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Poincaré α-series for classical Schottky groups and its applications

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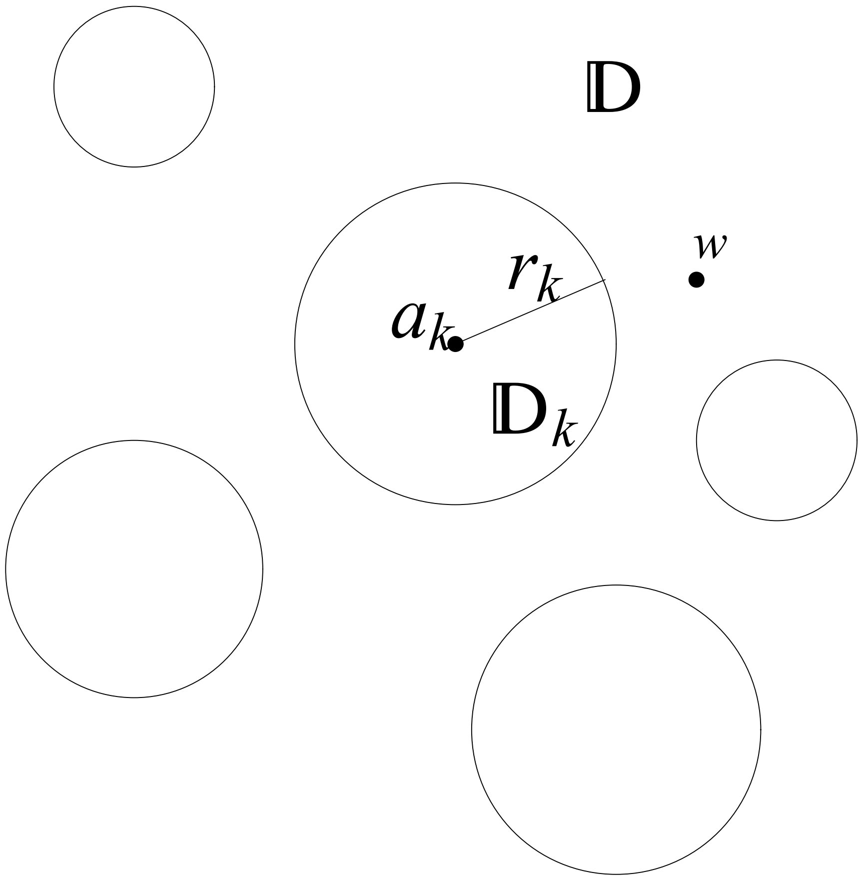
Abstract:

The Poincaré α-series ($\alpha \in \mathbb{R}^n$) for classical Schottky groups is introduced and used to solve Riemann-Hilbert problems for n-connected circular domains. The classical Poincaré θ_2-series can be obtained from the α-series by the substitution $\alpha = 0 \in \mathbb{R}^n$. The real Jacobi inversion problem and its generalisations are investigated via the Poincaré α-series. In particular, it is shown that the Riemann theta-function coincides with the Poincaré α -series. Relations to conformal mappings to slit domains and the Schottky-Klein prime function are established. A fast algorithm to compute Poincaré series for disks close to each other is outlined.

Geometry



Poincaré series

Let $H(z)$ be a meromorphic function. The θ_2-Poincaré series

$$\theta_2(z) := \sum_{j=0}^{\infty} H[\gamma_j(z)]\,(c_j\,z + d_j)^{-2} \quad (1)$$

is associated with a group $\mathcal{E}$ of inversions $\gamma_j(z) = z^*_{(k_p,k_{p-1},\ldots,k_1,k)}$ with respect to $|\,t - a_{k_m}\,| = r_{k_m}$.

The series (1) can be either absolutely convergent or absolutely divergent. Poincaré proposed to investigate the absolute convergence by comparison with the series $\sum_{j=0}^{\infty} (\,|\,c_j\,|)^{-2}$.

W. Burnside, On a Class of Automorphic Functions, Proc. London Math. Soc. 23 (1891), 49-88.

Myrberg P.J., Zur Theorie der Konvergenz der Poincaréschen Reihen, Ann. Acad. Sci. Fennicae, A9, No. 4 (1916), 1-75.

T.Akaza, K.Inoue, Limit sets of geometrically finite free Kleinian groups, Tohoku Math J 36 (1984).

Separation restriction :

$$\Delta = \max_{k \neq m} \frac{r_k + r_m}{|a_k - a_m|} < \frac{1}{(n-1)^{\frac{1}{4}}}.$$

Theorem . Let a rational function $H(z)$ has poles only at regular points of K. Then the Poincaré θ_2 – series converges uniformly in every compact subset of $D \big/ \{\text{limit points of } \mathcal{E}\}$.

Mityushev V. Convergence of the Poincare series for classical Schottky groups, Proc. AMS, 126, 8, 2399-2406, 1998

Riemann-Hilbert problem

$\operatorname{Re} \overline{\lambda(t)}\, \phi(t) = f(t)$ on $| t - a_k | = r_k,\ k = 1, 2, \ldots, n$.

Factorization method $\Rightarrow$ $\operatorname{Re} e^{-i\alpha_k} \phi(t) = g(t)$ on $| t - a_k | = r_k$.
The generalized real Jacobi inversion problem

$\sum_{m=1}^{n-1} \operatorname{Im} w_k(z_m) \equiv \frac{1}{2\pi i} \int_{\partial D} \gamma(t)\, d w_m(t),\ k = 1, 2, \ldots, n-1$

Poincaré α-series

$$\Theta_1(z;\boldsymbol{\alpha}) = H(z) - \sum_{k=1}^{n} e^{2i\alpha_k}\,\overline{H\left(z^*_{(k)}\right)}\,\left(\overline{z^*_{(k)}}\right)' + \sum_{k,k_1} e^{2i(\alpha_k-\alpha_{k_1})}\,H\left(z^*_{(k_1\,k)}\right)\left(z^*_{(k_1\,k)}\right)' - \sum_{k,k_1,k_2} e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2})}\,\overline{H\left(z^*_{(k_2,k_1\,k)}\right)}\,\left(\overline{z^*_{(k_2,k_1\,k)}}\right)' + \ldots,$$

$$\Theta_2(z;\boldsymbol{\alpha}) = H(z) + \sum_{k=1}^{n} e^{2i\alpha_k}\,\overline{H\left(z^*_{(k)}\right)}\,\left(\overline{z^*_{(k)}}\right)' + \sum_{k,k_1} e^{2i(\alpha_k-\alpha_{k_1})}\,H\left(z^*_{(k_1\,k)}\right)\left(z^*_{(k_1\,k)}\right)' + \sum_{k,k_1,k_2} e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2})}\,\overline{H\left(z^*_{(k_2,k_1\,k)}\right)}\,\left(\overline{z^*_{(k_2,k_1\,k)}}\right)' \ldots,$$

$$\theta_2(z;\boldsymbol{\alpha}) = \frac{1}{2}\left[\Theta_1(z;\boldsymbol{\alpha}) + \Theta_2(z;\boldsymbol{\alpha})\right].$$

Schwarz-Christoffel Formula: $f(z) = \int^z \exp[\omega(\zeta)]\, d\zeta$,
[T. K. DeLillo, A. R. Elcrat and J. A. Pfaltzgraff, J. d'Analyse Math. 94 (2004) 17-47]

$$\exp[\omega(\zeta)] = \prod_{m=1}^{n} \prod_{l=1}^{M_m} \left[\prod_{\gamma_o \in O'_m} \frac{z-\gamma_o(\overline{z_{l\,m}})}{z-\gamma_o(\overline{a_m})} \prod_{\gamma_e \in \mathcal{E}'_m} \frac{z-\gamma_e(z_{l\,m})}{z-\gamma_e(a_m)}\right]^{\beta_{l\,m}}$$

[V. Mityushev, CMFT, 12, (2012) 449 - 463]:

$$\exp[\omega(\zeta)] = \prod_{m=1}^{n} \prod_{l=1}^{M_m} \left\{\left(\frac{z_{l\,m}-z}{z_{l\,m}-w}\right)^{\frac{\beta_{l\,m}}{2}} \left[\prod_{k=1}^{n} \frac{\overline{z_{l\,m}-z^*_{(k)}}}{\overline{z_{l\,m}-w^*_{(k)}}}\right]^{\frac{\beta_{l\,m}}{2}} \left[\prod_{k,k_1} \frac{z_{l\,m}-z^*_{(k_1\,k)}}{z_{l\,m}-w^*_{(k_1\,k)}}\right]^{\frac{\beta_{l\,m}}{2}} \ldots\right\} \times$$
$$\left(\prod_{k=1}^{n} \frac{a_k-w}{a_k-z}\right) \left(\prod_{k,k_1} \frac{\overline{a_{k_1}-w^*_{(k)}}}{\overline{a_{k_1}-z^*_{(k)}}}\right) \left(\prod_{k,k_1,k_2} \frac{a_{k_1}-w^*_{(k_1\,k)}}{a_{k_1}-z^*_{(k_1\,k)}}\right) \ldots$$

$$\int_w^z \sum_{k=1}^{\infty} \frac{1}{(n-t)^2}\, dt = \sum_{k=1}^{\infty} \left(\frac{1}{n-z} - \frac{1}{n-w}\right)$$

■ Schottky-Klein prime function

Crowdy, D. The Schottky-Klein Prime Function on the Schottky Double of Planar Domains. Comput. Methods Funct. Theory 10, n. 2, 501-517 (2010)

Let ζ and w be fixed points of $D \bigcup \partial D \setminus \{\infty\}$. The following functions was introduced [Mityushev 2000]

$$\omega(z, \zeta, w) = \ln \prod_{j=0}^{\infty} \mu_j(z, \zeta, w), \quad (2)$$

where

$$\mu_j(z, \zeta, w) = \begin{cases} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}, & \text{if } \gamma_j \in \mathcal{E} \\ \frac{\overline{\zeta - \gamma_j(\overline{z})}}{\overline{\zeta - \gamma_j(\overline{w})}}, & \text{if } \gamma_j \in \mathcal{O} \end{cases}$$

The infinite product (2) converges uniformly in z in every compact subset of $D \bigcup \partial D \setminus \{\infty, \zeta, w\}$.

The following infinite product is correctly defined for z not equal to ∞, ζ, w:

$$\Omega(z, \zeta, w) = \prod_{\gamma_j \in \mathcal{E},\ j \neq 0} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}$$

The Schottky-Klein function

$$S(z, \zeta) = (\zeta - z)\, \Omega(\zeta, z, z)\, \Omega(z, \zeta, \zeta) =$$

$$= (\zeta - z) \prod_{\gamma_j \in \mathcal{E},\ j \neq 0} \frac{z - \gamma_j(\zeta)}{z - \gamma_j(z)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(\zeta)}.$$

The correspondence between j and k_p, k_{p-1}, ..., k_1, k is established via the numeration of the elements of $\mathcal{E}$, i.e., via the relation $\gamma_j(z) = z^*_{(k_p, k_{p-1}, \ldots, k_1, k)}$.

Similar to $S(z, \zeta, w)$ one can introduce $\boldsymbol{\alpha}$-prime functions

$$S(z, \zeta, \boldsymbol{\alpha}) = (\zeta - z)\,\Omega(\zeta, z, z)\,\Omega(z, \zeta, \zeta) =$$

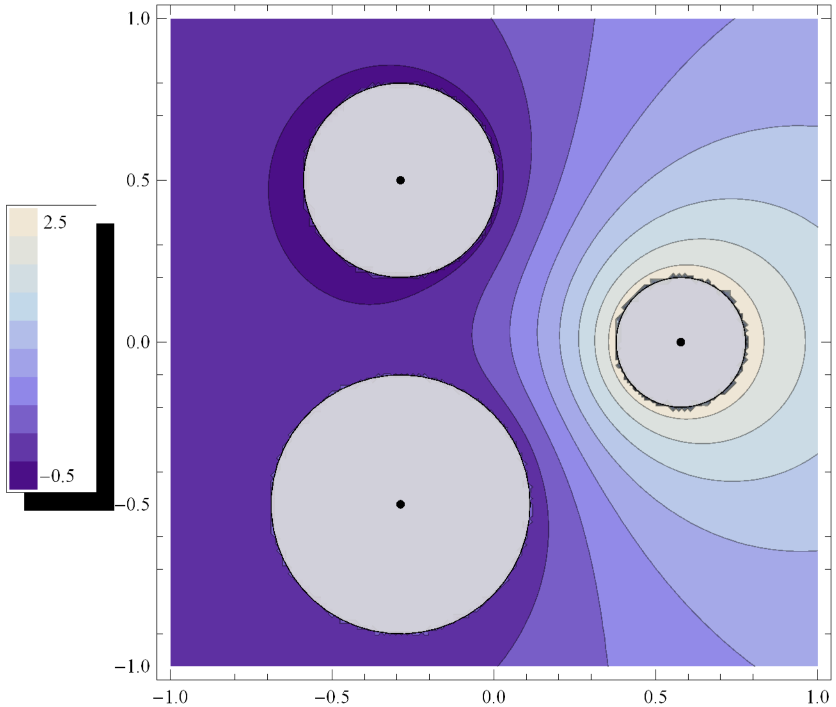
$$= (\zeta - z) \prod_{\gamma_j \in \mathcal{E},\, j \neq 0} e^{2 i s_j(\boldsymbol{\alpha})} \frac{z-\gamma_j(\zeta)}{z-\gamma_j(z)} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(\zeta)},$$

where

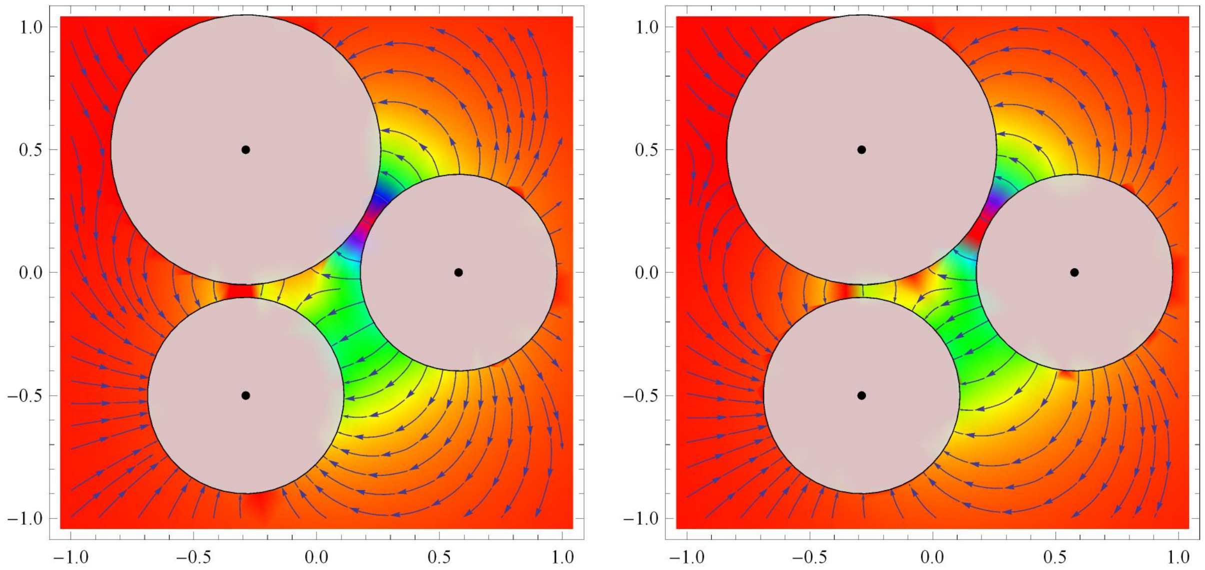
$$s_j(\boldsymbol{\alpha}) = \alpha_k - \alpha_{k_1} + \ldots + \alpha_{k_{p-1}} - \alpha_{k_p}.$$

Fast algorithm

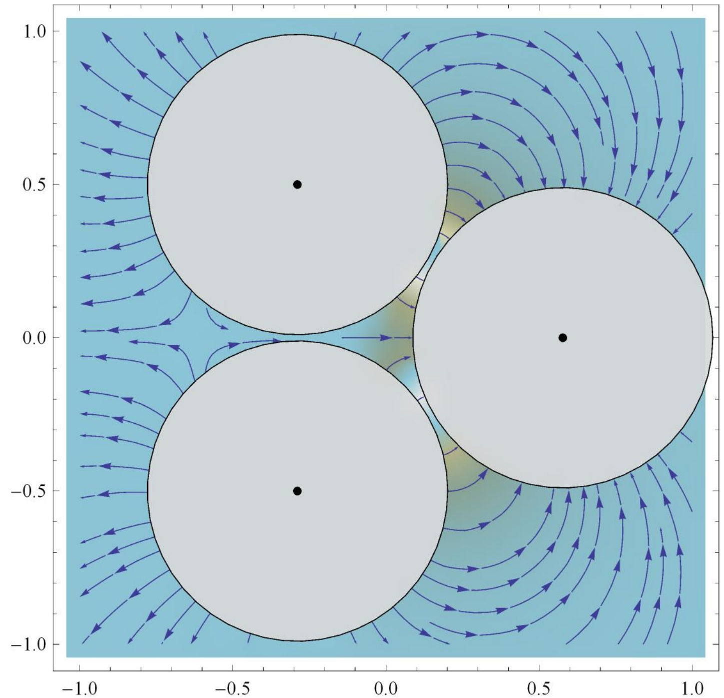
Mityushev V., Rylko N.: A fast algorithm for computing the flux around non-overlapping disks on the plane, Mathematical and Computer Modelling, doi:10.1016/j.mcm.2012.11.019 (2013)



$u_1 = 2.82$, $u_2 = -1.12$, $u_3 = -0.85$; 2 iterations



1 and 2 iterations



Potential $u\left(a_3 + 0.49\, e^{i\theta}\right)$ on the third circle; 6 iterations

θ	*fast*	*classical*
-3	-0.84566	-1.35578
-2	-0.84608	-1.03438
-1	-0.84633	-0.60140
0	-0.84648	0.08666
1	-0.84605	-0.69489
2	-0.84457	-1.60638
3	-0.84548	-1.43547