SIAM J. APPL. MATH.
Vol. 65, No. 3, pp. 941–963

EXACT SOLUTIONS FOR THE EVOLUTION OF A BUBBLE IN STOKES FLOW: A CAUCHY TRANSFORM APPROACH*

DARREN CROWDY† AND MICHAEL SIEGEL‡

Abstract. A Cauchy transform approach to the problem of determining the free surface evolution of a single bubble in Stokes flow is developed. A number of exact solutions to a class of problems have been derived in the literature using conformal mapping theory, and these solutions are retrieved and further generalized using the new formulation. Certain quantities which are conserved by the dynamics are also identified, the existence of which had not previously been pointed out. A principal purpose of this paper is to use the new formulation to understand when it is possible to externally specify the evolution of the bubble area in such classes of exact solution. It is found to be possible only for certain types of far-field boundary conditions.

Key words. bubble, Stokes flow, complex variables, Cauchy transform

AMS subject classifications. 76D27, 30E25

DOI. 10.1137/S0036139903430847

1. Introduction.

It is a well-known yet remarkable fact that large classes of exact solutions can be found for unsteady two-dimensional (2-D) Stokes flow with a free surface, both with and without surface tension. The solutions follow from the application of powerful complex variable methods; most often these methods involve the use of a conformal mapping $z(\zeta, t)$ to reformulate the free boundary problem as a boundary value problem on a fixed domain in the ζ-plane (assumed in this paper to be a unit disk). The free boundary evolution is then conveniently described by the functional form of the map $z(\zeta, t)$.

Among the first results from the application of complex variable methods to free boundary problems for 2-D Stokes bubbles are the steady bubble solutions of Richardson [18, 19]. Antanovskii [2] later constructed exact unsteady solutions in the case when the asymptotic form of the far-field flow is given by an mth order irrotational straining flow. Independently, Tanveer and Vasconcelos [23] derived explicit unsteady solutions in the form of polynomial mappings in the case when the far-field flow is purely linear, e.g., pure straining flow or simple shear flow with $m = 1$. They also constructed new exact solutions for an expanding/contracting bubble in a quiescent flow. Antanovskii [3] also constructed explicit steady solutions in the case of nonlinear[1] and rotational far-field conditions. The latter solutions were applied as a simple 2-D model for flow in Taylor's four roller mill. Siegel [22] later generalized these results to include explicit unsteady solutions for certain nonlinear rotational far-field conditions, including the time-dependent evolution of the solutions in [3]. Additionally, there is a large amount of literature describing related developments for the evolution of viscous blobs in 2-D Stokes flow (see, e.g., Howison and Richardson [14] and the

*Received by the editors July 1, 2003; accepted for publication (in revised form) July 15, 2004; published electronically March 31, 2005.
http://www.siam.org/journals/siap/65-3/43084.html

†Department of Mathematics, Imperial College of Science and Technology, London SW7 2AZ, England (d.crowdy@imperial.ac.uk). The research of this author was partially supported by EPSRC grant GR/R40104/01.

‡Department of Mathematical Sciences, NJIT, Newark, NJ 07102 (misieg@m.njit.edu). The research of this author was supported in part by NSF grant DMS-0104350.

[1] A precise definition of the term "nonlinear" in this context is given in section 3.3.1.

references therein). Complex variable methods have also been applied to study the evolution of bubbles in Hele–Shaw flow. Cummings, Howison, and King [10] provide a comparison of the developments in Hele–Shaw flow and 2-D Stokes flows. Aside from their intrinsic mathematical interest, these investigations have led to improved understanding of the formation of cusp singularities in free surface flow. The exact 2-D solutions are also useful as an important component of the leading-order solution to three-dimensional flow in slender geometries [9].

The aforementioned exact solutions for 2-D Stokes flow and Hele–Shaw flow all take the form of rational conformal mappings which may be written as

$$z(\zeta,t)=\frac{a_0(t)+a_1(t)\zeta+\cdots+a_N(t)\zeta^N}{\zeta(1+b_1(t)\zeta+\cdots+b_M(t)\zeta^M)}. \tag{1}$$

For convenience we have chosen $b_0(t)=1$ (this is always possible through a redefinition of the other coefficients); if $b_i(t)\equiv 0$, then the mapping reduces to a simpler polynomial form. Note that the extra ζ term in the denominator of (1) is due to the mapping of the inside of the unit disk to the exterior of the bubble. When the dynamics preserves the form (1), the free boundary evolution reduces from an infinite-dimensional dynamical system (namely, the original governing PDEs) to a finite system of ODEs from which one can compute the $N+M+1$ parameters of the conformal mapping from given initial data and external flow.[2]

In essence, demonstration of the existence of a solution of the form (1) for a given free boundary problem involves two key steps. First, it must be shown that the form (1) is preserved in time; i.e., if $z(\zeta,0)$ is a rational function of the form (1) for some M,N, then, as long as the solution exists, $z(\zeta,t)$ remains a rational function with the same M,N. In this paper we shall refer to this requirement as the closure condition. In particular, this implies that the number and type of (pole) singularities in the complex plane (i.e., $|\zeta|>1$) is invariant with time. Second, it is generally required that solutions $z(\zeta,t)$ do not generate any flow singularities (i.e., sources or sinks) in the *finite* fluid domain, although we do allow sources or sinks at infinity, corresponding to expanding/contracting bubble area.

For the solutions described in [2, 23] the closure condition is automatically satisfied, i.e., without any restrictions on the map coefficients $a_i,\ b_i$. The governing ODEs for the coefficients $a_i,\ b_i$ therefore come strictly from the second requirement, which gives $N+M$ conditions for $N+M+1$ unknowns. As a final condition, one is free to specify the time rate of change of the bubble area or, in other words, to specify the existence of a time-dependent source or sink fixed at infinity.

In contrast, for the solutions derived in [22] the closure condition imposes a constraint on the map coefficients $a_i,\ b_i$, in addition to the $N+M$ constraints supplied by the second condition above. Thus, seemingly, one is not free to arbitrarily specify the bubble area. It therefore seems rather fortuitous that the constant area condition employed in [22] is correct, i.e., that the dynamics actually preserves bubble area. However, it is not at all clear from the discussion there how to ascertain if or when this is the case.

In this paper, we provide a general discussion of when it is possible to obtain exact solutions to a very broad class of problems which encompasses and expands the class investigated in [2, 22, 23]. Additionally, we specify precisely when it is possible to externally control the bubble area and, in cases for which the area is not controllable,

[2]Solutions to such a system of ODEs are commonly referred to as exact solutions.

provide a simple means of ascertaining its time dependence. Our analysis involves a different approach from the one given in [2, 22, 23], namely, that emphasis is placed on consideration of the Cauchy transform which we define here as

$$\frac{1}{2\pi i}\oint_{\partial D(t)} \frac{\bar{z}'}{z'-z}dz', \tag{2}$$

where $\partial D(t)$ denotes the boundary of the fluid domain $D(t)$. When z is inside the bubble, the line integral (2) defines an analytic function, say $C(z,t)$. The analytic continuation of $C(z,t)$ outside the bubble (and into the fluid region $D(t)$) contains a great deal of information about the bubble shape. In many respects, $C(z,t)$ is a more natural mathematical object to consider; indeed, given the functional form of $C(z,t)$ at each instant it is possible to reconstruct the relevant conformal mapping. The evolution equation for $C(z,t)$ also has a convenient mathematical form which is easy to analyze. In particular, using the new formulation presented here it is possible to resolve such issues as the question of bubble area evolution, a question which often involves formidable calculation in the usual conformal map approach. It is appropriate to mention that the Cauchy transform has been used to great effect by many previous authors in the study of Hele–Shaw free boundary problems [12, 20, 21, 24].

The rest of this paper is organized as follows. In section 2 we introduce the Cauchy transform formulation for the problem of Stokes flow for a single bubble and prove its equivalence to the usual Stokes flow formulation. We also demonstrate the existence of certain conserved quantities which are useful in constructing exact solutions. In section 3 we use the new formulation to retrieve the exact solutions which have been previously presented in the literature and show how new classes of solutions may be derived. In particular, with the new approach the closure condition is easily verified, in contrast to the much more involved calculations that are necessary using the conformal map approach. More importantly, we use our formulation to investigate when it is possible to externally specify the evolution of bubble area. Some concluding remarks are presented in section 4.

2. Mathematical formulation.

2.1. Stokes flow problem. Consider the quasi-steady evolution of a single bubble in an ambient Stokes flow. The fluid inside the bubble is assumed to have zero viscosity, implying that it is a passive fluid with spatially constant pressure, which for convenience is set to zero. We denote the fluid region exterior to the bubble by $D(t)$, and the bubble is denoted by $D_c(t)$. The flow is allowed to be singular at infinity, although we do not consider any flow singularities (such as sources or sinks) in the finite flow domain. Figure 1 gives a schematic. In view of the incompressibility of the flow, it is convenient to introduce a streamfunction $\psi(x,y)$ which satisfies

$$\mathbf{u} = \nabla^{\perp}\psi. \tag{3}$$

It is easily seen that

$$\nabla^4\psi = 0 \quad \text{in } D(t). \tag{4}$$

We assume that surface tension acts on the bubble boundary so that the stress condition is

$$-pn_j + 2e_{jk}n_k = \kappa n_j, \tag{5}$$

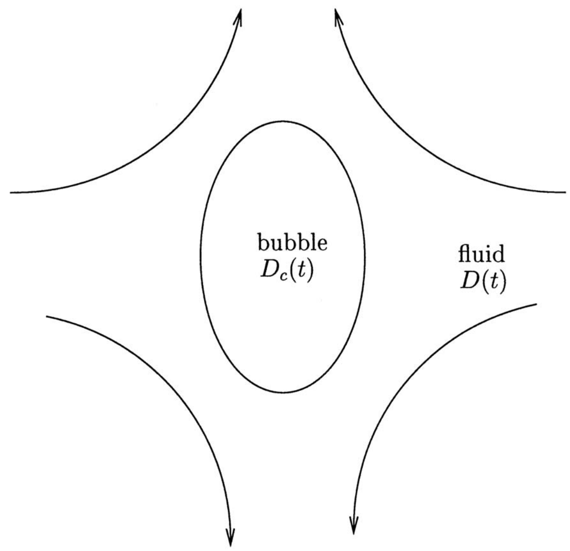

FIG. 1. *Schematic illustrating the bubble region $D_c(t)$ and the fluid region $D(t)$. There is a singular flow field at infinity.*

where κ is the surface curvature (assumed positive for a convex surface), p is the pressure exterior to the bubble, $\mathbf{n}$ is a unit normal pointing outward from the bubble, and

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$$

is the rate of strain tensor. In (5) we have assumed that velocities are nondimensionalized by σ/μ, where σ is the surface tension, lengths are nondimensionalized by R, the undeformed bubble radius, and p is nondimensionalized by σ/R. Additionally, time is nondimensionalized by $R\mu/\sigma$. The kinematic condition is that

$$\mathbf{u} \cdot \mathbf{n} = V_n \tag{6}$$

at each point on the interface.

The problem is now reformulated as a problem in analytic function theory following the formulation of Tanveer and Vasconcelos [23]. The general solution of (4) at each instant has the form

$$\psi = \mathrm{Im}[\bar{z} f(z,t) + g(z,t)], \tag{7}$$

where $z = x + iy$ and the overbar denotes complex conjugate. Here $f(z,t)$ and $g(z,t)$ are the *Goursat functions* which are analytic everywhere in the fluid region $D(t)$. In terms of the Goursat functions, the following relations can easily be established:

$$\begin{aligned} p - i\omega &= 4f'(z,t), \\ u + iv &= -f(z,t) + z\overline{f'(z,t)} + \overline{g'(z,t)}, \\ e_{11} + ie_{12} &= z\overline{f''(z,t)} + \overline{g''(z,t)}, \end{aligned} \tag{8}$$

where ω is the vorticity and u, v are the x- and y-components of velocity. Defining s to be the arclength traversed in a counterclockwise direction around the bubble boundary $z(s,t)$, the stress boundary condition can be written in the form

$$f(z,t) + z\overline{f'(z,t)} + \overline{g'(z,t)} = -i\frac{z_s}{2} \quad \text{on } \partial D(t). \tag{9}$$

Using the second equation of (8) and (9), the kinematic condition can be written as

$$\operatorname{Im}\left[(z_t + 2f)\,\bar{z}_s\right] = -\frac{1}{2} \quad \text{on } \partial D(t). \tag{10}$$

Equations (9) and (10) are supplemented by boundary conditions on $f(z,t)$ and $g'(z,t)$ at infinity. In this paper we consider far-field conditions of the form

$$f(z,t) = f_n z^n + \cdots + f_0 + O\left(\frac{1}{z}\right), \tag{11}$$

$$g'(z,t) = g_m z^m + \cdots + g_0 + O\left(\frac{1}{z}\right). \tag{12}$$

2.2. Evolution of the Cauchy transform. Given the above formulation, it is instructive to consider the evolution of the Cauchy transform $C(z,t)$ introduced in (2). The following result is central to the subsequent developments in this paper.

THEOREM 2.1 (evolution of the Cauchy transform). *The Stokes flow problem described in section* 2.1 *is equivalent to the equation*

$$\frac{\partial C(z,t)}{\partial t} + \frac{\partial I(z,t)}{\partial z} = R(z,t) \tag{13}$$

together with (10) *and boundary condition* (11). *Here $C(z,t)$ and $I(z,t)$ are defined for $z \in D_c(t)$ (i.e., inside the bubble) by*

$$\begin{aligned} C(z,t) &= \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{\bar{z}'}{z'-z}\,dz', \\ I(z,t) &= \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{-2f(z',t)\bar{z}'}{z'-z}\,dz' \end{aligned} \tag{14}$$

and where the forcing term $R(z,t)$ is given by

$$R(z,t) = 2\left(g_m z^m + g_{m-1}z^{m-1} + \cdots + g_0\right). \tag{15}$$

Remark. The viscous sintering problem, which involves the evolution of fluid drops (rather than bubbles), has been investigated using the Cauchy transform approach by Crowdy [5]. There, the Cauchy transform of the domain was defined by the area integral

$$C(z,t) \equiv \frac{1}{\pi}\int\int_{D(t)} \frac{dx'dy'}{z'-z}. \tag{16}$$

In the case of a bounded fluid region, (16) is equivalent to our current definition (2), by the complex form of Green's theorem [1]. Note also that, in the viscous sintering problem, the inhomogeneous term $R(z,t)$ is absent since there are no singularities present to drive the flow; i.e., the flow is driven purely by surface tension. For the

Stokes bubbles considered here, the term $R(z,t)$ represents the forcing due to the singular behavior of the flow at infinity.

Remark. In the viscous sintering problem, [5] provides a proof of the forward result for Theorem 2.1, i.e., that a solution to the Stokes flow problem satisfies (13) (with $R(z,t)=0$). This paper provides the first proof of the "backward" result, and hence the equivalence of the two formulations. Although the proof here is for bubbles, it may easily be modified to the case of fluid drops.

Proof of Theorem 2.1. We first prove the "backward" result; i.e., we assume the existence of a solution $C(z,t)$, $f(z,t)$ to (10), (13) (with $f(z,t)$ analytic in $D(t)$ and having far-field behavior (11)) and show that (9) is satisfied for an appropriately defined $g'(z,t)$ that is analytic in the fluid domain $D(t)$ and satisfies (12). By direct differentiation we have

$$\frac{\partial C(z,t)}{\partial t} = -\frac{1}{2\pi i}\oint_{\partial D(t)} \frac{z'_t \bar{z}' dz'}{(z'-z)^2} + \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{(\bar{z}'dz')_t}{z'-z}. \tag{17}$$

Next, the kinematic condition (10) is rewritten in the form

$$(\bar{z}dz)_t = z_t d\bar{z} + \bar{z}(dz_t) + 2fd\bar{z} - 2\bar{f}dz + ids. \tag{18}$$

Substituting (18) into (17) gives the equation

$$\begin{aligned}\frac{\partial C(z,t)}{\partial t} &= \frac{1}{2\pi i}\oint_{\partial D(t)} \left[\frac{-z'_t\bar{z}'dz'}{(z'-z)^2} + \frac{z'_t d\bar{z}' + \bar{z}'(dz')_t}{z'-z}\right] \\ &\quad + \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2f(z',t)d\bar{z}' - 2\overline{f}(z',t)dz' + ids}{z'-z}.\end{aligned} \tag{19}$$

The terms in square brackets represent a total (spatial) differential of $z'_t\bar{z}'/(z'-z)$ which is assumed to be single-valued. Therefore the first integral term is zero. After replacing $\partial C/\partial t$ using (13), we obtain

$$\begin{aligned}&R(z,t) + \frac{1}{2\pi i}\frac{\partial}{\partial z}\oint_{\partial D(t)} \frac{2f(z',t)\bar{z}'}{z'-z}\, dz' \\ &\quad - \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2f(z',t)d\bar{z}' - 2\overline{f}(z',t)dz' + ids}{z'-z} = 0.\end{aligned} \tag{20}$$

But

$$\begin{aligned}\frac{1}{2\pi i}\frac{\partial}{\partial z}\oint_{\partial D(t)} \frac{f(z',t)\bar{z}'}{z'-z}\, dz' &= \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{f(z',t)\bar{z}'}{(z'-z)^2}\, dz' \\ &= \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{f_{z'}(z',t)\bar{z}' + f(z',t)\bar{z}'_{z'}}{z'-z}\, dz',\end{aligned} \tag{21}$$

where the subscript z' denotes partial differentiation, and the latter equality follows after integration by parts. Substituting (21) into (20) then yields

$$R(z,t) + \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{\left(2f_{z'}(z',t)\bar{z}' + 2\overline{f}(z',t)\right) - ids}{z'-z} = 0. \tag{22}$$

It is convenient to introduce the notation

$$\tilde{f}(z,t) = f(z,t) - (f_n z^n + \cdots + f_0)$$

so that $\tilde{f}(z,t)$ represents the component of $f(z,t)$ that decays to zero as $z \to \infty$. By the well-known properties of Cauchy integrals, we have for $z \in D_c(t)$

$$\frac{1}{2\pi i} \oint_{\partial D(t)} \frac{\tilde{f}(z',t)}{z'-z}\, dz' = 0, \tag{23}$$

$$\frac{1}{2\pi i} \frac{\partial}{\partial z} \oint_{\partial D(t)} \frac{\tilde{f}(z',t)}{z'-z}\, dz' = 0, \tag{24}$$

$$\frac{1}{2\pi i} \oint_{\partial D(t)} \frac{R(z',t)}{z'-z}\, dz' = R(z,t). \tag{25}$$

Equation (22) is now modified by adding twice the conjugate of (23) and the product of $2\bar{z}$ with (24) to it. After employing (25), the modified equation is written as

$$\overline{\Phi}(z,t) + \bar{z}\Phi'(z,t) + \Psi(z,t) = 0 \tag{26}$$

for $z \in D_c(t)$, where

$$\Phi(z,t) = \frac{1}{2\pi i} \oint_{\partial D(t)} \frac{2\tilde{f}(z',t)}{z'-z}\, dz', \tag{27}$$

$$\Psi(z,t) = \frac{1}{2\pi i} \oint_{\partial D(t)} \frac{R(z',t) + 2f_{z'}(z',t)\bar{z}' + 2\overline{f}(z',t)}{z'-z}\, dz', - \frac{1}{2\pi i} \oint_{\partial D(t)} \frac{ids}{z'-z}. \tag{28}$$

Additionally, upon taking the limit of (26) as $z \to \tau$, a point on the boundary of $D_c(t)$, we obtain the relation

$$\overline{\Phi}(\tau,t) + \bar{\tau}\Phi'(\tau,t) + \Psi(\tau,t) = 0. \tag{29}$$

The functions $\Phi(z,t)$ and $\Psi(z,t)$ are analytic in $D_c(t)$, and the boundary condition (29) is identical to the one satisfied by the Goursat functions in the plane problem of the theory of elasticity for the region $D_c(t)$, under the assumption that the boundary of the region is free from the action of external forces. It follows from the theorem of uniqueness [15] of the solution to the plane problem of elasticity that

$$\Phi(z,t) = i\alpha z + \beta,$$
$$\Psi(z,t) = -\bar{\beta},$$

where α is a real and β is a complex constant. But from (23), $\Phi(z) = 0$ in $D_c(t)$, implying that $\alpha = \beta = 0$. Hence

$$\Phi(z,t) = 0, \qquad \Psi(z,t) = 0. \tag{30}$$

From the second identity of (30) it is concluded that the following function may be analytically continued to $D(t)$:

$$\chi(z,t) = g_m z^m + \cdots + g_0 + \overline{f}(z,t) + \bar{z}f'(z,t) - \frac{i}{2z_s}.$$

Upon associating $g'(z,t)$ with $g_m z^m + \cdots + g_0 - \chi(z,t)$, it is concluded that (10)–(13) determine $f(z,t)$ and $g'(z,t)$ as analytic functions in $D(t)$ that satisfy the boundary condition (9) and far-field condition (12). This proves the "backward" result.

We next consider the "forward" problem; i.e., we show that the Stokes problem (9)–(12) implies (13). First, take the conjugate of (9) and use the fact that $z_s \bar{z}_s = 1$ to write the stress condition in the form

$$2\bar{f}dz + 2\bar{z}df + 2dg = ids.$$

This equation is combined with the kinematic condition (10) to give

$$(\bar{z}dz)_t = 2dg + 2d(\bar{z}f) + \bar{z}dz_t + z_t d\bar{z}. \tag{31}$$

Substituting (31) into (17) then yields the equation

$$\frac{\partial C(z,t)}{\partial t} = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2g'(z',t)dz'}{z'-z} + \frac{2d(\bar{z}'f)}{z'-z} + \left[-\frac{z'_t \bar{z}' dz'}{(z'-z)^2} + \frac{\bar{z}'(dz')_t}{z'-z} + \frac{z'_t d\bar{z}'}{z'-z}\right].$$

The terms in square brackets represent a total (spatial) differential and therefore give zero total contribution to the integral. The first integral term on the right-hand side is rewritten using

$$\frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2g'(z',t)}{z'-z}\, dz' = 2(g_m z^m + \cdots + g_0) = R(z,t).$$

Finally, using integration by parts on the second integral term and rearranging, we obtain (13), which completes the proof. □

2.3. Analytic continuation inside the fluid region. The functions $C(z,t)$, $I(z,t)$, and $R(z,t)$ defined by the integrals (14) and (15) are all analytic inside the bubble. We now consider the analytic continuations of these functions inside the fluid domain $D(t)$, i.e., exterior to the bubble. By the continuation principle, (13) is also the equation relating these (analytically continued) functions inside $D(t)$.

It is assumed that the bubble boundary $\partial D(t)$ is an analytic curve. This implies that there exists a (unique) function (known as the *Schwarz function* of the curve [11]) analytic inside an annular domain containing the curve $\partial D(t)$ which satisfies the equation

$$\bar{z} = S(z,t) \tag{32}$$

everywhere on the curve $\partial D(t)$. But, by the Plemelj formulae,

$$S(z,t) = C(z,t) - C_i(z,t), \tag{33}$$

$$-2f(z,t)S(z,t) = I(z,t) - I_i(z,t), \tag{34}$$

$$2g'(z,t) = R(z,t) - R_i(z,t), \tag{35}$$

where, for $z \in D(t)$, the functions $C_i(z,t)$, $I_i(z,t)$, and $R_i(z,t)$ are given by the integrals

$$C_i(z,t) = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{\bar{z}'}{z'-z} dz', \tag{36}$$

$$I_i(z,t) = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{-2f(z',t)\bar{z}'}{z'-z}dz', \tag{37}$$

$$R_i(z,t) = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2g'(z',t)}{z'-z}dz'. \tag{38}$$

The formulae (33)–(35) provide expressions for the analytic continuations of $C(z,t)$, $I(z,t)$, and $R(z,t)$ into the fluid domain $D(t)$.

2.4. Conservation of finite poles of $C(z,t)$. The following result will be useful in constructing exact solutions to the Stokes flow problem (10)–(13).

Let $C(z,t)$, $f(z,t)$ be solutions to (10)–(13). If $C(z,0)$ initially has a pole at a finite point $z_j(0)$ inside the fluid domain, then (provided the solution exists) $C(z,t)$ continues to have a pole at the point $z_j(t)$, where $z_j(t)$ satisfies the ODE

$$\dot{z}_j(t) = -2f(z_j(t),t). \tag{39}$$

To derive the result, first note that the complex form of Green's theorem [1] can be used to write $I(z,t)$ in the form

$$I(z,t) = -\frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2f(z',t)-2f(z,t)}{z'-z}\ \bar{z}'dz' - 2f(z,t)C(z,t). \tag{40}$$

Define the function $\Sigma(z,t)$ as

$$\Sigma(z,t) = -\frac{1}{2\pi i}\oint_{\partial D(t)} \frac{2f(z',t)-2f(z,t)}{z'-z}\ \bar{z}'dz'. \tag{41}$$

The singularity of the integrand in (41) is removable, and $\Sigma(z,t)$ is therefore analytic in $D(t)$. Thus,

$$I(z,t) = \Sigma(z,t) - 2f(z,t)C(z,t), \tag{42}$$

which, when substituted into (13), yields the following PDE for the Cauchy transform:

$$\frac{\partial C}{\partial t} - 2f(z,t)\frac{\partial C}{\partial z} - 2\frac{\partial f(z,t)}{\partial z}C(z,t) + \frac{\partial \Sigma}{\partial z}(z,t) = R(z,t). \tag{43}$$

This equation also governs the analytic continuation of $C(z,t)$ inside $D(t)$. But inside $D(t)$, (43) has the form of a first-order linear equation for $C(z,t)$ with coefficients that are known a priori to be analytic in $D(t)$. Thus, provided they exist, solutions for $C(z,t)$ will have the same analytic structure inside $D(t)$ as solutions of a first-order linear PDE with analytic coefficients. Using the well-known theory of such equations, it is deduced that the pole singularities are preserved and move on characteristics. In this case,

$$-\dot{z}_j(t) - 2f(z_j(t),t) = 0, \tag{44}$$

and the result is demonstrated.

Remark. Note that the above result says nothing about any singularities of $C(z,t)$ at infinity. In the present application, the coefficient function $f(z,t)$ is singular at infinity (and, in the language of ODEs [13], is therefore a *fixed singularity* of (43)).

Moreover, depending on the far-field flow, $R(z,t)$ may also be singular at infinity. The behavior of $C(z,t)$ at $z \to \infty$ must be examined by a local analysis of (13) in each case. Such an analysis will be seen to be crucial in determining whether $C(z,t)$ can consistently preserve a rational function form under evolution, i.e., satisfy the closure condition. When this is the case, the pole at infinity does not move into the finite complex plane. These issues are discussed further in section 3.

Remark. The results of this section are closely related to the results of section 4.5 of Cummings, Howison, and King [10], where rather different arguments are used.

2.5. Circulation theorem and conserved quantities. In the context of the viscous sintering problem, Crowdy [5] derived a circulation theorem which provides the existence of conserved quantities associated with certain choices of initial conditions. This theorem can be extended to the case of a bubble in an infinite flow.

Let the curve γ_j be a *fixed* closed curve surrounding the isolated pole singularity $z_j(t) \in D(t)$. Because $z_j(t)$ moves at finite speed (see (39)), γ_j can always be chosen so that it continues to enclose the (moving) singularity $z_j(t)$, at least for sufficiently short times. Consider the circulation-type quantity

$$\oint_{\gamma_j} C(z,t)dz. \tag{45}$$

Because γ_j is assumed fixed, we have

$$\frac{d}{dt}\oint_{\gamma_j} C(z,t)dz = \oint_{\gamma_j} \frac{\partial C(z,t)}{\partial t}dz = \oint_{\gamma_j}\left(-\frac{\partial I(z,t)}{\partial z} + 2g'(z,t) + R_i(z,t)\right)dz. \tag{46}$$

But $R_i(z,t)$ is analytic everywhere inside γ_j, as is $g'(z,t)$ if the flow has no singularities in the finite plane. Use of Cauchy's theorem then implies that

$$\frac{d}{dt}\oint_{\gamma_j} C(z,t)dz = -\left[I(z,t)\right]_{\gamma_j}, \tag{47}$$

where the square bracket denotes the change in $I(z,t)$ around the contour γ_j. But, by (33) and (34), while $I(z,t)$ has a simple pole at $z_j(t)$, it is single-valued so that the right-hand side of (47) vanishes. This implies that

$$\oint_{\gamma_j} C(z,t)dz = A_j(t), \qquad j = 1, \ldots, N, \tag{48}$$

are constants of the motion, i.e.,

$$A_j(t) = A_j(0). \tag{49}$$

Note that in (48) we have used the fact that the simple pole of $C(z,t)$ at $z_j(t)$ is preserved in time.

The existence of the above conserved quantities has not been pointed out in any previous studies of bubbles in ambient Stokes flows. It is, however, closely related to a similar result that has been observed in the context of evolving viscous blobs [5, 17, 6].

3. Exact solutions. The advantage of considering the Cauchy transform $C(z,t)$ is that its evolution equation is particularly simple. In certain situations, it will be seen that $C(z,t)$ can retain a rational function form under evolution. In this case, the corresponding solutions will be called *exact* in the sense that the evolution depends on just a *finite set* of time-evolving parameters. As an example of the simplicity of the evolution equations, suppose $C(z,0)$ is rational with a finite distribution of simple pole singularities. Suppose too that it is known that $C(z,t)$ preserves this functional form under evolution. Then, if $C(z,0)$ has the form

$$C(z,0) = \sum_{j=1}^{N} \frac{A_j(0)}{z - z_j(0)}, \tag{50}$$

then, by the results above, $C(z,t)$ has the form

$$C(z,t) = \sum_{j=1}^{N} \frac{A_j(0)}{z - z_j(t)}, \tag{51}$$

where

$$\dot{z}_j(t) = -2f(z_j(t),t), \qquad j = 1,\dots,N. \tag{52}$$

With the singularities under control at all regular points of the evolution equation (13), the question of whether $C(z,t)$ can consistently preserve a given rational function form must be determined by a local analysis of (13) at the fixed singularity at infinity. This will be illustrated in the context of the examples considered in section 3.1.

Even supposing that $C(z,t)$ indeed evolves as a rational function, it still remains to reconstruct the corresponding time-evolving domain from knowledge of the Cauchy transform. This can be done using conformal maps, but it is important to observe that in the case of unbounded domains such as here, knowledge of the Cauchy transform does not uniquely determine the unbounded domain. Rather, it determines it only up to a real degree of freedom which can be associated with the freedom to specify the bubble area. The appendix provides a discussion of this point in terms of the inverse problem of 2-D potential theory.

None of the authors Antanovskii [2], Siegel [22], or Tanveer and Vasconcelos [23] use the above formulation but instead make direct use of a conformal map formulation in a parametric ζ-plane. It is instructive to retrieve the conformal mapping solutions of [2, 22, 23] using the above perspective. In this way, certain advantages of the Cauchy transform formulation will become apparent. In particular, we gain important insight into the bubble area evolution in each case and see exactly when it is possible to specify it externally.

We note that, in the process of retrieving the conformal mapping solutions of [2, 22, 23], the values of $f(z_j(t),t)$ are computed with the aid of the conformal map. One can bypass the introduction of a conformal map and compute f from the kinematic condition (10) by using an alternative representation of the boundary (e.g., algebraic curves; see [8]). Although this may require the use of a boundary integral numerical calculation, one still has the advantage of a finite/exact representation of the interface, since the preserved rational function form of the Cauchy transform implies that the shape can be described by a small finite number of parameters.

3.1. Exact solutions of Tanveer and Vasconcelos [23].

3.1.1. The Cauchy transform. Tanveer and Vasconcelos assume that the far-field form of $f(z,t)$ and $g'(z,t)$ are of the form

$$f(z,t) \sim f_1 z + \mathcal{O}(1), \tag{53}$$

$$g'(z,t) \sim g_1 z + \mathcal{O}(1). \tag{54}$$

Specifically, in the notation of Tanveer and Vasconcelos [23],

$$f_1 = \frac{1}{4}\left(\frac{p_\infty(t)}{\mu} - i\omega_0\right) \tag{55}$$

and

$$g_1 = \frac{1}{2}\left(\alpha_0 - i\beta_0\right). \tag{56}$$

$p_\infty(t)$ and ω_0 are the fluid pressure and vorticity in the far-field, respectively, while α_0 and β_0 characterize the strain rates of a linear straining flow at infinity.

As an illustrative example, we take the first case considered by Tanveer and Vasconcelos [23], that is, a bubble in a shear flow of the form

$$\mathbf{u} = (\Gamma y, 0). \tag{57}$$

This corresponds to the far-field values $\alpha_0 = 0$ and $\beta_0 = -\omega_0 = \Gamma$ in the notation of (55) and (56), or equivalently

$$f_1 = \frac{i\Gamma}{4}, \qquad g_1 = -\frac{i\Gamma}{2} \tag{58}$$

in the notation of (53) and (54).

Tanveer and Vasconcelos [23] show the existence of exact solutions that are polynomial maps, i.e., of the form (1) with $b_i = 0$. To retrieve these, let us seek solutions in which $C(z,t)$ has the rational function form

$$C(z,t) = A(t)z; \tag{59}$$

that is, the only singularity of $C(z,t)$ in the fluid region is a single simple pole singularity at infinity. It is necessary to check that this is a consistent solution of (13). To do this, we analyze (13) in the neighborhood of infinity and find that we require

$$\frac{\partial}{\partial t}\left(A(t)z\right) + \frac{\partial}{\partial z}\left(-2f_1 z(A(t)z)\right) + o(z) = 2g_1 z + o(z). \tag{60}$$

This is consistent, provided $A(t)$ satisfies

$$\dot{A}(t) - 4f_1 A(t) = 2g_1, \tag{61}$$

where this equation comes from equating coefficients of z in (60).

3.1.2. Conformal mapping. We now consider the conformal maps for which the corresponding Cauchy transform $C(z,t)$ has the form (16). Consider, for example, the mapping from the unit ζ-disc given by

$$z(\zeta,t) = \frac{a}{\zeta} + b\zeta, \tag{62}$$

where a and b are functions of time and where a can be assumed real (using a rotational degree of freedom of the Riemann mapping theorem). Note that, on the unit ζ-circle,

$$\bar{z} = a\zeta + \frac{\bar{b}}{\zeta}. \tag{63}$$

From (62) we have that

$$\frac{1}{\zeta} = \frac{z}{a} - \frac{b}{a}\zeta. \tag{64}$$

Using (64) in (63) we obtain

$$\bar{z} = \frac{\bar{b}}{a}z + \left(a - \frac{|b|^2}{a}\right)\zeta, \tag{65}$$

which is valid on the unit ζ-circle. By comparison with (33) we make the identifications

$$C(z,t) = \frac{\bar{b}}{a}z, \qquad C_i(z,t) = -\left(a - \frac{|b|^2}{a}\right)\zeta(z,t), \tag{66}$$

where we have also used the fact that $C_i(z,t)$ decays at infinity. This shows that all conformal maps of the form (62) have corresponding Cauchy transforms of the form (16) with

$$A = \frac{\bar{b}}{a}. \tag{67}$$

Note that a and b can be multiplied by any real constant (corresponding to changing the area of the elliptical bubble) and the Cauchy transform (16) remains unchanged. A degree of freedom is therefore available, and this is used up, following Tanveer and Vasconcelos [23], by specifying the area of the bubble to be fixed in time, i.e.,

$$a^2 - |b|^2 = R^2, \tag{68}$$

where R is constant. (Alternatively, one can specify the bubble area to be an arbitrary function of time, corresponding to the presence of a source or sink at infinity.) This gives one equation relating a and b as derived by Tanveer and Vasconcelos [23]. The second equation obtained by them is

$$\frac{d(ab)}{dt} = -(2I_0 + i\Gamma)ab + i\Gamma a^2, \tag{69}$$

where $I_0 = I(0,t)$ and

$$I(\zeta,t) = \frac{1}{4\pi i}\oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'}\left[\frac{\zeta+\zeta'}{\zeta'-\zeta}\right]\frac{1}{|z_\zeta(\zeta',t)|}. \tag{70}$$

This equation must, of course, be equivalent to (61). To see this, note that upon applying the Poisson integral formula [1] to the kinematic condition it can be shown that

$$z_t(\zeta,t) + 2f(z(\zeta,t),t) = \zeta\left[I(\zeta,t) + \frac{i\omega_0}{2}\right] z_\zeta(\zeta,t) \tag{71}$$

(see Tanveer and Vasconcelos [23]). Substituting the conformal map (62) into this equation and equating powers of ζ^{-1} provides

$$\dot{a} + 2f_1 a = -a\left(I_0 + \frac{i\omega_0}{2}\right). \tag{72}$$

Eliminating f_1 in (61) using (72), as well as using (67) and (58), gives the required (69).

3.2. Exact solutions of Antanovskii [2]. Antanovskii [2] assumes the far-field asymptotic form of $f(z,t)$ and $g'(z,t)$ to be

$$f(z,t) \sim f_1 z + \mathcal{O}(1/z), \tag{73}$$

$$g'(z,t) \sim g_m z^m + \mathcal{O}(1/z), \tag{74}$$

where $m \geq 1$ is some positive integer and f_1 is some real time-dependent function that is independent of z. This gives rise to a situation in which the far-field flow is irrotational and given by an mth order straining flow.

Anticipating the form of the singularity in $C(z,t)$ at infinity in order to satisfy the closure condition, we seek solutions in which $C(z,t)$ is an mth order polynomial, i.e.,

$$C(z,t) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z. \tag{75}$$

Analyzing the singularity of (13) at infinity gives the equations

$$\begin{aligned} \dot{A}_m - 2(m+1)f_1 A_m &= 2g_m, \\ \dot{A}_{m-1} - 2mf_1 A_{m-1} &= 0, \\ \dot{A}_{m-2} - 2(m-1)f_1 A_{m-2} &= 0, \\ &\cdots \\ \dot{A}_1 - 2f_1 A_1 &= 0. \end{aligned} \tag{76}$$

First, it is immediately clear that if $A_{m-1}(0) = A_{m-2}(0) = \cdots = A_1(0) = 0$, then $A_{m-1}(t) = A_{m-2}(t) = \cdots = A_1(t) = 0$. It therefore remains only to satisfy the equation for A_m, viz.,

$$\dot{A}_m - 2(m+1)f_1 A_m = 2g_m. \tag{77}$$

3.2.1. Conformal mapping. Now consider the class of conformal maps given by

$$z(\zeta,t) = \frac{a}{\zeta} + b\zeta^m, \tag{78}$$

where a is again assumed to be real. On the unit ζ-circle we have

$$\overline{z}(\zeta^{-1}, t) = a\zeta + \frac{\bar{b}}{\zeta^m}. \tag{79}$$

But from (78),

$$\frac{1}{\zeta} = \frac{z}{a} - \frac{b}{a}\zeta^m, \tag{80}$$

which, when substituted into (79), gives

$$S(z(\zeta)) = \overline{z}(\zeta^{-1}, t) = a\zeta + \bar{b}\left(\frac{z}{a} - \frac{b}{a}\zeta^m\right)^m \tag{81}$$

from which, by comparison with (33), we deduce that

$$C(z,t) = \frac{\bar{b}}{a^m} z^m. \tag{82}$$

Thus, all maps of the form (78) yield Cauchy transforms of the form $C(z,t) = A_m z^m$ with

$$A_m = \frac{\bar{b}}{a^m}. \tag{83}$$

Note again that b and a can be multiplied by a real number without changing $C(z,t)$, while the only equation to be satisfied is (77). Thus, the relevant evolution equations for the two parameters a and b are (77) along with an area evolution equation which may be arbitrarily specified. These can be shown to be equivalent to those given in equation (37) of Antanovskii [2].

3.3. Exact solutions of Siegel [22].

3.3.1. The Cauchy transform. Siegel [22] assumes the far-field asymptotic form of $f(z,t)$ and $g'(z,t)$ to be

$$f(z,t) \sim f_3 z^3 + f_1 z + \mathcal{O}(1/z), \tag{84}$$

$$g'(z,t) \sim g_1 z + \mathcal{O}(1/z). \tag{85}$$

f_3 and g_1 are externally specifiable and will be taken to be constant in time (g_1 gives a measure of an imposed irrotational straining flow at infinity, while f_3 produces a rotational far-field component). Such boundary conditions in which $f(z)$ is a nonlinear function of z in the far field will be referred to as *nonlinear* far field conditions. Siegel [22] makes the choices

$$g_1 = 1, \qquad f_3 = \frac{\epsilon}{2}. \tag{86}$$

Antanovskii [3] derived exact *steady* solution for the bubble shape subject to the above far-field conditions. The analysis of Siegel [22] essentially generalizes the results of Antanovskii [3] to the case where the bubble evolves in a quasi-steady, time-dependent manner.

Suppose we seek solutions to (13) in which $C(z,t)$ evolves as

$$C(z,t) = \sum_{j=1}^{N} \frac{A_j(t)}{z - z_j(t)} \tag{87}$$

for all times. It is clear that

$$C(z,t) \sim \frac{B(t)}{z} \quad \text{as } z \to \infty, \tag{88}$$

where

$$B(t) = \sum_{j=1}^{N} A_j(t). \tag{89}$$

By the circulation theorem, all the quantities $\{A_j(t)\}$ are constants of the motion. This implies, by (89), that $B(t)$ is also a constant of the motion. Moreover, all the poles $\{z_j(t)\}$ move according to (52). The Cauchy transform is determined at each instant by these equations.

As mentioned earlier, it is important to verify that the solution (87) is a consistent solution of (13) at the fixed singularity at the point at infinity. To see this, note from (36) that

$$C_i(z,t) \sim -\frac{1}{2\pi i}\left(\oint_{\partial D(t)} \bar{z}' dz'\right)\frac{1}{z} + \mathcal{O}(1/z^2), \tag{90}$$

while, from (37),

$$I_i(z,t) \sim \mathcal{O}\left(\frac{1}{z}\right) \tag{91}$$

as $z \to \infty$. But the area of the bubble (denoted $\mathcal{A}$) is precisely

$$\mathcal{A} = \frac{1}{2i}\oint_{\partial D(t)} \bar{z}' dz'. \tag{92}$$

Therefore, using (33) and (88),

$$S(z) = \frac{B(t) + \mathcal{A}/\pi}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \tag{93}$$

as $z \to \infty$. Inside the fluid domain $D(t)$, (13) takes the form

$$\frac{\partial C(z,t)}{\partial t} + \frac{\partial}{\partial z}\left[-2f(z,t)S(z,t) + I_i(z,t)\right] = 2g'(z,t) + R_i(z,t). \tag{94}$$

A local analysis of this equation as $z \to \infty$ therefore implies that

$$\frac{\partial}{\partial z}\left(-2f_3 z^3\left(\frac{B}{z} + \frac{\mathcal{A}/\pi}{z}\right)\right) + \mathcal{O}(1/z) = 2g_1 z + \mathcal{O}(1/z) \tag{95}$$

so that, equating coefficients of the singularity at $\mathcal{O}(z)$, we have

$$-2(\mathcal{A}/\pi + B)f_3 = g_1. \tag{96}$$

This is a necessary condition if the solution (87) is to be a consistent solution of (13).

It is crucial to note that if $C(z,t)$ is determined at each instant, then, in contrast to the previous examples of Tanveer and Vasconcelos [23] and Antanovskii [2], (96) represents an *additional* constraint on the free boundary evolution. In this case, there is no freedom to externally specify the bubble area evolution. Rather, it is determined implicitly by the exact solution itself. Since A has been shown to be a constant of the motion, (96) implies that, for solutions of the form (87), the area $\mathcal{A}$ of the bubble is necessarily fixed in time if f_3 and g_1 are constant (independent of time), which has been assumed to be the case, i.e.,

$$\mathcal{A} = -\pi \left(B + \frac{g_1}{2f_3} \right). \tag{97}$$

The initial area of the bubble therefore dictates the value of B.

In summary, solutions of (13) in which $C(z,t)$ is rational with a finite set of simple pole singularities (and for which $f(z,t)$ and $g'(z,t)$ have the far-field form (53) and (54)) are admitted, and, in these solutions, the bubble area remains constant. If we seek solutions in which the bubble area is not fixed in time (but varies, for example, at some externally specified rate Q), the solutions will not be such that the Cauchy transform has the far-field behavior (88) for all times $t > 0$. In that case, the functional form of the Cauchy transform will not be of the proposed simple rational character and will not lend itself to exact solutions.

3.3.2. Conformal maps. The corresponding conformal maps from a unit ζ-disk to fluid domains whose Cauchy transforms have a finite distribution of simple pole singularities with the far-field behavior (88) are given by rational functions of the form

$$z(\zeta,t) = \frac{C}{\zeta}\left(\frac{\prod_{j=1}^{N}(\zeta - \eta_j(t))}{\prod_{j=1}^{N}(\zeta - \zeta_j(t))} \right), \tag{98}$$

where C can be assumed real (to use up the rotational degree of freedom in the Riemann mapping theorem). It follows that

$$\overline{z}(\zeta^{-1},t) = C\zeta \left(\frac{\prod_{j=1}^{N}(1 - \zeta\bar{\eta}_j(t))}{\prod_{j=1}^{N}(1 - \zeta\bar{\zeta}_j(t))} \right). \tag{99}$$

It is clear that because $S(z,t) = \bar{z}$, this class of domains is such that the far-field asymptotic form of $C(z,t)$ is of the form in (88). It is also clear that

$$z_j(t) = \overline{z(\zeta_j,t)} \tag{100}$$

so that the N equations (39) can be viewed as providing evolution equations for the parameters $\{\zeta_j(t) | j = 1,\dots,N\}$. The N constants of motion derived from the circulation theorem (by taking a contour γ_j around each distinct pole $z_j(t)$) provide N additional equations for the parameters $\{\eta_j(t) | j = 1,\dots,N\}$. Finally, the parameter C is determined by condition (96). Equivalently, one could determine C by ensuring that the bubble area is conserved under evolution.

Siegel [22] considers one of the above solutions in detail. This solution has a mapping given by the special choice

$$z(\zeta,t) = \frac{1}{\zeta}\frac{\gamma_0 + \gamma_1\zeta^2}{1 - \gamma_2\zeta^2}. \tag{101}$$

It is straightforward to see that this map corresponds to domains with the Cauchy transform

$$C(z,t) = \frac{E(t)}{z - z_0(t)} + \frac{E(t)}{z + z_0(t)}, \tag{102}$$

where

$$z_0(t) = z(\sqrt{\gamma_2}, t). \tag{103}$$

In this case,

$$C(z,t) \sim \frac{2E(t)}{z} \quad \text{as } z \to \infty. \tag{104}$$

The relevant equations of motion for this solution are

$$\dot{z}_0(t) = -2f(z_0(t), t), \tag{105}$$

$$E(t) = E(0), \tag{106}$$

$$-\left(\frac{\mathcal{A}}{\pi} + 2E\right) = \frac{g_1}{2f_3}, \tag{107}$$

where the last equation is just (96). Equations (105) and (106) determine $C(z,t)$ at each instant, while (107) additionally constrains the bubble area to be fixed in time. This additional constraint is absent in the solutions of [23] and [2].

The three equations obtained by Siegel [22] are

$$\dot{\gamma}_2 = -2\gamma_2 \left[I(\sqrt{\gamma_2}, t) - \epsilon \frac{\gamma_0^2}{\gamma_2} \right], \tag{108}$$

$$\gamma_1 = \frac{\gamma_2}{\epsilon \gamma_0}, \tag{109}$$

$$\gamma_0 = \left[\frac{2(1+\gamma_1^2)}{c_1 + (c_1^2 - c_2)^{1/2}} \right]^{1/2}, \tag{110}$$

where condition (110) derives from the fact that the bubble area is taken to be constant and equal to π. c_1 and c_2 are defined as

$$c_1 = 1 - \epsilon\gamma_1^2 \left(2(1-\epsilon) + \epsilon\gamma_1^2\right) \quad \text{and} \quad c_2 = 4\epsilon^2\gamma_1^2(1+\gamma_1^2)\left[3 + \epsilon(2+\epsilon)\gamma_1^2\right], \tag{111}$$

while

$$I(\zeta,t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta}\right] \left\{ \frac{1}{|z_\zeta|} + \mathrm{Re}\left[\frac{\epsilon\gamma_0^2}{\zeta^2}\right] \right\}. \tag{112}$$

We now indicate how (105)–(107) are equivalent to (108)–(110). First we use the fact, as derived by Siegel [22], that

$$z_t(\zeta,t) + 2f(z(\zeta,t),t) = \zeta z_\zeta(\zeta,t) \left(-\frac{\epsilon\gamma_0^2}{\zeta^2} + I(\zeta,t) \right) \tag{113}$$

to eliminate $f(z_0(t), t)$ from (105). This reproduces exactly (108). Next recall from (93) that as $z \to \infty$,

$$S(z) \sim \left(2E + \frac{\mathcal{A}}{\pi}\right)\frac{1}{z}. \tag{114}$$

But

$$\begin{aligned} S(z(\zeta,t)) = \overline{z(\zeta,t)} = &= \zeta\left(\frac{\gamma_0\zeta^2+\gamma_1}{\zeta^2-\gamma_2}\right) \\ &\sim -\frac{\gamma_1}{\gamma_2}\zeta \quad \text{as} \quad \zeta \to 0 \\ &\sim -\left(\frac{\gamma_1\gamma_0}{\gamma_2}\right)\frac{1}{z} \quad \text{as} \;\; z \to \infty, \end{aligned} \tag{115}$$

where we have used that fact that $\zeta \sim \frac{\gamma_0}{z}$ as $z \to \infty$. Together, (114), (115), (86), and (107) yield Siegel's second equation (109). Finally, the equation for $S(z(\zeta, t), t)$ in (115) also yields the following equation for $E(t)$ in terms of the conformal mapping parameters, viz.,

$$E(t) = \frac{\gamma_0\gamma_2+\gamma_1}{2} z_\zeta(\sqrt{\gamma_2}, t) = \frac{\gamma_0\gamma_2+\gamma_1}{2\gamma_2(1-\gamma_2^2)^2}\left(\gamma_1\gamma_2+\gamma_1\gamma_2^3-\gamma_0+3\gamma_0\gamma_2^2\right). \tag{116}$$

Using (116), (86), and the fact that $\mathcal{A} = \pi$ in (107) combine, after some further algebra, to retrieve Siegel's final equation (110).

Remark. The Cauchy transform formulation is crucial in proving that the bubble area is conserved for all maps of the form (98). This is because the bubble area $\mathcal{A}$ appears explicitly in the form of the far-field asymptotics of $C(z, t)$. To establish this general result using the direct conformal mapping approach of [23, 22] would be exceedingly difficult.

Remark. If the conformal map $z(\zeta, t)$ of the form (98) has N poles, the Cauchy transform formulation also immediately implies the existence of precisely N conserved quantities, one associated with each of the N poles.

3.4. A final example. We now give an example to show that the previous result is rather special and that, for general nonlinear flow conditions at infinity, while exact solutions to the problem exist, it is not generally possible to externally specify the bubble area evolution in these solutions.

A natural generalization of the far-field conditions considered in the previous three subsections is

$$f(z,t) \sim f_3 z^3 + f_1 z + \mathcal{O}(z^{-1}), \tag{117}$$

$$g'(z,t) \sim g_3 z^3 + g_1 z + \mathcal{O}(z^{-1}). \tag{118}$$

These far-field conditions are again nonlinear and are exactly those considered by Antanovskii [4] in his studies of the formation of cusped bubbles, although his analysis is restricted to the derivation of classes of exact *steady* solutions. The special case $g_3 = 0$ reduces to the far-field conditions considered in Antanovskii [3] and Siegel [22] (as well as in section 3.3). For this reason, one might expect the bubble area to again be constant under evolution. However, this is not the case, as will now be shown.

Suppose that we seek exact solutions in which $C(z,t)$ has the rational function form

$$C(z,t) = A_0(t)z + \sum_{k=1}^{N} \frac{A_k(t)}{z - z_k(t)} \tag{119}$$

for all times. As $z \to \infty$,

$$C(z,t) \sim A_0(t)z + \left(\sum_{k=1}^{N} A_k(t)\right)\frac{1}{z} + \dots, \tag{120}$$

and this far-field form is forced by (13), as will be shown. First, it is known immediately that the points $\{z_k(t)|k = 1, 2, \dots, N\}$ must satisfy

$$\dot{z}_k(t) = -2f(z_k(t), t), \qquad k = 1, \dots, N, \tag{121}$$

while the circulation theorem implies that

$$A_k(t) = A_k(0), \qquad k = 1, \dots, N. \tag{122}$$

It remains to determine the evolution of $A_0(t)$, as well as to ensure that (119) is a consistent solution of (13) at $z \to \infty$. Analyzing (13) as $z \to \infty$, at $\mathcal{O}(z^3)$ we get

$$-8f_3A_0(t) = 2g_3, \tag{123}$$

which implies that $A_0(t)$ must also be constant. $C(z,t)$ is now completely determined, but we expect to be able to specify another degree of freedom associated with the bubble area $\mathcal{A}$. But at $\mathcal{O}(z)$, we get another equation having the form

$$-4\left(f_1A_0 + f_3\mathcal{A}\right) = 2g_1. \tag{124}$$

Equation (124) is a further constraint on the solution and can be thought of as an equation governing the bubble area $\mathcal{A}$. It is clear from (124) that the bubble area is not constant this time. Rather, how the bubble area evolves is governed by the exact solution itself. To see this, recall that f_1 (which is related to the far-field pressure evaluated in the near field of the bubble) is a time-evolving quantity whose evolution is governed by (113) and is not externally controllable.

Note that if g_3 is taken equal to zero (so that the far-field conditions of Siegel [22] are retrieved), then necessarily $A_0(t) = 0$ (by (123)), and then $\mathcal{A}$ turns out to be constant (by (124)).

4. Conclusion. We have introduced a Cauchy transform formulation of the problem of Stokes flow for a single bubble and proven its equivalence to the usual formulation of free surface Stokes flow. The new formulation has been used to derive a very broad class of exact solutions (namely, the class for which the Cauchy transform takes the form of a rational function), generalizing the set of solutions which have heretofore appeared in the literature. Indeed, it is surmised that the class of solutions discussed here is maximal, i.e., that it includes all cases for which the evolution is reducible to a finite-dimensional system of ODEs. We use our formulation to investigate when it is possible to externally specify the evolution of bubble area. This issue is extremely difficult to address using other approaches. In general, it is found that when the Goursat function $f(z)$ has a nonlinear far-field behavior it is *not* possible to find exact solutions *and* specify the bubble area evolution. Instead the bubble area in such cases is determined by the exact solution. In the case of pure linear strain (so that $g'(z) \sim g_1 z$ as $z \to \infty$) it happens that the bubble area is constant, and this situation corresponds to a physically interesting case. However, this occurrence is somewhat coincidental.

Concerning generalizations, it is possible to extend the present formulation to the case of a more general compressible bubble with an externally specified equation of state relating its internal pressure (say, $p_B(t)$) to its area. Pozrikidis [16] has considered such problems using numerical boundary integral methods, while Crowdy [7] has generalized the solutions of Tanveer and Vasconcelos [23] to this case.

Appendix. Inverse problem of potential theory. In this appendix we explain why, in the case of a simply connected unbounded domain, knowledge of $C(z,t)$ determines the domain $D(t)$ up to a single real degree of freedom associated with specification of the bubble area.

Consider a smooth family of bounded, time-evolving, simply connected planar domains $D(t)$ in some time interval $t \in [0,T)$. Define the harmonic moments of these domains to be the integrals of a basis of all functions harmonic in $D(t)$ at time t. Suppose all the moments of $D(t)$ for $t \in [0,T)$ are known. It is a well-known result of the inverse problem of 2-D potential theory that the domains $D(t)$ can be uniquely reconstructed from knowledge of all these harmonic moments. Varchenko and Etingof [24] discuss this result in detail. For a bounded domain, if the Cauchy transform is defined as

$$C(z,t) = \frac{1}{\pi}\int\int_{D(t)} \frac{dx'dy'}{z'-z} = \frac{1}{2\pi i}\oint_{\partial D(t)} \frac{\bar{z}'dz'}{z'-z}, \tag{125}$$

then it is a generating function for the harmonic moments because, Laurent expanding for large $|z|$,

$$C(z,t) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}, \tag{126}$$

where

$$M_n = \frac{1}{\pi}\int\int_{D(t)} z'^{n dx'dy'}. \tag{127}$$

The harmonic functions

$$\left\{ \mathrm{Re}\Big[z^n\Big],\ \mathrm{Im}\Big[z^n\Big] \Big| n = 0,1,2,\dots \right\} \tag{128}$$

span the space of functions harmonic in $D(t)$. The real and imaginary parts of the set of complex moments (127) generate all the harmonic moments of $D(t)$.

Exactly the same result pertains to the case of unbounded domains $D(t)$, except that if the moments are defined in terms of area integrals over $D(t)$, some of them do not exist, owing to the unboundedness of the domain. This is the reason for our choice of defining the Cauchy transform, from the outset, as the line integral

$$\frac{1}{2\pi i}\oint_{\partial D} \frac{\bar{z}'dz'}{z'-z}. \tag{129}$$

If $z \in D_c$, the Cauchy transform defines an analytic function $C(z,t)$, say. Assume D_c contains the origin. Then $C(z,t)$ has a Taylor expansion

$$C(z,t) = \sum_{n=0}^{\infty} M_n z^n, \tag{130}$$

where

$$M_n = \frac{1}{2\pi i} \oint_{\partial D} \frac{\bar{z}' dz'}{z'^{n+1}}. \tag{131}$$

Suppose now that $C(z,t)$ is known. This is equivalent to knowledge of the moments $M_n, n = 0, 1, \ldots$, from which it is possible to infer the values of the harmonic moments associated with the set of functions

$$\left\{ \mathrm{Re}\left[\frac{1}{z^{n+1}}\right],\ \mathrm{Im}\left[\frac{1}{z^{n+1}}\right] \middle| n = 0, 1, , \ldots \right\}. \tag{132}$$

This is a subspace of codimension one in the space of functions harmonic in $D(t)$ because it excludes the constant function 1, which is also harmonic in $D(t)$. Thus, knowledge of $C(z,t)$ is not quite enough to determine all harmonic moments—just one more moment is needed. Generalizing the set (131), the "moment" corresponding to the constant function 1 is

$$\frac{1}{2\pi i} \oint_{\partial D} \bar{z}' dz', \tag{133}$$

which is proportional to the area of the bubble. Thus, in the case of an unbounded simply connected domain $D(t)$, the Cauchy transform $C(z,t)$ determines the domain up to a single real degree of freedom associated with the area of the complement of $D(t)$ (here, the area of the bubble).

Acknowledgments. The authors wish to thank B. Gustafsson and S. Tanveer for useful discussions.

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