zeta)$ has the following properties:

(i) if ζ is on $C^{(0)}$ or $C^{(1)}$, then $\overline{\mathcal{T}(\zeta)} = \mathcal{T}(\zeta)$,
(ii) it has a second-order pole at each of the points $\{\alpha_k^{(j)} | k=1, \ldots, n_j;\ j=0,1\}$ with associated strengths $\{[\alpha_k^{(j)}]^2(1-[\beta_k^{(j)}]^2)/2 | k=1, \ldots, n_j;\ j=0,1\}$, and
(iii) it is a loxodromic function.

Indeed, (i) is just a restatement of equation (5.4). Regarding (ii), we note that local arguments imply that near the prevertex $\alpha_k^{(j)}$, equation (4.20) is still valid. Then, a simple calculation (Nehari 1952; Ablowitz & Fokas 1997) shows that

$$\{z(\zeta), \zeta\} = \frac{1}{2} \frac{1 - \left[\beta_k^{(j)}\right]^2}{\left(\zeta - \alpha_k^{(j)}\right)^2} + \mathcal{O}\left(\left(\zeta - \alpha_k^{(j)}\right)^{-1}\right). \tag{5.23}$$

Regarding (iii), if we denote the Schwarz conjugate of $\mathcal{T}(\zeta)$ by $\hat{\mathcal{T}}(\zeta)$, i.e.

$$\hat{\mathcal{T}}(\zeta) = \overline{\mathcal{T}(\bar{\zeta})}, \tag{5.24}$$

then equation (5.4) can be written as $\hat{\mathcal{T}}(\bar{\zeta}) = \mathcal{T}(\zeta)$. Hence,

$$\begin{aligned} \hat{\mathcal{T}}(\zeta^{-1}) &= \mathcal{T}(\zeta), \quad \text{on } C^{(0)}, \\ \hat{\mathcal{T}}(\rho^2\zeta^{-1}) &= \mathcal{T}(\zeta), \quad \text{on } C^{(1)}. \end{aligned} \tag{5.25}$$

Analytic continuation of these equations into the annulus $\rho < |\zeta| < 1$ yields $\hat{\mathcal{T}}(\zeta^{-1}) = \hat{\mathcal{T}}(\rho^2\zeta^{-1})$ and hence $\mathcal{T}(\rho^2\zeta) = \mathcal{T}(\zeta)$. This condition, together with the fact that the only singularities of $\mathcal{T}(\zeta)$ are the second-order poles at the prevertices $\{\alpha_k^{(j)} | k=1, \ldots, n_j;\ j=0,1\}$ imply that $\mathcal{T}(\zeta)$ is a loxodromic function.

Let $\mathcal{T}_C(\zeta)$ defined by

$$\mathcal{T}_C(\zeta) = \sum_{k=1}^{n_0}\left(\frac{\left[\beta_k^{(0)}\right]^2-1}{2}\right)L\left(\zeta/\alpha_k^{(0)}\right) + \mathrm{i}\gamma_k^{(0)}K\left(\zeta/\alpha_k^{(0)}\right)$$
$$+\sum_{k=1}^{n_1}\left(\frac{\left[\beta_k^{(1)}\right]^2-1}{2}\right)L\left(\zeta/\alpha_k^{(1)}\right) + \mathrm{i}\gamma_k^{(1)}K\left(\zeta/\alpha_k^{(1)}\right) + c, \tag{5.26}$$

be a candidate function for the required function $\mathcal{T}(\zeta)$. First, from equation (3.7), it is easy to show that $\mathcal{T}_C(\zeta)$ satisfies (ii). Next, equations (3.4) and (3.6) and the imposed conditions (5.21) can be used to verify that $\mathcal{T}_C(\zeta)$ is a loxodromic function and therefore satisfies condition (iii).

Finally, note that if ζ is on $C^{(0)}$, then

$$\overline{\mathcal{T}_C}(\zeta^{-1}) = \sum_{k=1}^{n_0}\left(\frac{\left[\beta_k^{(0)}\right]^2-1}{2}\right)L\left(\alpha_k^{(0)}/\zeta\right) - \mathrm{i}\gamma_k^{(0)}K\left(\alpha_k^{(0)}/\zeta\right)$$
$$+\sum_{k=1}^{n_1}\left(\frac{\left[\beta_k^{(1)}\right]^2-1}{2}\right)L\left(\alpha_k^{(1)}/(\rho^2\zeta)\right) - \mathrm{i}\gamma_k^{(1)}K\left(\alpha_k^{(1)}/(\rho^2\zeta)\right) + \bar{c},$$
$$= \sum_{k=1}^{n_0}\left(\frac{\left[\beta_k^{(0)}\right]^2-1}{2}\right)L\left(\zeta/\alpha_k^{(0)}\right) + \mathrm{i}\gamma_k^{(0)}K\left(\zeta/\alpha_k^{(0)}\right) - \mathrm{i}\sum_{k=1}^{n_0}\gamma_k^{(0)}$$
$$+\sum_{k=1}^{n_1}\left(\frac{\left[\beta_k^{(1)}\right]^2-1}{2}\right)L\left(\zeta/\alpha_k^{(1)}\right) + \mathrm{i}\gamma_k^{(1)}K\left(\zeta/\alpha_k^{(1)}\right) - 2\mathrm{i}\sum_{k=1}^{n_1}\gamma_k^{(1)} + \bar{c},$$
$$= \mathcal{T}_C(\zeta) - c + \bar{c} - \mathrm{i}\sum_{k=1}^{n_0}\gamma_k^{(0)} - 2\mathrm{i}\sum_{k=1}^{n_1}\gamma_k^{(1)},$$
$$= \mathcal{T}_C(\zeta), \tag{5.27}$$

where we have made use of equations (3.4), (3.6), (5.21) and (5.22). A similar calculation shows that $\overline{\mathcal{T}_C}(\rho^2\zeta^{-1}) = \mathcal{T}_C(\zeta)$. The candidate function $\mathcal{T}_C(\zeta)$ therefore satisfies all conditions (i)–(iii) required of $\mathcal{T}(\zeta)$. Finally, an application of the generalized Liouville theorem (for loxodromic functions) yields the result (5.20). This completes the proof of proposition 5.2. ■

6. The parameter problem

Proposition 5.2 reduces the construction of the required conformal mapping to solving the third-order nonlinear differential equation

$$\zeta^2\{z(\zeta),\zeta\} = \mathcal{T}(\zeta) \tag{6.1}$$

with the functional form of $\mathcal{T}(\zeta)$ given in equation (5.20) up to a finite set of accessory parameters.

However, there are additional conditions on the parameters appearing in the expression for $\mathcal{T}(\zeta)$. To see this, first note that a polycircular region with a total of N sides depends on $3N$ real parameters (one can specify, for example, the centre and radius of each circular arc). On the other hand, the ordinary differential equation (6.1) is invariant under the arbitrary Möbius mappings of the solution $z(\zeta)$, i.e. if

$$Z(\zeta) = \frac{az(\zeta)+b}{cz(\zeta)+d}, \quad a,b,c,d\in\mathbb{C}, \quad ad-bc=1, \tag{6.2}$$

then the equation satisfied by $Z(\zeta)$ is the same as that satisfied by $z(\zeta)$, i.e.

$$\zeta^2\{Z(\zeta),\zeta\} = \mathcal{T}(\zeta). \tag{6.3}$$

There are six real degrees of freedom associated with the Möbius mapping (6.2); these are precisely the six real constants needed to specify (initial) conditions on, say, $z(1)$, $z_\zeta(1)$ and $z_{\zeta\zeta}(1)$ (where subscripts denote derivatives) in order to solve the third-order ordinary differential equation (6.1). It follows that equation (6.1) should depend on precisely $3N-6$ real parameters in order to describe a given polycircular arc region with N sides.

However, the parameters appearing in the representation (5.20) are

$$\left\{\alpha_i^{(0)},\alpha_j^{(1)}|i=1,\dots,n_0;\ j=1,\dots,n_1\right\}, \quad \left\{\beta_i^{(0)},\beta_j^{(1)}|i=1,\dots,n_0;\ j=1,\dots,n_1\right\},$$
$$\{\gamma_i^{(0)},\gamma_j^{(1)}|i=1,\dots,n_0;\ j=1,\dots,n_1\}, \quad c\in\mathbb{C} \text{ and } \rho\in\mathbb{R}. \tag{6.4}$$

The total number of sides N is n_0+n_1 and, counting all the parameters in equation (6.4), it follows that $\mathcal{T}(\zeta)$ depends on a total of $3N+3$ real parameters.

Now, these parameters are constrained by the two real relations (5.21) and (5.22), which reduce the count to $3N+1$ real parameters. In addition, there remains a single rotational degree of freedom associated with the Riemann mapping theorem (the fact that we have specified that P_1 corresponds to C_1, centred at the origin, has used up two of the usual three degrees of freedom associated with the Riemann mapping theorem). Taking this freedom into account reduces the count to $3N$ parameters. It is clear that we have still to determine six real constraints on the parameters appearing in $\mathcal{T}(\zeta)$.

Before presenting these constraints, we first consider the simply connected case. Nehari (1952) also discusses the question of the 'parameter count' in the case of simply connected mappings from an upper-half plane.

(*a*) *Simply connected case*

The simply connected case corresponds to $\rho \to 0$. Then, we have

$$L(\zeta) = -\frac{\zeta}{(1-\zeta)^2}, \quad K(\zeta) = -\frac{\zeta}{(1-\zeta)}. \tag{6.5}$$

Thus, equation (5.20) becomes

$$\mathcal{T}(\zeta) = \sum_{k=1}^{n_0} \left(\frac{1-\left[\beta_k^{(0)}\right]^2}{2} \right) \frac{\zeta \alpha_k^{(0)}}{\left(\zeta - \alpha_k^{(0)}\right)^2} + \sum_{k=1}^{n_0} \frac{\mathrm{i}\gamma_k \zeta}{\left(\zeta - \alpha_k^{(0)}\right)} + c. \tag{6.6}$$

In this case, condition (5.21) disappears while equation (5.22) assumes the form

$$-c + \bar{c} - \mathrm{i} \sum_{n=1}^{n_0} \gamma_k^{(0)} = 0. \tag{6.7}$$

In this case, the total number of sides is $N = n_0$ and the total number of parameters in equation (6.6) is $3N+2$. Equation (6.7) is a single real constraint and there are also the usual three real degrees of freedom associated with the Riemann mapping theorem. This reduces the count to $3N-2$. Therefore, there must exist four additional real conditions on the parameters in equation (6.6). These four conditions follow from the two complex conditions

$$\mathcal{T}(0) = \mathcal{T}_\zeta(0) = 0. \tag{6.8}$$

These are a direct consequence of equation (5.1): since $z(\zeta)$ is analytic at $\zeta = 0$, it follows that the function $\{z(\zeta), \zeta\}$ has a finite value at $\zeta = 0$; hence, by equation (5.1), $\mathcal{T}(\zeta)$ has a double zero there. This, together with the fact that $\mathcal{T}(\zeta)$ is analytic at $\zeta = 0$, leads to conditions (6.8).

From equation (6.6), the condition $\mathcal{T}(0) = 0$ clearly requires that $c = 0$. Hence

$$\sum_{n=1}^{n_0} \gamma_k^{(0)} = 0. \tag{6.9}$$

Now equation (6.6) can be written as

$$\mathcal{T}(\zeta) = \zeta^2 \left(\sum_{k=1}^{n_0} \left(\frac{1-[\beta_k^{(0)}]^2}{2} \right) \frac{\alpha_k^{(0)}}{\zeta\left(\zeta - \alpha_k^{(0)}\right)^2} + \sum_{k=1}^{n_0} \frac{\mathrm{i}\gamma_k}{\zeta\left(\zeta - \alpha_k^{(0)}\right)} \right). \tag{6.10}$$

The condition $\mathcal{T}_\zeta(0) = 0$ yields

$$\sum_{n=1}^{n_0} \left[\frac{1-[\beta_k^{(0)}]^2}{2\alpha_k^{(0)}} - \frac{\mathrm{i}\gamma_k^{(0)}}{\alpha_k^{(0)}} \right] = 0. \tag{6.11}$$

Hence, equation (6.10) becomes

$$\mathcal{T}(\zeta)=\zeta^2\left(\sum_{k=1}^{n_0}\left(\frac{1-\left[\beta_k^{(0)}\right]^2}{2}\right)\frac{\left(2-\zeta/\alpha_k^{(0)}\right)}{\left(\zeta-\alpha_k^{(0)}\right)^2}+\frac{\mathrm{i}\gamma_k^{(0)}}{\alpha_k^{(0)}\left(\zeta-\alpha_k^{(0)}\right)}\right)$$

$$=\zeta^2\left(\sum_{k=1}^{n_0}\left(\frac{1-\left[\beta_k^{(0)}\right]^2}{2\left(\zeta-\alpha_k^{(0)}\right)^2}\right)-\left(\frac{1-\left[\beta_k^{(0)}\right]^2}{2\alpha_k^{(0)}\left(\zeta-\alpha_k^{(0)}\right)}\right)-\frac{\mathrm{i}\gamma_k^{(0)}}{\alpha_k^{(0)}\left(\zeta-\alpha_k^{(0)}\right)}\right). \qquad (6.12)$$

In summary, we deduce that

$$\{z(\zeta),\zeta\}=\frac{1}{2}\left[\sum_{k=1}^{n_0}\frac{1-\left[\beta_k^{(0)}\right]^2}{\left(\zeta-\alpha_k^{(0)}\right)^2}-\frac{1-\left[\beta_k^{(0)}\right]^2}{\alpha_k^{(0)}\left(\zeta-\alpha_k^{(0)}\right)}-\frac{2\mathrm{i}\gamma_k^{(0)}}{\alpha_k^{(0)}\left(\zeta-\alpha_k^{(0)}\right)}\right], \qquad (6.13)$$

with the parameters satisfying the constraints

$$\sum_{n=1}^{n_0}\gamma_k^{(0)}=0,\quad \sum_{n=1}^{n_0}\left[\frac{1-\left[\beta_k^{(0)}\right]^2}{2\alpha_k^{(0)}}-\frac{\mathrm{i}\gamma_k^{(0)}}{\alpha_k^{(0)}}\right]=0. \qquad (6.14)$$

After appropriate changes in notation, equations (6.13) and (6.14) are exactly the results stated in theorem 4.1 of Howell (1993) on mapping the unit ζ-disc to a simply connected polycircular arc domain.

(*b*) *The doubly connected case*

We have already established that $\mathcal{T}(\zeta)$, as given in equation (5.20), depends on $3N$ real parameters. The six real constraints satisfied by these parameters are characterized by the following three complex equations:

$$\oint_C\left[\mathcal{T}(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}+\frac{3}{2}\zeta^2\frac{\mathrm{d}^2z}{\mathrm{d}\zeta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{-1}\right]\frac{\mathrm{d}\zeta}{\zeta^{n+1}}=0,\quad n=-1,0,1, \qquad (6.15)$$

where C is a simple closed contour in the annulus $\rho<|\zeta|<1$. Indeed, in this annulus, $z(\zeta)$ has a convergent Laurent expansion of the form

$$z(\zeta)=\cdots+\frac{a_{-2}}{\zeta^2}+\frac{a_{-1}}{\zeta}+a_0+a_1\zeta+a_2\zeta^2+\cdots \qquad (6.16)$$

Hence, $\zeta^2(\mathrm{d}^3z/\mathrm{d}\zeta^3)$ admits a Laurent series expansion where the coefficients of ζ^n for $n=-1$, 0, 1 vanish. Using equations (5.1) and (5.2), the latter conditions imply conditions (6.15).

In summary, we have now derived the functional form of a nonlinear equation (6.1), up to a finite set of parameters, for the required conformal mapping function $z(\zeta)$. There are a number of constraints on these parameters, the nature of which we have determined. In the next section, we consider two examples where symmetry consideration simplifies the parameter problem considerably and allows us to numerically solve equation (6.1) to construct the required conformal map.

7. Examples

To illustrate the constructive method, we consider some examples. First, we present an analytical check of the general method by constructing a special exact solution for a polycircular arc mapping using independent considerations. Then, in §7*b*, we use the new method to construct a mapping to a polycircular arc domain arising in a problem considered by Lord Rayleigh (1892).

In order to avoid complications associated with solving the parameter problem (Driscoll & Trefethen 2002), we consider examples that have both a rotational symmetry as well as a reflectional symmetry about the real z-axis. If the boundaries $P^{(0)}$ and $P^{(1)}$ of D_z have an n-fold rotational symmetry about the origin $z=0$, then the positions of the prevertices on $C^{(0)}$ and $C^{(1)}$ will also have this symmetry. The solutions we seek will then be of the general functional form

$$z(\zeta) = \zeta h(\zeta^n) \tag{7.1}$$

for some function h. Using this general form, we can show that $\mathcal{T}(\zeta)$ is invariant under the transformation $\zeta \mapsto \zeta\, e^{2\pi i/n}$. This implies that the coefficients $\{\gamma_k^{(j)} | k=1, \dots, n_j; j=0,1\}$ must be equal for all k. On the other hand, reflectional symmetry about the real axis implies that $\hat{z}(\zeta)=z(\zeta)$ (where $\hat{z}(\zeta)$ denotes the Schwarz conjugate to $z(\zeta)$) and hence $\hat{\mathcal{T}}(\zeta)=\mathcal{T}(\zeta)$ (it also implies that the prevertices must appear in complex conjugate pairs). This relation implies that the coefficients $\gamma_k^{(j)}$ corresponding to complex conjugate pairs of prevertices must be complex conjugates. Combining these facts, we conclude that all coefficients $\{\gamma_k^{(j)} | k=1, \dots, n_j;\ j=0,1\}$ for such rotationally and reflectionally symmetric domains must vanish.

(*a*) *An analytical check*

It is shown in appendix B, using independent considerations, that the conformal mapping from the annulus $\rho<|\zeta|<1$ to the unit disc with a symmetrical slit along the real axis (figure 5) is given by

$$z(\zeta) = \frac{P(-\zeta) - P(\zeta)}{P(-\zeta) + P(\zeta)}. \tag{7.2}$$

In what follows, as an independent verification of our method, we will compare formula (7.2) with the formula obtained using proposition 5.2. In this case, proposition 5.2 implies

$$\mathcal{T}(\zeta) = \frac{3}{2}\left(L(\zeta\rho^{-1}) + L(-\zeta\rho^{-1})\right) + R_1 \tag{7.3}$$

for some appropriately chosen real constant R_1. Here, we have used the fact that $\alpha_1^{(1)}=\rho$, $\alpha_2^{(1)}=-\rho$, $\beta_1^{(1)}=\beta_2^{(1)}=1$.

In practice, it would be necessary to solve for the accessory parameter R_1 numerically. Here, however, we already know that the relevant map is equation (7.2). We can therefore check that the result of equation (7.2) is consistent with equation (7.3) by picking an arbitrary value of ζ and evaluating the left-hand side of equation (7.3) at this point using the known function (7.2). This determines R_1. With this value of R_1, it can then be verified that the functional relation (7.3) holds for *any* choice of ζ. This was tested numerically and was found to be the case, thus providing a check on the general analysis.

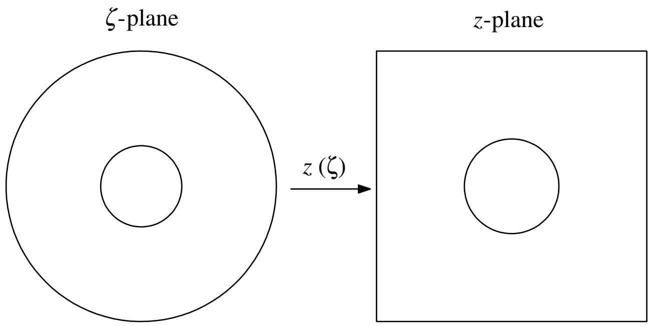

Figure 2. Schematic of the mapping from an annulus to a single period window of the problem considered by Lord Rayleigh (1892).

(b) *Rayleigh's example*

We now consider a classical example that is of great interest for applications. Lord Rayleigh (1892) analysed the problem of potential flow past a doubly periodic array of cylindrical objects. One approach to the analysis of such problems is to construct a conformal mapping from the annulus $\rho<|\zeta|<1$ to a single period window of this array (see figure 2 for a schematic of this target domain). It is clear that such a domain is a polycircular arc region. Rayleigh's example is also studied by Bjørstad & Grosse (1987), but they exploit the fourfold rotational symmetry of the target domain to find a mapping from a simply connected preimage domain to one quarter of the full domain. Here, in contrast, we seek a mapping from the preimage annulus to the full doubly connected domain.

Let the unit circle $|\zeta|=1$ map to the square outer boundary of the target domain. By the fourfold rotational symmetry, the preimage points on the unit ζ can be expected to be at the points

$$\alpha_k^{(0)} = \mathrm{e}^{\mathrm{i}\pi(2k-1)/4}. \tag{7.4}$$

The function $\mathcal{T}(\zeta)$ has second-order poles at the points given in equation (7.4). In order to encode the fourfold rotational symmetry, it is natural to consider the function

$$P_4(\zeta) = \prod_{k=1}^{4} P\left(\zeta/\alpha_k^{(0)}\right) = (1+\zeta^4)\prod_{j=1}^{\infty}\left(1+\rho^{8j}\zeta^4\right)\left(1+\rho^{8j}\right)/\zeta^4\big), \tag{7.5}$$

which is simply a product of four prime functions defined in equation (3.1), but with arguments chosen so that the zeros of $P_4(\zeta)$ are exactly the points $\{\alpha_k^{(0)}\}$.

By analogy with $K(\zeta)$ and $L(\zeta)$, now define

$$K_4(\zeta) \equiv \zeta\frac{P_4'(\zeta)}{P_4(\zeta)}, \quad L_4(\zeta) \equiv \zeta K_4'(\zeta). \tag{7.6}$$

It is easy to verify that $K_4(\zeta)$ has simple poles, and $L_4(\zeta)$ has second-order poles, at the points (7.4). From the results of proposition 5.2, we find that

$$\mathcal{T}(\zeta) = -\frac{3}{8}L_4(\zeta) + R_1. \tag{7.7}$$

The single real accessory parameter R_1 must be determined as part of the solution.

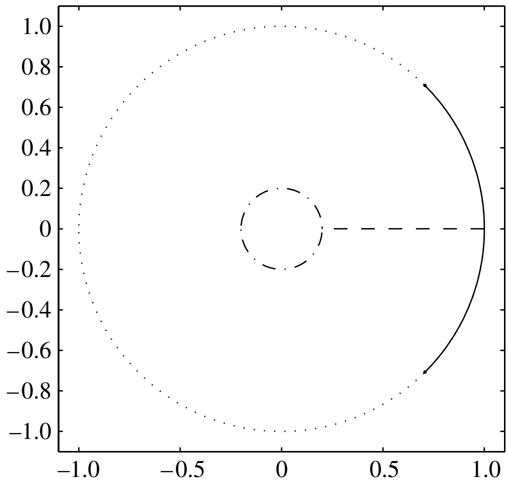

Figure 3. The annulus $0.2<|\zeta|<1$. The solid, dashed and dot-dashed curves are the three curves described in equation (7.12).

In order to evaluate the conformal mapping and the value of R_1 numerically, equation (6.1) is written in the form

$$\frac{d^3z}{d\zeta^3} = \frac{3}{2}\left(\frac{d^2z}{d\zeta^2}\right)^2 \bigg/ \frac{dz}{d\zeta} + \frac{\mathcal{T}(\zeta)}{\zeta^2}\frac{dz}{d\zeta}, \tag{7.8}$$

with $\mathcal{T}(\zeta)$ given by equation (7.7). Defining $p=dz/d\zeta$ and $q=d^2z/d\zeta^2$, equation (7.8) can be written as the following system for (z, p, q):

$$\frac{dz}{d\zeta} = p, \quad \frac{dp}{d\zeta} = q, \quad \frac{dq}{d\zeta} = \frac{3}{2}\frac{q^2}{p} + \frac{\mathcal{T}(\zeta)p}{\zeta^2}. \tag{7.9}$$

The initial conditions are chosen as follows. We choose $z=0$ to be the image of $\zeta=1$. Since $z=0$ is on a straight-line segment, we demand that the curvature at this point vanishes. It is easy to show that the curvature κ is given by the formula

$$\kappa = \left|\frac{d\zeta}{dz}\right| \mathrm{Re}\left[1 + \zeta\frac{d^2z}{d\zeta^2} \bigg/ \frac{dz}{d\zeta}\right]. \tag{7.10}$$

Hence, we take initial conditions

$$z(1) = 0, \quad p(1) = r, \quad q(1) = -r, \tag{7.11}$$

for some real parameter r. The value of r simply determines the scale of the target domain, so we arbitrarily set $r=1$. We also choose $\rho=0.2$.

With these initial conditions, and an arbitrary choice of R_1, the system (7.9) was integrated numerically along the following three curves in the ζ-plane:

$$\{|\zeta| = 1, -\pi/4 \le \arg[\zeta] \le \pi/4\}, \quad \{\rho \le \zeta \le 1, \arg[\zeta] = 0\}, \quad |\zeta| = \rho. \tag{7.12}$$

These curves are shown as solid, dashed and dot-dashed curves, respectively, in figure 3. This integration procedure was repeated for different values of R_1 until the image of these three curves was as required. It is found (numerically) that

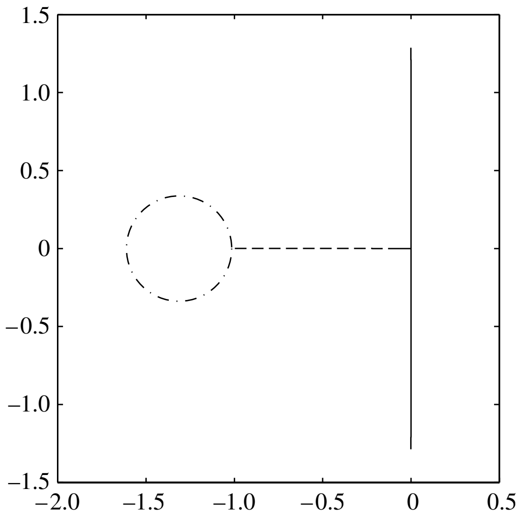

Figure 4. The images of the solid, dashed and dot-dashed curves of figure 3. Note that $z(1)=0$. The radius of the inner enclosed circle is found to be centred at -1.311 and to have radius 0.283. The length of each square side is found to be 2.575.

$R_1=0$ is the appropriate value of this accessory parameter. The numerically computed images of these three curves, when $R_1=0$, are shown in figure 4 (also as solid, dashed and dot-dashed curves). Of course, the rest of the shape will follow from the fourfold symmetry of the mapping. It was also checked numerically that, with $R_1=0$, different values of r simply produce rescaled versions of the same image.

In summary, the conformal mapping $z(\zeta)$ from the annulus $\rho<|\zeta|<1$ to the polycircular arc domain relevant to Rayleigh's problem (figure 2) is the solution of the following explicit nonlinear ordinary differential equation

$$\frac{\mathrm{d}^3 z}{\mathrm{d}\zeta^3}=\frac{3}{2}\left(\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}\right)^2 \Big/ \frac{\mathrm{d}z}{\mathrm{d}\zeta}-\frac{3}{8\zeta}L_4(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}, \tag{7.13}$$

where $L_4(\zeta)$ is defined by equations (7.5) and (7.6).

8. Discussion

This paper has presented a new procedure for constructing conformal maps to doubly connected polycircular arc domains. Some simple illustrative examples of the method have also been presented. It is expected that the formulation here will form the basis of a general numerical scheme for constructing conformal mappings to doubly connected polycircular arc domains.

It is worth pointing out that Hu (1998) has already developed software to numerically construct S–C mappings to doubly connected polygonal domains based on the doubly connected S–C formula. It would be of interest to examine whether such software can be extended to doubly connected polycircular arc domains using the new formulae presented in this paper. Actually, there is a wider issue of whether existing software (such as the S–C Toolbox www.math.udel.edu/-driscoll/SC) can be enhanced to incorporate mappings to multiply connected polygonal and polycircular arc domains.

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Appendix A. Properties of $P(\zeta)$

For the sake of completeness, we derive equations (3.2), (3.4) and (3.6). The definition (3.1) of $P(\zeta)$ implies

$$P(\rho^2\zeta) = (1-\rho^2\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\rho^2\zeta)(1-\rho^{2k}\rho^{-2}\zeta^{-1}). \tag{A 1}$$

On use of the identities

$$\begin{aligned}\prod_{k=1}^{\infty}(1-\rho^{2k}\rho^{-2}\zeta^{-1}) &= (1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1}),\\ (1-\rho^2\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\rho^2\zeta) &= \prod_{k=1}^{\infty}(1-\rho^{2k}\zeta),\end{aligned} \tag{A 2}$$

it can be shown that the right-hand side of (A 1) becomes

$$(1-\zeta^{-1})\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta^{-1})(1-\rho^{2k}\zeta), \tag{A 3}$$

and the first identity of equation (3.2) follows.

The second identity in equation (3.2) is a direct consequence of the following invariance:

$$\frac{P(\zeta)}{1-\zeta} = \frac{P(\zeta^{-1})}{1-\zeta^{-1}}. \tag{A 4}$$

Differentiation of the first of the identities (3.2) with respect to ζ yields

$$\rho^2 P'(\rho^2\zeta) = \zeta^{-2}P(\zeta) - \zeta^{-1}P'(\zeta), \tag{A 5}$$

which, on division by the equation

$$\zeta^{-1}P(\rho^2\zeta) = -\zeta^{-2}P(\zeta), \tag{A 6}$$

leads to the first of the identities in equation (3.4). Differentiation of the second of the identities (3.2) with respect to ζ implies

$$-\zeta^{-2}P'(\zeta^{-1}) = \zeta^{-2}P(\zeta) - \zeta^{-1}P'(\zeta), \tag{A 7}$$

which, on division by the equation

$$-\zeta^{-1}P(\zeta^{-1}) = \zeta^{-2}P(\zeta), \tag{A 8}$$

yields the second of the identities in equation (3.4).

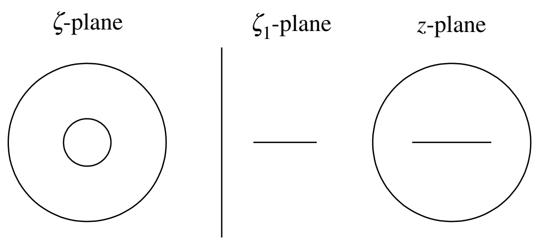

Figure 5. Schematic of the sequence of conformal maps (B 1) leading to formula (7.2).

Finally, the identities (3.6) follow by differentiating equation (3.4) with respect to ζ.

By employing appropriate transformations of the parameters and of the independent variable, the functions $P(\zeta)$, $K(\zeta)$ and $L(\zeta)$ can be identified with the Weierstrass-σ, Weierstrass-ζ and Weierstrass-$\mathcal{P}$ functions, respectively (Valiron 1947).

Appendix B. An exact polycircular arc mapping

This appendix gives a brief derivation of equation (7.2), which maps the annulus $\rho<|\zeta|<1$ to the doubly connected polycircular arc region consisting of the unit disc with a slit along the real axis (figure 5).

Consider the sequence of two conformal mappings given by

$$\zeta_1(\zeta) = \frac{P(\zeta)}{P(-\zeta)}, \quad z(\zeta_1) = \frac{1-\zeta_1}{1+\zeta_1}. \tag{B 1}$$

The composition of these two mappings gives equation (7.2). Consider the first of these mappings. On $|\zeta|=1$,

$$\overline{\zeta_1(\zeta)} = \frac{P(\zeta^{-1})}{P(-\zeta^{-1})} = \frac{-\zeta^{-1}P(\zeta)}{\zeta^{-1}P(-\zeta)} = -\zeta_1(\zeta), \tag{B 2}$$

where we have used equation (3.2). This shows that $|\zeta|=1$ maps to the imaginary ζ_1-axis. Similarly, on $|\zeta|=\rho$,

$$\overline{\zeta_1(\zeta)} = \frac{P(\rho^2\zeta^{-1})}{P(-\rho^2\zeta^{-1})} = \zeta_1(\zeta), \tag{B 3}$$

where we have again used equation (3.2). The circle $C^{(1)}$ therefore maps to the real ζ_1-axis. Since the map is single-valued and does not become infinite, it is clear that there must be (at least) two points on $C^{(1)}$ where $d\zeta_1(\zeta)/d\zeta=0$ (so that the image turns through a corner of angle 2π as ζ passes through such a point). In fact, the argument principle can be used to show that the map is univalent and therefore that there are only two such points of non-conformality on $C^{(1)}$.

Finally, the second Möbius mapping in (B 1) is the one that takes the right-half ζ_1-plane to the interior of the unit z-disc. Thus, the imaginary ζ_1-axis maps to $|z|=1$, while any finite segment on the positive real ζ_1-axis will map to a segment on the real z-axis inside the unit z-disc. Figure 5 shows a schematic of this construction.

It is worth noting that a different derivation of this map, using elliptic function theory, can be found in Nehari (1952, pp. 293–296).

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