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# Computing the Schottky-Klein Prime Function on the Schottky Double of Planar Domains

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**Abstract.** A numerical algorithm is presented for the computation of the Schottky-Klein prime function on the Schottky double of multiply connected circular domains in the plane. While there exist classical formulae for the Schottky-Klein prime function in the form of infinite products over a Schottky group, such products are not convergent for all choices of multiply connected circular domains. The prime function itself, however, is a well-defined function for any multiply connected circular domain. The present algorithm facilitates the evaluation of this prime function when the planar domains are such that the classical infinite product representation is either not convergent or so slowly convergent as to be impracticable.

**Keywords.** Schottky-Klein prime function, Schottky double, multiply connected.

**2000 MSC.** 30F10, 30F15.

## 1. Introduction

The Schottky-Klein prime function is an important transcendental function with a primary role in solving problems involving multiply connected domains. It has, however, received almost no attention in the applied mathematics literature until relatively recently.

The prime function is documented in Chapter 12 of H. Baker's now classic 19th century monograph [1] and arises again in the memoir by Hejhal [16]. In terms of applications, it has recently been demonstrated that the prime function arises naturally in a variety of important applied mathematical problems. For example, Crowdy and Marshall [7] have shown how to construct a class of domains called multiply connected *quadrature domains* — a class of domains that arise, for example, in the field of fluid dynamics [5] — by expressing the conformal

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mapping $z(\zeta)$ from a circular preimage domain to these domains as a ratio of products of Schottky-Klein prime functions, i.e.,

$$z(\zeta) = R\frac{\prod_{k=1}^{N}\omega(\zeta,\alpha_k)}{\prod_{k=1}^{N}\omega(\zeta,\beta_k)} \tag{1}$$

where $\{\alpha_k \mid k = 1,\dots,N\}$ are the zeros and $\{\beta_k \mid k = 1,\dots,N\}$ are the poles of the mapping. Provided the zeros and poles satisfy certain conditions, (1) is one way of representing a function that is automorphic with respect to a Schottky group (the notion of a Schottky group will be introduced later). Indeed, the very fact that meromorphic functions which are automorphic with respect to a Schottky group can be factorized into ratios of products of this basic transcendental function is precisely why it is dubbed the "prime function". It is the generalization, to higher genus Riemann surfaces, of the simple function $\omega(\zeta,\gamma) = (\zeta-\gamma)$ relevant to the (genus zero) Riemann sphere. It is well-known that polynomials and rational functions, which are meromorphic functions on the Riemann sphere, can be written as products (or ratios of products) of the prime function $\omega(\zeta,\gamma) = (\zeta-\gamma)$ and (1) is really a generalization of this result to higher-genus Riemann surfaces.

Another application of the Schottky-Klein prime function arises in the solution of a long-standing problem in classical function theory: the problem of finding a generalized Schwarz-Christoffel formula to a multiply connected polygonal region. Crowdy [4] has shown that the Schwarz-Christoffel mapping from a bounded $(M+1)$-connected circular region $D_\zeta$ to a bounded $(M+1)$-connected polygonal region can be written in the form

$$z(\zeta) = A + B\int^{\zeta} S_M(\zeta')\prod_{k=1}^{n_0}[\omega(\zeta',a_k^{(0)})]^{\beta_k^{(0)}}\prod_{j=1}^{M}\prod_{k=1}^{n_j}[\omega(\zeta',a_k^{(j)})]^{\beta_k^{(j)}}\,d\zeta'$$

where $\omega$ is Schottky-Klein prime function and

$$S_M(\zeta) = \begin{cases} 1 & M=0,\\ \dfrac{1}{\zeta^2} & M=1,\\ \dfrac{\omega_\zeta(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1}) - \omega_\zeta(\zeta,\bar{\alpha}^{-1})\omega(\zeta,\alpha)}{\prod_{j=1}^{M}\omega(\zeta,\gamma_1^{(j)})\omega(\zeta,\gamma_2^{(j)})} & M\ge 2. \end{cases}$$

DeLillo, Elcrat and Pfaltzgraff [12] have also considered this important problem although from a different, but related, perspective. Further, Crowdy and Marshall [10] have shown that there are elegant formulae, in terms of the prime function, for the conformal mappings from multiply connected circular domains to all the other canonical multiply connected slit domains appearing in the classical literature [18, 19].

Beyond conformal mapping theory, there are applications of the Schottky-Klein prime function in potential theory. For example, the classical *modified Green's* *function* $G_0(\zeta,\alpha)$ for Laplace's equation in a multiply connected circular domain can be written concisely in terms of the prime function. In [8] it is shown that

$$G_0(\zeta,\alpha) = -\log\left|\frac{\omega(\zeta,\alpha)}{\alpha\omega(\zeta,\bar{\alpha}^{-1})}\right|.$$

This function has important applications, for example, to the fluid dynamical problem of point vortex motion in geometrically complicated (multiply connected) domains [9]. Furthermore, the standard first-type Green's function in the domain, as well as the harmonic measures of the domain, also have explicit representations in terms of the prime function [11].

All this evidence points to the primary role played by the Schottky-Klein prime function when performing analysis involving multiply connected domains. It is therefore a matter of some considerable interest and importance to be able to readily compute this prime function. One way to define the prime function is by means of a classical infinite product formula recorded, for example, in Chapter 12 of Baker's monograph on Abelian functions [1]. However, this infinite product does not always converge; there are usually restrictions on the multiply connected domain (or the Schottky group) required in order to ensure convergence. It should be emphasized, however, that the prime function is a well-defined (indeed a uniquely defined) function for *any* multiply connected circular domain [16]. It is therefore a pressing matter to be able to find alternative ways to evaluate this important function and try to divorce ourselves from a dependence on an infinite product formula that does not always converge and, even when it does, can converge so slowly as to render its use in applications impracticable, especially if accuracy to a large number of digits is required.

This is the aim of the present paper. Here we present a robust numerical algorithm to compute the Schottky-Klein prime function that does not require the convergence of a product or sum over a Schottky group. Instead, the algorithm is based on representing the prime function (and some subsidiary functions, as will be seen) in terms of Fourier-Laurent expansions about the centres of the circles making up the circular domain. Numerical tests of the validity of our algorithm are presented in Section 5.

## 2. The Schottky-Klein prime function

Let $D_\zeta$ be the multiply connected circular domain consisting of the unit disk in the $\zeta$-plane with $M$ smaller circular disks excised. Let the unit circle be denoted $C_0$ and the boundaries of the $M$ enclosed circular disks be denoted $\{C_j \mid j = 1,\ldots,M\}$. Let the radius and centre of $C_j$ be denoted $q_j$ and $\delta_j$ respectively.

First define $M$ Möbius maps $\{\phi_j \mid j = 1,\ldots,M\}$ corresponding to the conjugation map for points on the circle $C_j$. That is, if $C_j$ has equation

$$|\zeta-\delta_j|^2 = (\zeta-\delta_j)(\bar{\zeta}-\bar{\delta}_j) = {q_j}^2$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{{q_j}^2}{\zeta - \delta_j}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{{q_j}^2}{\zeta - \delta_j}.$$

If $\zeta$ is a point on $C_j$ then its complex conjugate is given by $\bar{\zeta} = \phi_j(\zeta)$.
Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{{q_j}^2\zeta}{1 - \bar{\delta}_j\zeta}.$$

Let $C_j'$ be the circle obtained by reflection of the circle $C_j$ in the unit circle $|\zeta| = 1$ (i.e. the circle obtained by the transformation $\zeta \mapsto \bar{\zeta}^{-1}$). Figure 1 shows a schematic in a quadruply connected case. It is easily verified that the

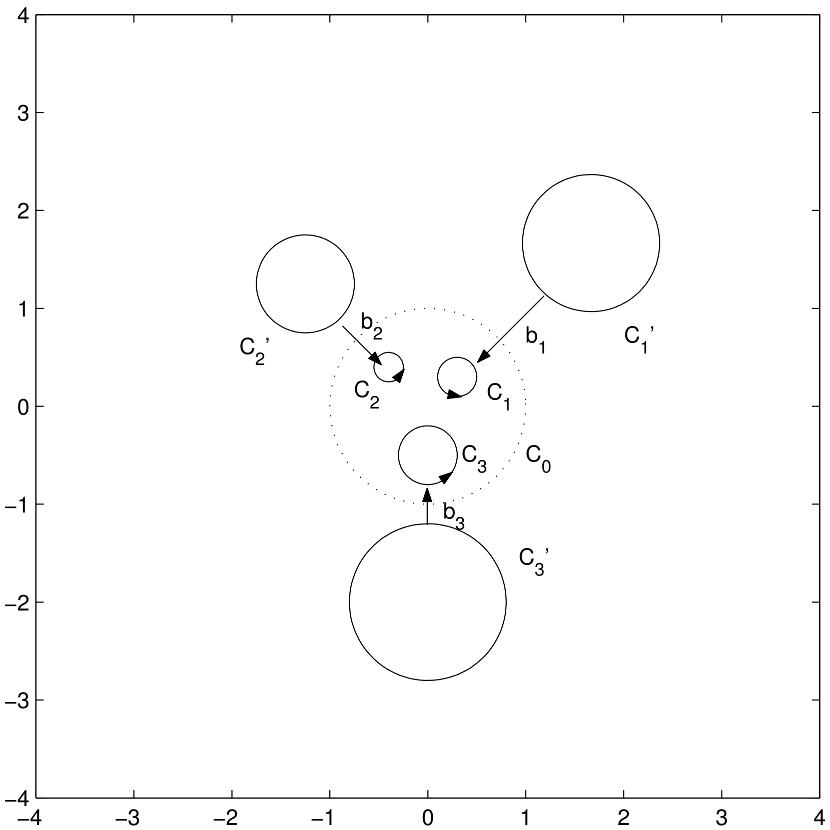


FIGURE 1. Schematic illustrating the circles $C_j$ and $C_j'$ in a quadruply connected case ($M = 3$). Each of the three circles $\{C_j \mid j = 1, 2, 3\}$ is an $a$-cycle. The three $b$-cycles are also shown.

image of the circle $C_j'$ under the transformation $\theta_j$ is the circle $C_j$. Thus, $\theta_j$

identifies circle $C'_j$ with circle $C_j$. Since the $M$ circles $\{C_j \mid j = 1, \ldots, M\}$ are non-overlapping, so are the $M$ circles $\{C'_j \mid j = 1, \ldots, M\}$. The (classical) *Schottky group* $\Theta$ is defined to be the infinite free group of mappings generated by compositions of the $M$ basic Möbius maps $\{\theta_j \mid j = 1, \ldots, M\}$ and their inverses $\{\theta_j^{-1} \mid j = 1, \ldots, M\}$ and including the identity map. Let the radius and centre of $C'_j$ be denoted $q'_j$ and $\delta'_j$ respectively. It is easy to show that

$$q'_j = \frac{q_j}{||\delta_j|^2 - q_j{}^2|}, \qquad \delta'_j = \frac{\delta_j}{|\delta_j|^2 - q_j{}^2}.$$

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j, C'_j \mid j = 1, \ldots, M\}$. Let this region be called $F$. $F$ is known as the *fundamental region* associated with the Schottky group generated by the Möbius maps $\{\theta_j \mid j = 1, \ldots, M\}$ and their inverses. This is because the entire plane is tesselated with copies of this fundamental region obtained by mapping $F$ by the elements of the Schottky group. This fundamental region can be understood as having two "halves" — the half that is inside the unit circle but exterior to the circles $C_j$ is the domain $D_\zeta$, the other half is the region outside the unit circle and exterior to the circles $C'_j$.

These two halves of $F$, one just a reflection through the unit circle of the other, can be viewed as a model of the two "sides" of a compact (symmetric) Riemann surface associated with $D_\zeta$ known as its *Schottky double*. The genus of this compact Riemann surface is $M$. Indeed, Baker [1] discusses how the circles $C_j$ (or, equivalently, the identified circles $C'_j$) can be understood, in the language of Riemann surface theory, as $M$ $a$-cycles on a genus-$M$ Riemann surface; further, any line joining a pair of identified points on $C_j$ and $C'_j$ can be viewed as a $b$-cycle (there are also $M$ of these). The schematic in Figure 1 illustrates the $a$- and $b$-cycles for the case shown (see Baker [1] or Farkas & Kra [13] for a definition of the $a$ and $b$-cycles associated with a compact Riemann surface). It is also well-known [13] that any compact Riemann surface of genus $M$ also possesses exactly $M$ holomorphic differentials which we shall here denote $\{dv_j(\zeta) \mid j = 1, \ldots, M\}$. The functions $\{v_j(\zeta) \mid j = 1, \ldots, M\}$ are the *integrals of the first kind* and each is uniquely determined, up to an additive constant, by their periods around the $a$- and $b$-cycles. These functions are analytic, but not single-valued, everywhere in $F$. Let $a_k$ denote the $k$-th $a$-cycle (which can be taken to be the circle $C_k$) and let $b_k$ denote the $k$-th $b$-cycle (which can be taken to be any line joining identified points on $C_k$ and $C'_k$). Here we normalize the holomorphic differentials so that

$$\oint_{a_k} dv_j = \delta_{jk}, \oint_{b_k} dv_j = \tau_{jk}$$

for some set of constants $\tau_{jk}$. Gustafsson [15] has considered the Schottky double in his analysis of multiply connected quadrature domains.

Armed with a normalized basis of $a$ and $b$-cycles, the $M$ integrals of the first kind and the Schottky group $\Theta$, we have now set up all the necessary machinery to be able to define the Schottky-Klein prime function. The following theorem is established in Hejhal [16]; it holds for any compact Riemann surface, not just the Schottky double of a planar domain considered here:

**Theorem.** *There is a unique function $X(\zeta,\gamma)$ defined by the properties:*

(i) *$X(\zeta,\gamma)$ is analytic everywhere in $F$.*
(ii) *For $\gamma \in F$, the function $X(\zeta,\gamma)$ has a second-order zero at each of the points $\{\theta(\gamma) \mid \theta \in \Theta\}$.*
(iii) *For $\gamma \in F$,*

$$\lim_{\zeta\to\gamma} \frac{X(\zeta,\gamma)}{(\zeta-\gamma)^2} = 1. \tag{2}$$

(iv) *For $j = 1,\dots,M$,*

$$X(\theta_j(\zeta),\gamma) = \exp(-2\pi i(2(v_j(\zeta)-v_j(\gamma))+\tau_{jj}))\,\frac{d\theta_j(\zeta)}{d\zeta}X(\zeta,\gamma). \tag{3}$$

Hejhal [16] then defines the *Klein prime function* $\omega(\zeta,\gamma)$ (or what we will call, following Baker [1], the *Schottky-Klein prime function*) as the square root of this function, i.e.,

$$\omega(\zeta,\gamma) = (X(\zeta,\gamma))^{1/2}$$

where the branch of the square root is chosen so that $\omega(\zeta,\gamma)$ behaves like $(\zeta-\gamma)$ as $\zeta\to\gamma$.

The challenge here is to find explicit representations of the $M$ integrals of the first kind $\{v_j(\zeta) \mid j = 1,\dots,M\}$. Given these, the quantities $\tau_{jj}$ can be determined and the theorem just stated used to determine $X(\zeta,\gamma)$, and hence $\omega(\zeta,\gamma)$.

## 3. Integrals of the first kind

Let $\{\Omega_k(\zeta,\bar\zeta) \mid k = 1,\dots,M\}$ be the $M$ harmonic measures associated with the circular domain $D_\zeta$. By definition, $\Omega_k(\zeta,\bar\zeta)$ is a harmonic function in $D_\zeta$ and satisfies the boundary conditions

$$\Omega_k(\zeta,\bar\zeta) = \begin{cases} 1 & \text{on } C_k, \\ 0 & \text{on } C_j,\ j\neq k. \end{cases}$$

Let the harmonic conjugate function to $\Omega_k(\zeta,\bar\zeta)$ be $H_k(\zeta,\bar\zeta)$. Then the $M$ functions $\hat v_k(\zeta)$ defined by

$$\hat v_k(\zeta) \equiv H_k(\zeta,\bar\zeta) + i\Omega_k(\zeta,\bar\zeta)$$

are analytic (but not single-valued) functions in $D_\zeta$.

Consider now some linear combination of the functions $\{\hat v_k(\zeta) \mid k = 1,\dots,M\}$ given by

$$v_k(\zeta) = \sum_{j=1}^{M} Q_{kj}\hat v_j(\zeta)$$

where the coefficients $\{Q_{kj}\}$ are real. Since $v_j$ is a (real) linear combination of the functions $\{\hat{v}_j(\zeta) \mid j = 1, \ldots, M\}$ it follows immediately that

$$\mathrm{Im}(v_j) = 0, \qquad \text{on } C_0.$$

Equivalently, on $C_0$,

$$\overline{v_j}(\zeta^{-1}) = v_j(\zeta) \tag{4}$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ on $C_0$. Being a relation between functions of $\zeta$, (4) can be analytically continued off $C_0$. In particular, (4) can be used to deduce that $v_j(\zeta)$ extends to an analytic function everywhere in the fundamental region.

We now examine whether it is possible to pick the coefficients $\{Q_{kj}\}$ in order to satisfy the normalization conditions that, for each $k = 1, \ldots, M$,

$$\oint_{C_m} dv_k = \delta_{km}, \qquad m = 1, \ldots, M \tag{5}$$

where $\delta_{km}$ denotes the Kronecker delta. Observe that, on use of the single-valuedness of $\Omega_k$ and the Cauchy-Riemann relations, conditions (5) can be rewritten in the following form:

$$\begin{aligned} \delta_{km} &= \oint_{C_m} dv_k = \oint_{C_m} \sum_{j=1}^{M} Q_{kj}\, d\hat{v}_j = \oint_{C_m} \sum_{j=1}^{M} Q_{kj}\, d(H_j + i\Omega_j) \\ &= \oint_{C_m} \sum_{j=1}^{M} Q_{kj} \frac{\partial H_j}{\partial s}\, ds = \oint_{C_m} \sum_{j=1}^{M} Q_{kj} \frac{\partial \Omega_j}{\partial n}\, ds \\ &= \sum_{j=1}^{M} Q_{kj} \oint_{C_m} \frac{\partial \Omega_j}{\partial n}\, ds = -2\pi Q_{kj} P_{jm} \end{aligned} \tag{6}$$

where the square matrix $\mathbf{P}$ has components

$$P_{jm} \equiv -\frac{1}{2\pi} \oint_{C_m} \frac{\partial \Omega_j}{\partial n}\, ds, \qquad j, m = 1, \ldots, M.$$

It follows from (6) that

$$-2\pi \mathbf{Q} = \mathbf{P}^{-1}. \tag{7}$$

The matrix $\mathbf{P}$ is precisely the same matrix introduced at the end of Schiffer [19, §1] who shows that it has a well-defined inverse and that it is positive definite. It follows that the coefficients $\{Q_{kj}\}$ for which (5) is satisfied exist and are uniquely defined by (7). As a result, the differentials $\{dv_j \mid j = 1, \ldots, M\}$ have the required normalization with respect to the $a$-cycles.

With the functions $\{v_j \mid j = 1, \ldots, M\}$ now well-defined, let us now consider the periods of these differentials around the $b$-cycles. The periods around the

$b$-cycles are given by

$$\oint_{b_m} dv_j = v_j(\theta_m(\zeta)) - v_j(\zeta) \tag{8}$$

where $\zeta$ is any point chosen on $C'_m$. In particular, the right hand side of (8) must be *constant* for all choices of $\zeta$ on $C'_m$. We must verify that the functions $\{v_j(\zeta) \mid j = 1, \ldots, M\}$ satisfy this condition. On use of the boundary properties of $\{\hat{v}_j(\zeta) \mid j = 1, \ldots, M\}$ it also follows that, on $C_m$,

$$\mathrm{Im}(v_j) = Q_{jm}. \tag{9}$$

Equivalently, (9) can be written

$$\overline{v_j}(\phi_m(\zeta)) - v_j(\zeta) = 2iQ_{jm}$$

or

$$\overline{v_j}(\overline{\theta_m}(\zeta^{-1})) - v_j(\zeta) = 2iQ_{jm}. \tag{10}$$

Being a relation between functions of $\zeta$, (10) can be analytically continued off $C_m$. Combining (4) and (10) we then deduce that

$$\overline{v_j}(\overline{\theta_m}(\zeta^{-1})) - \overline{v_j}(\zeta^{-1}) = 2iQ_{jm}$$

or, on taking the conjugate of this equation, that

$$v_j(\theta_m(\zeta)) - v_j(\zeta) = -2iQ_{jm}$$

It follows from (7) that we can identify the matrix $\tau$ with

$$\tau = \frac{i}{\pi}\mathbf{P}^{-1}.$$

The functions $\{v_j(\zeta) \mid j = 1, \ldots, M\}$ just constructed are precisely the required integrals of the first kind on the Riemann surface. They are linear combinations of the analytic extensions of the harmonic measures of the domain $D_\zeta$. This observation will be crucial in the numerical construction of the Schottky-Klein prime function.

Finally, for use later on, note that it is also easy to verify, from the identifications between $C_m$ and $C'_m$ that

$$\delta_{jm} = \oint_{C_m} dv_j = -\oint_{C'_m} dv_j. \tag{11}$$

## 4. Numerical algorithm

The numerical algorithm we propose to compute the Schottky-Klein prime function is a two-step algorithm. The key idea underlying the numerical construction is borrowed from Trefethen's "ten-digit algorithm" entitled `manydisks.m` which is a MATLAB script designed to compute the first-type Green's function for Laplace's equation in the domain exterior to a collection of circular disks in the plane. Trefethen's code is based on a least-squares method for solving a linear system for the coefficients for a Fourier-Laurent expansion of the (analytic extension) of the Green's function. The general ideas of this algorithm can be readily adapted to present circumstances.

The principal mathematical observation is that any function which is analytic and single-valued in the fundamental region associated with a multiply connected circular domain $D_\zeta$ — that is, in the region exterior to the $2M$ circles $\{C_j, C_j' \mid j = 1, \ldots, M\}$ — has a Fourier-Laurent expansion of the form

$$A_0 + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{A_m^{(k)} {q_k}^m}{(\zeta - \delta_k)^m} + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{B_m^{(k)} {Q_k}^m}{(\zeta - \delta_k')^m}.$$

It is an easy exercise to show that the existence of such a representation is a simple consequence of Cauchy's integral formula once the Cauchy kernel is expanded, as a geometric series, about the centres of the circles.

We now describe the two steps of the numerical algorithm to compute the Schottky-Klein prime function.

**Step 1.** First, motivated by the above observations, in order to explicitly construct the functions $\{v_j(\zeta) \mid j = 1, \ldots, M\}$, for each $j$ we seek a Fourier-Laurent representation of the form

$$v_j(\zeta) = \frac{1}{2\pi i} \log\left(\frac{\zeta - \delta_j}{\zeta - \delta_j'}\right) + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{a_m^{(j,k)} {q_k}^m}{(\zeta - \delta_k)^m} + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{b_m^{(j,k)} {\tilde{q}_k}^m}{(\zeta - \delta_k')^m} \tag{12}$$

where

$$\tilde{q}_k = \begin{cases} q_k' & \text{if } |q_k'| < 1, \\ 1 & \text{if } |q_k'| \geq 1. \end{cases}$$

This rescaling of the Fourier-Laurent coefficients ensures numerical stability of the algorithm. For each $j$, the coefficients $\{a_m^{(j,k)}, b_m^{(j,k)} \mid k = 1, \ldots, M; m = 1, \ldots\}$ are to be determined. A possible constant term in the representation (12) has been ignored because it turns out to be inconsequential in computing the Schottky-Klein prime function. The logarithmic term in (12) ensures that it automatically satisfies the $a$-cycle normalization conditions (11).

The crucial observation is that the unknown coefficients $\{a_m^{(j,k)}, b_m^{(j,k)}\}$ for each $j$ can be determined by requiring that the imaginary part of $v_j$ is constant on the circles $\{C_m \mid m = 1, \ldots, M\}$, i.e.,

$$\mathrm{Im}(v_j(\zeta)) = \gamma_{jm} \tag{13}$$

where $\gamma_{jm}$ are constants. These constants are not known *a priori* but must be determined as part of the numerical solution.

The numerical procedure, implemented in Matlab, that we have employed is as follows:

(1a) truncate each sum in the representation (12) at order $N_1$ (where $N_1$ is chosen to be large enough for the desired accuracy);

(1b) substitute this truncated representation into equation (13) and evaluate it at $N_2$ equally-spaced collocation points on each of the $2M$ circles $\{C_j, C'_j \mid j = 1, \ldots, M\}$. A simple count reveals that there are $4MN_1 + 2M$ real unknowns and the real equation (13) is to be evaluated at $N_2$ points on each of the $2M$ circles. It is clear that we must pick $N_2 \geq 2N_1 + 1$ in order to obtain an overdetermined linear system;

(1c) use a least-squares method to solve this overdetermined linear system for the unknowns $\{a_m^{(j,k)}, b_m^{(j,k)}, \gamma_{km}\}$.

**Step 2.** The second step of the algorithm, now that the functions $\{v_j\}$ have been determined, is to write

$$X(\zeta, \gamma) = (\zeta - \gamma)^2 \hat{X}(\zeta, \gamma)$$

where $X(\zeta, \gamma)$ is the square of the required Schottky-Klein prime function, i.e.,

$$X(\zeta, \gamma) = \omega^2(\zeta, \gamma)$$

and the function $\hat{X}(\zeta, \gamma)$ has the Fourier-Laurent expansion

$$\hat{X}(\zeta, \gamma) = A \left( 1 + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{c_m^{(k)} {q_k}^m}{(\zeta - \delta_k)^m} + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{d_m^{(k)} {\tilde{q}_k}^m}{(\zeta - \delta'_k)^m} \right) \tag{14}$$

which is valid for $\zeta$-values inside (or on the boundary) of the fundamental region $F$. The coefficients $\{c_m^{(k)}, d_m^{(k)} \mid k = 1, \ldots, M; m = 1, \ldots\}$ and the constant $A$ are to be determined. The second step in the algorithm proceeds as follows:

(2a) Truncate the sums in (14) at the same order $N_1$ as in step (1a) of the algorithm.

(2b) Determine the coefficients $\{c_m^{(k)}, d_m^{(k)} \mid k = 1, \ldots, M; m = 1, \ldots, N_1\}$ from the transformation properties (3) – note that, since there are $M$ distinct choices of the mapping $\theta_j$, there are $M$ different transformation properties. To do this, pick a value of $j$ and substitute the (truncated) representation (14) into (3) and evaluate this relation at $2N_2$ collocation points on the *exterior* circle $C'_j$. (Note that, in contrast to the choice of collocation points in step (1b), here we must choose collocation points only on the circles $\{C'_j \mid j = 1, \ldots, M\}$ and not the circles $\{C_j \mid j = 1, \ldots, M\}$. This is because in order to evaluate (3) we must evaluate representation (14) not only at a point $\zeta$ but also at $\theta_j(\zeta)$. Only if $\zeta$ is on a circle $C'_j$ are both $\zeta$ and $\theta_j(\zeta)$ in the closure of the fundamental domain $F$ where (14) is valid). Repeat this for each $j = 1, \ldots, M$. Combining all these equations produces a linear system in the unknown coefficients $\{c_m^{(k)}, d_m^{(k)} \mid k = 1, \ldots, M; m = 1, \ldots, N_1\}$ which can be solved using a least-squares algorithm. Note that the transformation relations (3) are all independent of the value of $A$, so $A$ is not determined at this stage.

(2c) With the coefficients $\{c_m^{(k)}, d_m^{(k)} \mid k = 1, \ldots, M; m = 1, \ldots, N_1\}$ now known, $A$ can be determined by enforcing the normalization condition (2). This just means that $A$ must be chosen so that $\hat{X}(\gamma, \gamma) = 1$.

Two remarks are in order. First, if some application of the prime function requires that $\omega(\zeta, \gamma)$ be evaluated for many different values of $\zeta$ or $\gamma$ but in a fixed domain $D_\zeta$ then the first step of the algorithm should be performed only once (it can be viewed as a "setup" step). This is because this first step only involves the computation of the first integrals $\{v_j(\zeta) \mid j = 1, \ldots, M\}$ which depend only on the domain $D_\zeta$ (and not on either $\zeta$ or $\gamma$). Second, suppose an application involves a fixed domain $D_\zeta$ and requires multiple evaluations of $\omega(\zeta, \gamma)$ for different values of $\zeta$ but for a fixed value of $\gamma$. Then it is only necessary to perform both Step 1 *and* Step 2 once (again, as a setup step).

## 5. Verification of algorithm

The first test of the algorithm is to compare the numerical values of $X(\zeta, \gamma)$ obtained from the scheme above to the values given by the infinite product formula recorded by Baker [1]. This is only sensible, of course, for choices of $D_\zeta$ such that the infinite product formula is convergent. Following Baker [1], the Schottky-Klein prime function is defined as

$$\omega(\zeta, \gamma) = (\zeta - \gamma)\hat{\omega}(\zeta, \gamma) \tag{15}$$

where

$$\hat{\omega}(\zeta, \gamma) \equiv \prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta) - \gamma)(\theta_i(\gamma) - \zeta)}{(\theta_i(\zeta) - \zeta)(\theta_i(\gamma) - \gamma)} \tag{16}$$

and where the product is over all mappings $\theta_i$ in the set $\Theta''$ which denotes all mappings in the Schottky group $\Theta$ excluding the identity and all inverse maps. This means that if $\theta_1\theta_2$ is included, say, then $\theta_2^{-1}\theta_1^{-1}$ (its inverse) must be excluded.

A natural way to truncate the infinite product in equation (16) is by the *level* of the mappings. The identity map is the single level-zero map. The maps $\{\theta_k, \theta_k^{-1} \mid k = 1, \ldots, M\}$ are the $2M$ level-one maps. Any composition of these level-one maps that does not reduce to the identity is a level-two map. By extension, a composition of any *three* of the level-one maps that does not reduce to a lower level map is called a level-three map, and so on.

As a random example, a triply connected domain with interior circles both of radius 0.1 and centres at 0.5 and $0.5i$ were chosen. The value of $X(\zeta, \gamma)$ with arbitrarily chosen values $\zeta = -0.5 - 0.5i$ and $\gamma = 1$ were computed using both the infinite product formula (15) and (16) retaining all maps in the Schottky group up to different levels (in fact, up to level 9) and using the new algorithm described above. Table 1 shows the results, with truncation at different levels, using the infinite product while Table 2 gives the results from the numerical algorithm for different values of the truncation parameter $N_1$.

| Truncation level | $X(-0.5-0.5i,1)$ |
|---|---|
| level 4 | 2.39754711380740 + 1.76164303455525i |
| level 5 | 2.39754807522356 + 1.76164374094987i |
| level 6 | 2.39754812001040 + 1.76164377385126i |
| level 7 | 2.39754812211353 + 1.76164377539660i |
| level 8 | 2.39754812221309 + 1.76164377546950i |
| level 9 | 2.39754812221900 + 1.76164377547293i |

TABLE 1. Evaluation of $X(-0.5-0.5i,1)$ for a given triply connected domain using the infinite product formula from Baker [1] truncated at different levels.

| $N_1$ | $X(-0.5-0.5i,1)$ |
|---|---|
| 10 | 2.39754812225980 + 1.76164377550332i |
| 15 | 2.39754812221763 + 1.76164377547306i |
| 20 | 2.39754812221763 + 1.76164377547306i |
| 25 | 2.39754812221763 + 1.76164377547306i |

TABLE 2. Evaluation of $X(-0.5-0.5i,1)$ for a given triply connected domain using the new algorithm with differing values of the truncation $N_1$.

It is important to note from Table 2 that, with just $N_1 = 15$ terms retained in the Fourier-Laurent expansions, the algorithm has clearly converged to a definite value (increasing $N_1$ no longer affects the result to the number of digits shown). In contrast, while even a few levels are enough to obtain several digits of accuracy in the infinite product (probably enough for most applications), as the number of levels in the infinite product is increased, the calculation converges to a definite value (reassuringly, the same value given by the numerical algorithm) but the convergence is arguably slow if accuracy to a large number of digits is required. Moreover recall that, for an $M$-connected domain, there are $2M(2M-1)^{p-1}$ elements of the Schottky group at level $p$ and exactly half of these are needed in the infinite product. This means that, even for just a triply connected domain ($M = 2$), there are $2.3^8 = 13122$ level-9 terms to be included in the infinite product formula.

It is appropriate to devise a more global test of the numerical algorithm, one which involves computation of the prime function at multiple points in the domain $D_\zeta$ in order to arrive at the required result. To do so, we exploit a recent result of Crowdy [6] who has derived an integral formula, in terms of the Schottky-Klein prime function, for the solution of the modified Schwarz problem in multiply connected circular domain $D_\zeta$. More specifically, suppose that just the real part $\phi$ of some function $f_s(\zeta)$ that is analytic and single-valued in $D_\zeta$ is given at all points on the boundary $\partial D_\zeta$ of $D_\zeta$ (the subscript highlights the fact that $f_s(\zeta)$ must be single-valued in $D_\zeta$). Then, $f_s(\zeta)$ can be evaluated everywhere *inside* $D_\zeta$ by making use of the formulae, derived in [6], given by

$$\begin{aligned} f_s(\alpha) = {} & \frac{1}{2\pi i} \oint_{C_0} \phi\big(d\log\omega(\zeta,\alpha) + d\log\overline{\omega}(\zeta^{-1},\alpha^{-1})\big) \\ & - \sum_{j=1}^{M} \frac{1}{2\pi i} \oint_{C_j} \phi\big(d\log\omega(\zeta,\alpha) + d\log\overline{\omega}(\overline{\theta_j}(\zeta^{-1}),\alpha^{-1})\big) + iC \end{aligned} \tag{17}$$

where $C$ is some real constant. (17) is the solution of the modified Schwarz problem in a multiply connected circular domain $D_\zeta$ with the kernel functions expressed in terms of the Schottky-Klein prime function $\omega(\zeta,\alpha)$ of the domain. This formula can be used as a test of the accuracy of the algorithm for computing $\omega(\zeta,\gamma)$. First, we make an arbitrary choice of $f_s(\zeta)$. The values of the real part of $f_s(\zeta)$ on the boundary circles can then be fed into the right side of (17). The accuracy to which (17) can retrieve (known) values of $f_s(\zeta)$ *inside* the domain $D_\zeta$ can then be tested.

Figure 2 shows a quadruply connected circular domain in which the centres of the three interior circular disks, each of radius 0.2, were chosen arbitrarily to be fixed at 0.5, $-0.1+0.35i$ and $-0.4i$. The function $f_s(\zeta)=\zeta$ is chosen as a test function. It is a single-valued analytic function in the domain shown in Figure 2. First, an arbitrary value of $\alpha$ was picked and the real part of $f_s(\zeta)$ on the boundary circles of the domain was fed into formula (17) and the value of the integral determined using the trapezoidal rule. The disparity between the imaginary part of the result and the imaginary part of the original choice of $\alpha$ was then computed in order to evaluate the imaginary constant $C$ appearing in (17). Next, the value $\alpha = 0.5+0.5i$ was chosen arbitrarily and formula (17) again computed with this $\alpha$ and the value of $C$ just computed. The result was then compared with the value $0.5+0.5i$ (which is the expected result). The value obtained from formula (17) where the prime function was evaluated using the numerical algorithm above is $0.5000000135+0.4999999918i$ so that the absolute error is $0.0000000135-0.0000000082i$. To compute this, each of the Fourier-Laurent expansions is truncated at $N_1=20$ terms. $N_2$ is taken equal to 50 while 400 points are taken on each boundary circle in a trapezoidal rule computation of the contour integrals in (17). It is clear that, with these choices, between 8 and 9 digits of accuracy is obtained which should be ample for most applications.

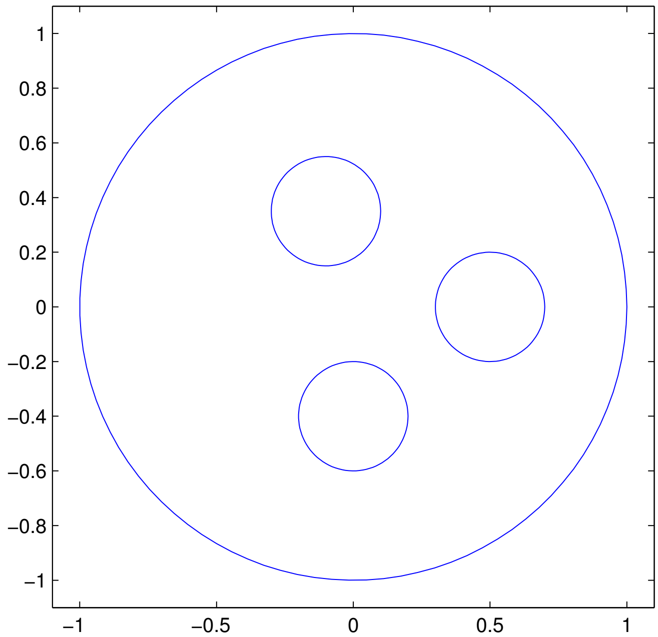


Figure 2. A quadruply connected circular domains with the centres of the interior circular discs fixed at 0.5, $-0.1+0.35i$ and $-0.4i$ and each with radius 0.2.

Finally, it is worth mentioning that the solution (17) of the Schwarz problem in multiply connected circular domains can be used to write down solutions to problems arising in applications such as fluid dynamics. For example, it can be used to study problems involving fluid stirrers as described in Price, Mullin & Kobine [17] or Finn, Cox & Bryne [14].

## 6. Discussion

This paper has proposed a novel numerical technique for the computation of the Schottky-Klein prime function on the Schottky double of planar multiply connected circular domains. It can be used to compute the prime function when alternative representations (such as infinite product formula (15) and (16) over the Schottky group) are not valid. In the Introduction, a survey of many new results involving the Schottky-Klein prime function were documented. The algorithm presented here can be used to compute the prime function for use in all those various applications whenever the infinite product formula does not converge or is too slowly convergent for practical use.

The use of this algorithm obviates the need to be concerned about the convergence properties of other representations of the prime function as products (or sums) over a Schottky group. Moreover, such products (and sums) must necessarily be truncated if they are to be evaluated numerically. The number of maps in the Schottky group at any given level of truncation grows exponentially with the connectivity of the domain which means, in practice, that evaluating such products and sums for domains of even moderately high connectivity can quickly become expensive. The present numerical algorithm is immune to this circumstance; if the connectivity increases by one, it simply means there are $2N_1$ more coefficients to find from the Fourier-Laurent expansion of the various functions about the two new circle centres. This is much less expensive computationally.

The only limitation of the new method appears to be that if the domain is such that the circles are very close together, the Fourier-Laurent expansions converge more and more slowly. However, it seems likely that even this limitation can potentially be overcome by using more sophisticated hybrid techniques such as those already developed in the context of two-dimensional electrostatics problems involving close-to-touching conductors [3]. Work on adapting such ideas to refine our computation of the prime function in such special cases is already in progress.

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