Two-Dimensional Shapes and Lemniscates

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University of South Florida

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Outline

1. Introduction

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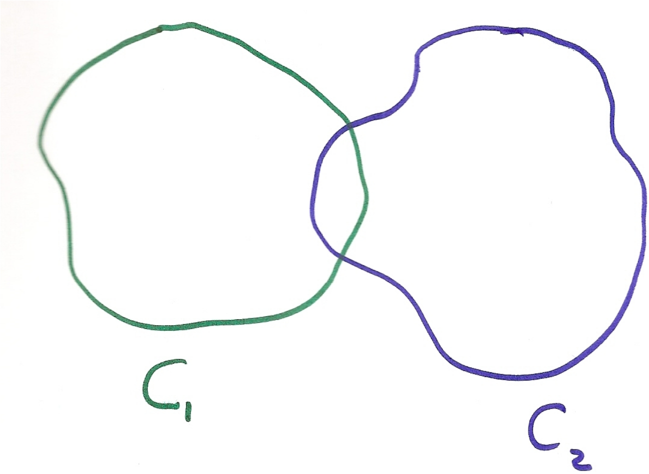

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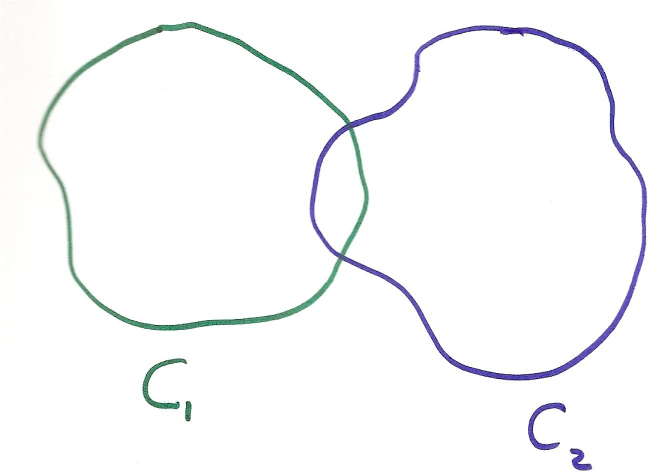

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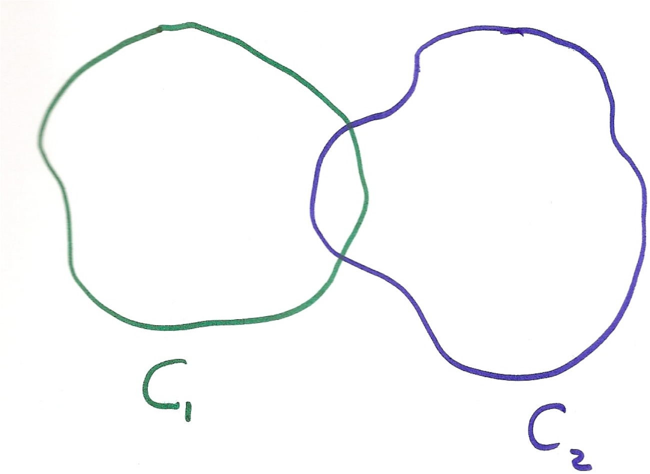

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$\rightsquigarrow$ an orientation preserving diffeo of the circle $\mathbb{T}$.

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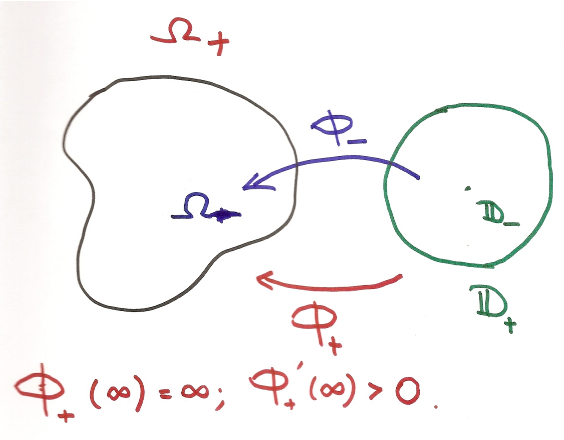

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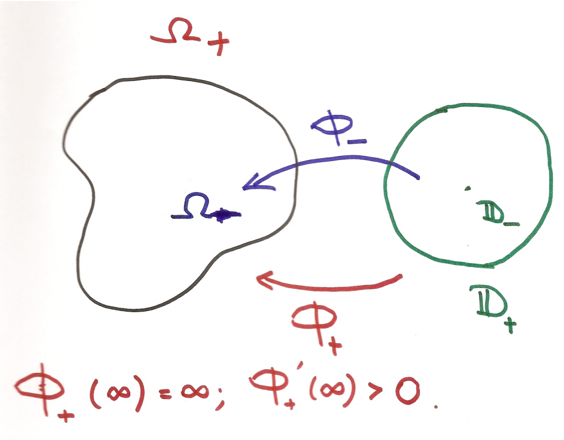

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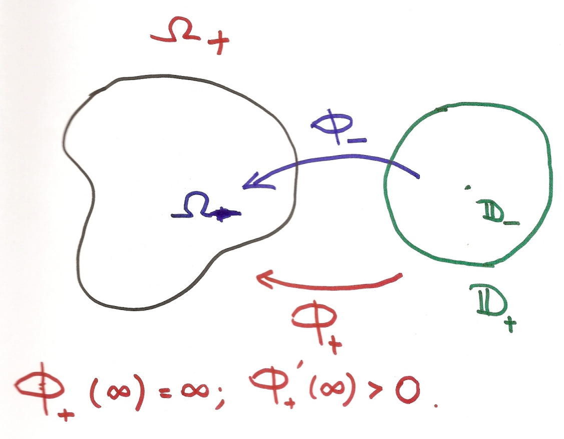

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Note: The statement is false if we replace $\text{Diff}_+(\mathbb{T})$ by $\text{Homeo}_+(\mathbb{T})$, ($\mathfrak{F}$ is neither 1-1 , nor onto).

D. Mumford - E. Sharon, 2004

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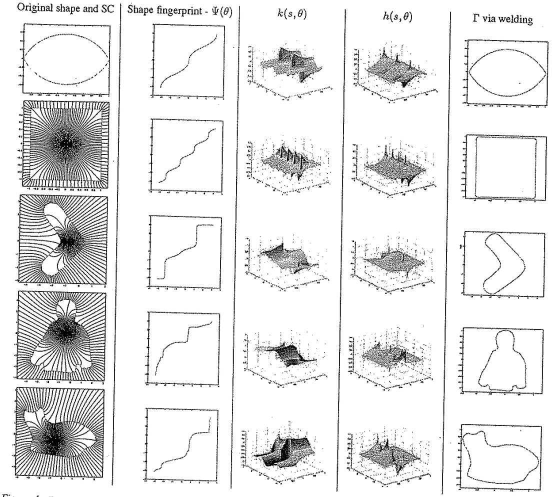
- For $\mathfrak{F}$, $\Phi_{-,+}$ are approximated by the Schwarz - Christoffel integrals.
- For $\mathfrak{F}^{-1}$, $\Phi_{-,+}$ are found via a series of renormalizations and by solving a Riemann - Hilbert type problem.

Mumford - Sharon Data, Examples



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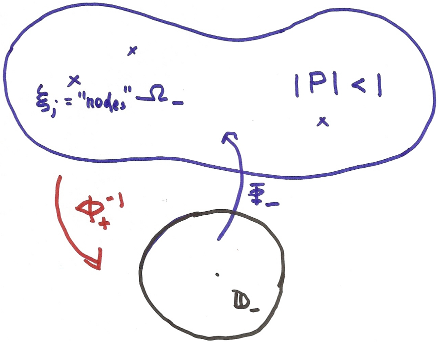

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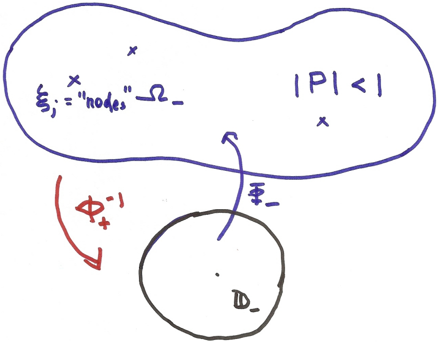

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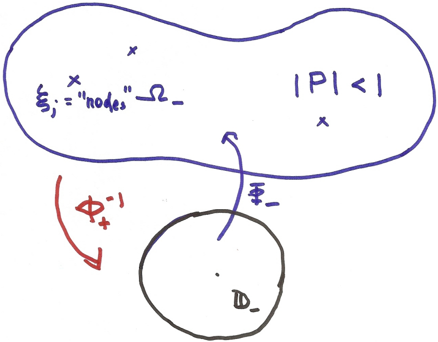

- Ω_- is connected
- All zeros ξ_j, $j = 1, \ldots, n$ and critical points of P lie inside Ω_-

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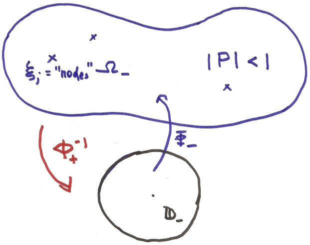

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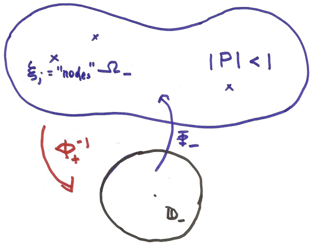

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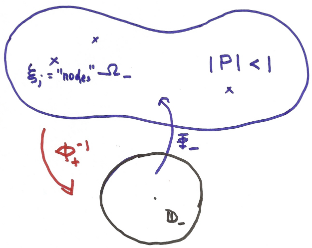

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\begin{aligned}
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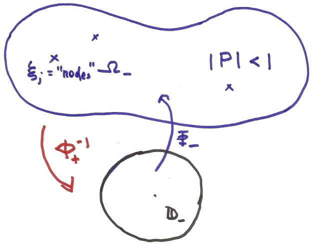

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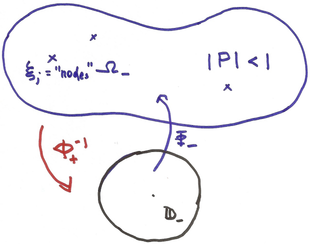

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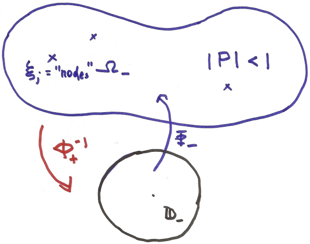

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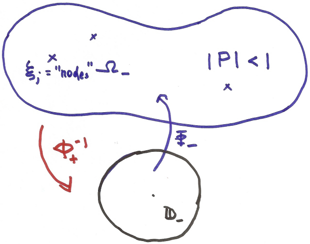

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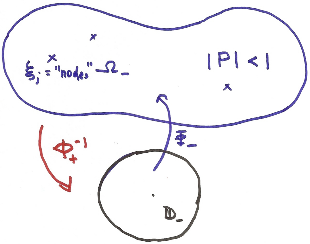

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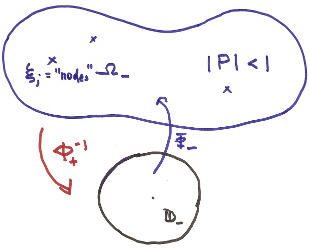

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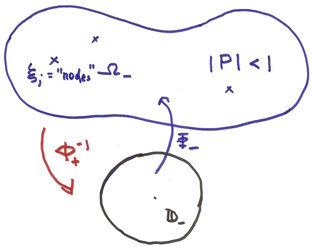

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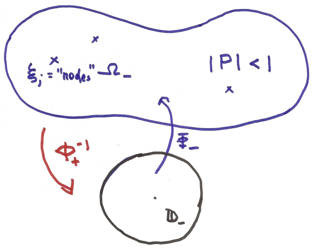

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Theorem

The fingerprint of the lemniscate $\Gamma := \partial\Omega$ *equals*

$$k := \mathbb{T} \to \mathbb{T},\ k = \Phi_+^{-1} \circ \Phi_- = \sqrt[n]{B_1(z)}.$$

Evolution of Bernoulli's Lemniscates

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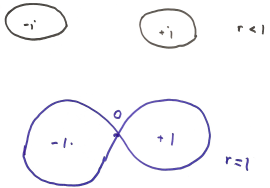

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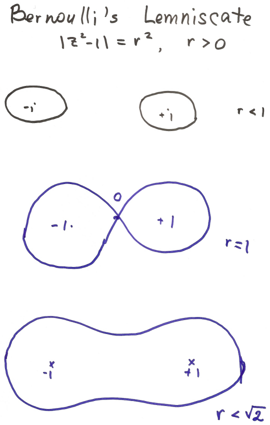

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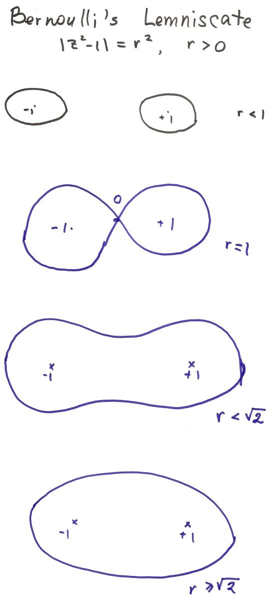

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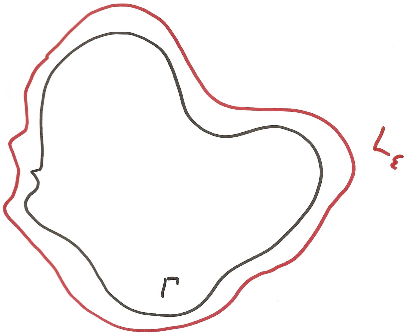

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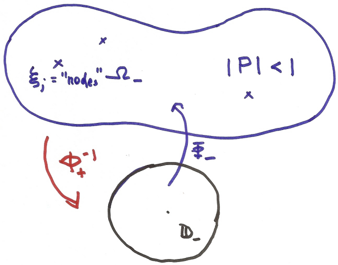

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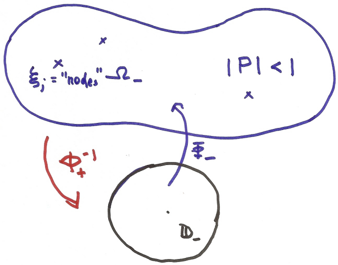

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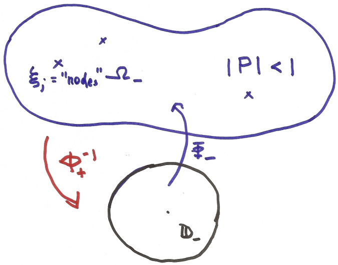

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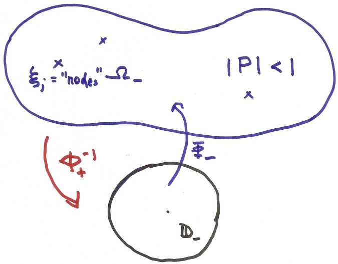

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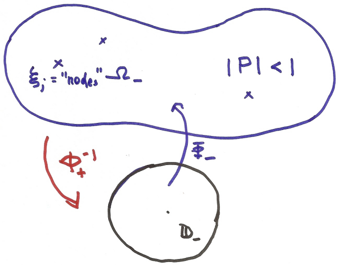

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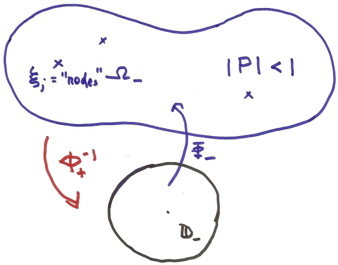

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(ii) Does each such k "fingerprint" a polynomial lemniscate?

Results: Ebenfelt - DK - Shapiro, 2011

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Theorem (I)

Algebraic diffeomorphisms of the unit circle

$$k = \sqrt[n]{B(z)}, \;\; B = e^{i\theta} \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \;\; |a_j| < 1,$$

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Theorem (II)

Every diffeomorphism $k = \sqrt[n]{B(z)}$ *of* $\mathbb{T}$*, where* B *is a Blaschke product of degree* n*, represents the fingerprint of a polynomial lemniscate* $\Gamma := \{|P| = 1, \deg P = n\}$.

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- Perform "balayage inward"

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$$\frac{d}{d\theta}\left(\frac{1}{n}\arg B(e^{i\theta})\right) = \frac{1}{n}\sum_{j=1}^{n} P(e^{i\theta}, a_j), \qquad (1)$$

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Injectivity of $\mathfrak{F}$: "Rigidity" Theorem

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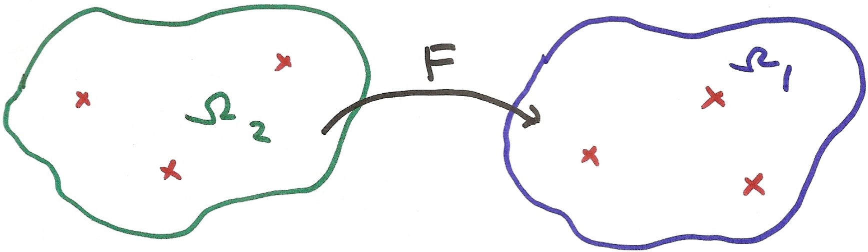

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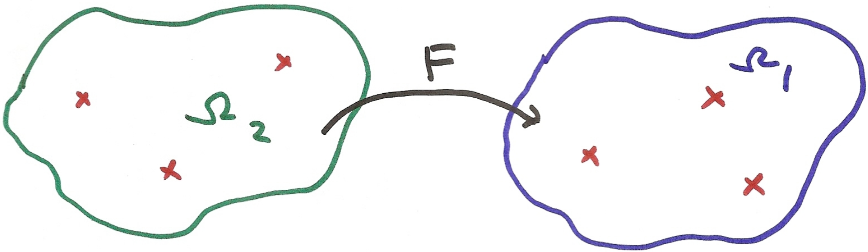

Theorem (III)

Let Ω_1, Ω_2 be (connected) n-lemniscates $\{|P| < 1\}, \{|Q| < 1\}$. If $F : \Omega_2 \to \Omega_1$ is a conformal mapping that maps nodes into nodes, then F is an affine mapping, i.e., $F = Aw + B$.

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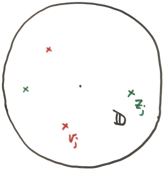

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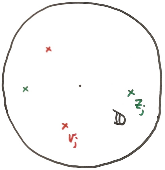

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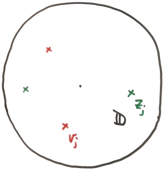

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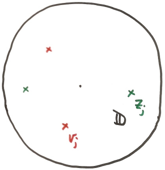

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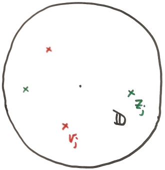

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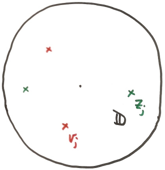

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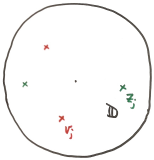

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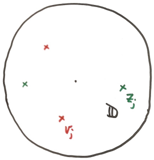

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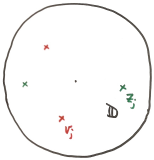

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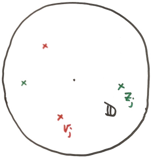

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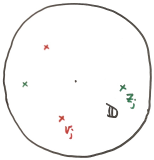

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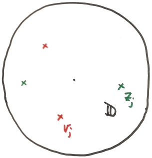

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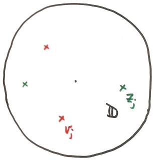

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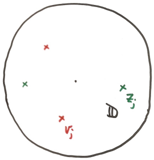

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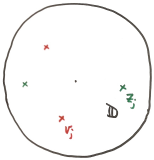

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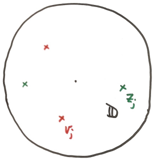

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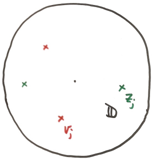

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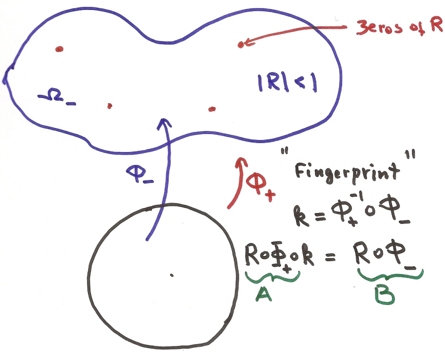

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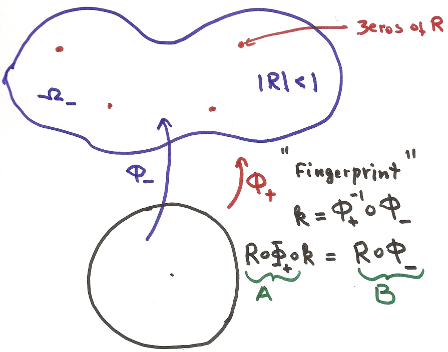

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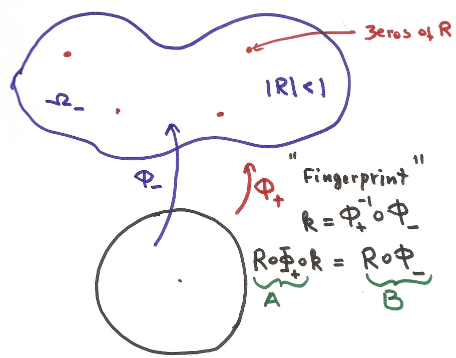

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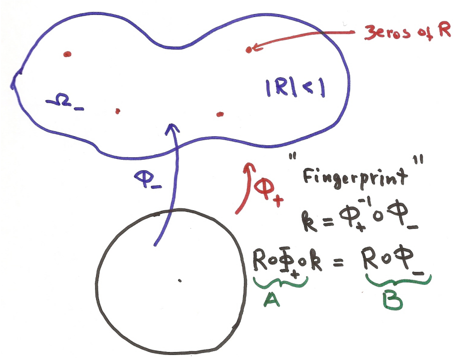

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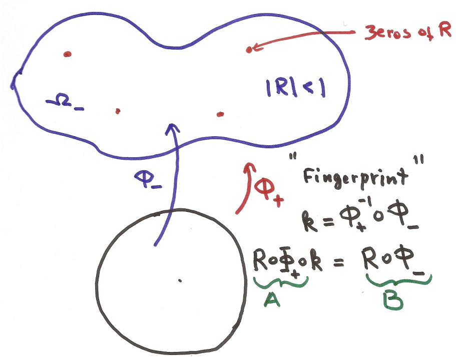

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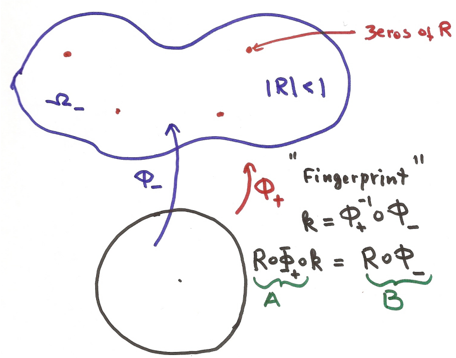

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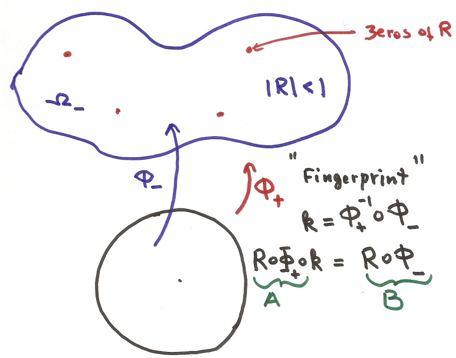

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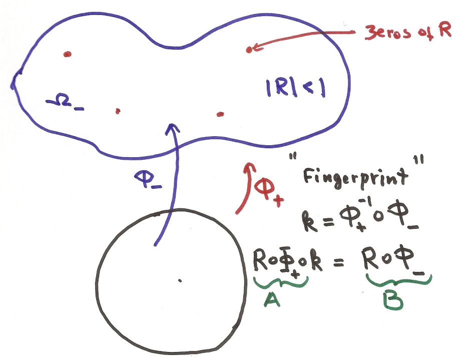

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There is no known direct proof of that fact.

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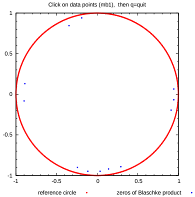
- $k = \sqrt[n]{B(z)}$ $\dashrightarrow$ Lemniscate $= \{|P| < 1\}$. How to find P knowing B?
- How effectively follow this scheme numerically? It looks "doable" numerically at least for $n \leq 4$.
- **Rational Lemniscates:** How to characterize analytically $\Omega_- := \{|R| < 1$, where R is a rational function$\}$? For polynomials NASC are that all the critical values are in the unit disk (E - K - S).
- Differential-geometric properties of lemniscates vs. placement of zeros of Blaschke products forming their fingerprints.

Courtesy of D. E. Marshall: Marshall's "zipping" algorithm

First Blaschke product B_1

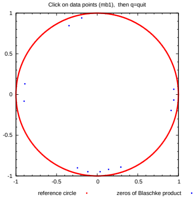
Courtesy of D. E. Marshall: Marshall's "zipping" algorithm

First Blaschke product B_1



Courtesy of D. E. Marshall: Marshall's "zipping" algorithm

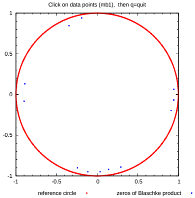
First Blaschke product B_1



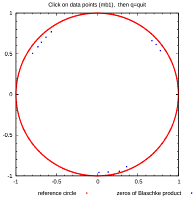
Second Blaschke product B_2

Courtesy of D. E. Marshall: Marshall's "zipping" algorithm

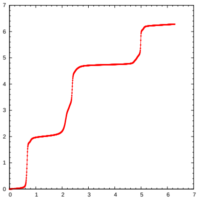
First Blaschke product B_1



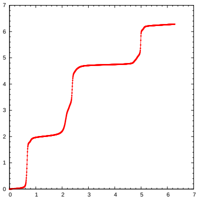
Second Blaschke product B_2



Fingerprint $k = B_2^{-1} \circ B_1$

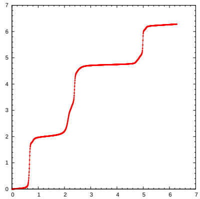


Fingerprint $k = B_2^{-1} \circ B_1$

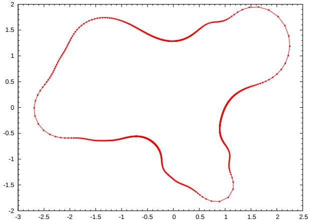


The Rational Lemniscate

Fingerprint $k = B_2^{-1} \circ B_1$



The Rational Lemniscate



THANK YOU!