Computational methods in conformal geometry based on Hodge theory and Ricci flow

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Conformal geometry in mapping, imaging & sensing,
Imperial College, London

Thanks

Thanks for the invitation.

Collaborators

The work is collaborated with Shing-Tung Yau, Feng Luo, Tony Chan, Ronald Lok Ming Lui, Paul Thompson, Yalin Wang, Hong Qin, Dimitris Samaras, Jie Gao, Arie Kaufman, and many other mathematicians, computer scientists and medical doctors.

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal Geometric methods have merits:

1. Unification: All the surfaces in real life can be eventually unified to one of three canonical shapes, the sphere, the plane or the hyperbolic disk.
2. Dimension Reduction: All 3D geometric processing problems are converted to 2D image processing problems.
3. Information Preservation: All the deformation preserves the intrinsic geometric information.
4. General Transformation: Capable of modeling all the mappings among surfaces.

Practical View

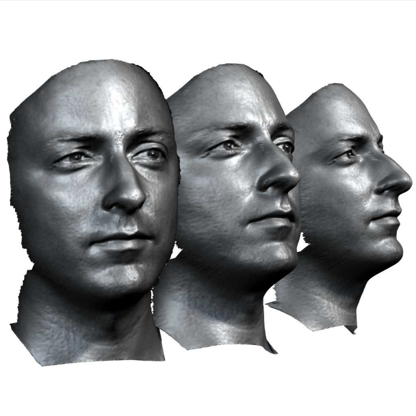
Conformal geometry offers the theoretic frameworks for

- Shape Space
- Mapping Space

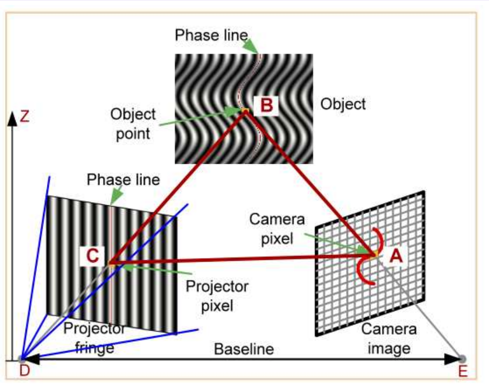
Reasons for Booming

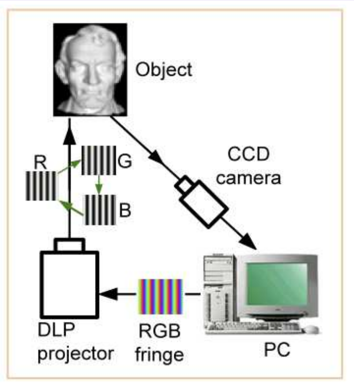
Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.

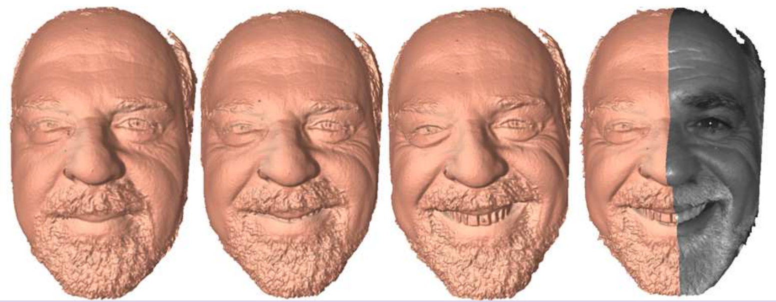


System Layout

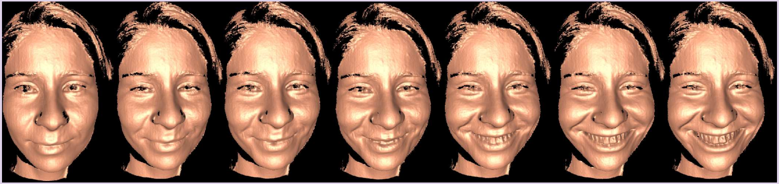




3D Scanning Results



3D Scanning Results



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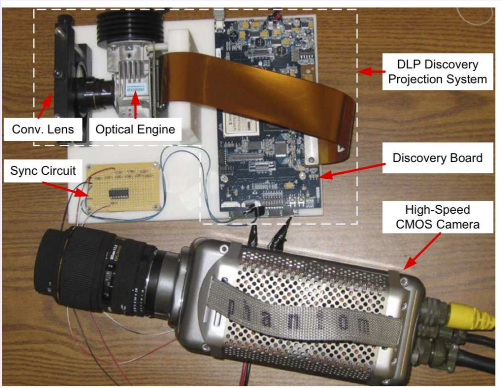
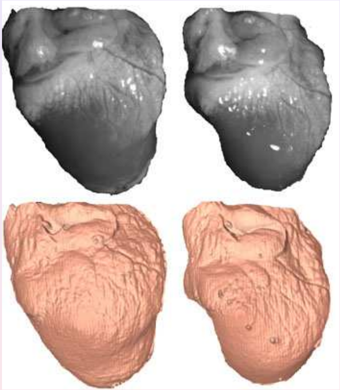




Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE's much easier.

Fundamental Problems

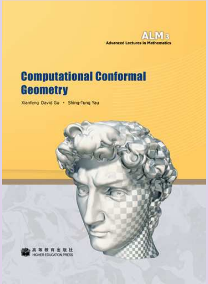
1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

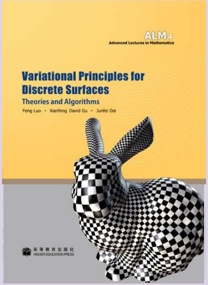
Complete Tools

Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

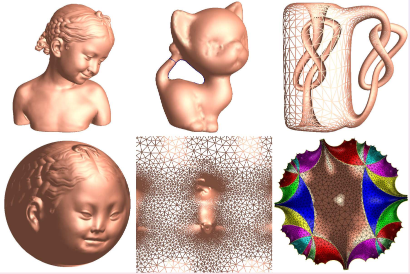
The theory, algorithms and sample code can be found in the following books.

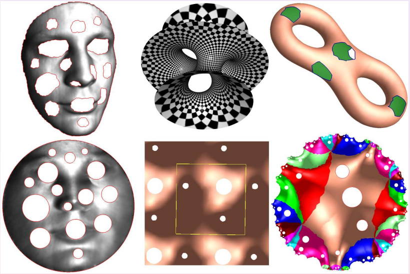




You can find them in the book store.

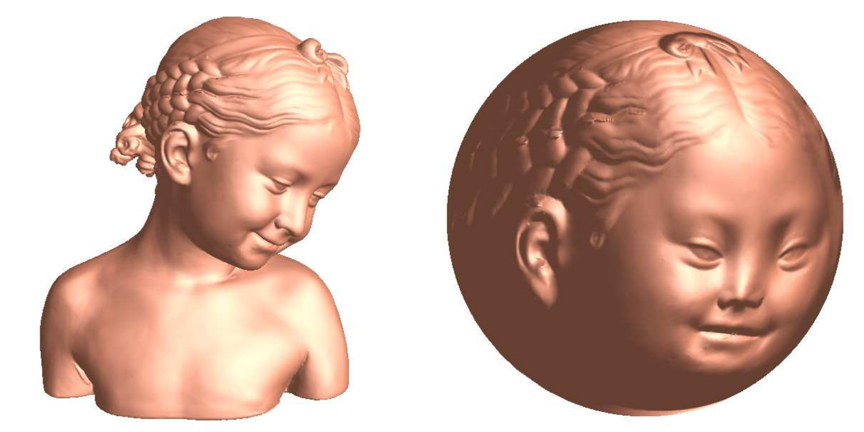
Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.





Computational Method - Harmonic Mapping

Spherical harmonic map



Let (M,g) and (N,h) be Riemannian manifolds, $u : M \to N$ is a C^1 mapping.

$$ds_M^2 = \sum g_{\alpha\beta} dx^\alpha dx^\beta, ds_N^2 = \sum h_{ij}(u(x)) du^i du^j.$$

The pull back metric of h induced by u is $u^*(ds_N^2)$ is a symmetric bilinear form

$$u^*(dS_N^2) = \sum_{\alpha,\beta} (\sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}) dx^\alpha dx^\beta.$$

The *energy density* of mapping u is defined as

$$|du|^2 = \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Equivalently, choose an orthogonal frame field under $u^*(ds_N^2)$, each basis vector field is of unit length under **g**, the dual 1-forms are $\{\omega_1, \omega_2, \cdots, \omega_n\}$, such that

$$u^*(ds_N^2) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega_\alpha)^2.$$

The the energy density of the mapping u is given by

$$|du|^2 = \sum_{\alpha=1}^{n} \lambda_\alpha.$$

Harmonic Energy and Harmonic Mapping

Definition (Harmonic Energy)

The harmonic energy functional $E(u)$ is defined as

$$E(u) = \int_M |du|^2 dv_M,$$

where $dv_M = (detg)^{\frac{1}{2}} dx$ is the volume element of M.

Definition (Harmonic Mapping)

In the space of mappings, the critical points of $E(u)$ are called harmonic mappings.

Suppose u is a mapping from a surface (S,g) to (N,h).
Suppose $\tilde{g} = e^{2\lambda} g$ is another metric of S, conformal to g, then

$$|\tilde{d}u|^2 = e^{-2\lambda}|du|^2, \sqrt{det\tilde{g}} = e^{2\lambda}\sqrt{detg},$$

Then $\tilde{g} = g$. Harmonic energy is invariant under conformal metric transformation.

Theorem

Harmonic energy only depends on the conformal structure of the surface, independent of the choice of Riemannian metric.

Harmonic Mapping

Suppose N is embedded in $\mathbb{R}^3$, $u : S \to N$ is a harmonic mapping, then

$$\Delta_g u^{T_u N} \equiv 0.$$

where $\Delta_g u = (\Delta_g u_1, \Delta_g u_2, \Delta_g u_3)$. Namely, $\Delta_g u$ is orthogonal to the tangent plane at the target space.

Heat Flow method

Definition (Heat flow)

Let $u : S \to N \subset \mathbb{R}^3$, the heat flow is given by

$$\frac{du(x,t)}{dt} = -(\Delta_g u)^{T_{u(x)}N}$$

The heat flow method will deform a mapping to the harmonic mapping under special normalization conditions.

Heat Flow method

Theorem

Harmonic mapping from a genus zero closed surface to the unit sphere must be a conformal mapping.

Proof.

Let $u : S \to \mathbb{S}^2$. Choose isothermal coordinates of both surfaces, define

$$\phi(z) = \langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \rangle$$

then

$$\phi(z) = \frac{1}{4}(|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle).$$

if $\phi(z) = 0$, then the mapping is conformal.
On the other hand, $\frac{\partial \phi(z)}{\partial \bar{z}} = 0$, then $\phi(z)$ is holomorphic.
$\phi(z)dz^2$ is globally defined, the so-called Hopf differential.
Sphere has no non-zero holomorphic quadratic differentials.

Möbius transformation

Theorem

The conformal automorphism from a sphere to itself must be a Möbius transformation

$$z \to \frac{az+b}{cz+d}, ad-bc=1, a,b,c,d \in \mathbb{C}.$$

Rado's theorem

Theorem (Rado)

Let $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary. For any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic mapping $u : D \to \Omega$, such that $u|_\partial D = \phi$, furthermore, u is a diffeomorphism.

Discrete Approximation

We use piecewise linear triangle mesh to approximate the original surface. suppose $u : M \to \mathbb{R}$ the harmonic energy is given by

$$E(u) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

The discrete Laplace-Beltrami operator is given by

$$\Delta f(v_i) = \sum_j w_{ij} (f(v_j) - f(v_i)).$$

where w_{ij} is the cotangent formula.

Suppose a map $w:(M,\sigma|dz|^2)\to(N,\rho|dw|^2)$ is given, the energy density is given by

$$e(w;\sigma,\rho)=\frac{\rho(w(z))}{\sigma(z)}\left(|w_z|^2+|w_{\bar{z}}|^2\right)$$

Harmonic energy is given

$$E(w;\sigma,\rho)=\int e(w;\sigma,\rho)\frac{1}{2i}\sigma dz\wedge d\bar{z}$$

The Euler-Lagrange equation is

$$w_{z\bar{z}} + (\log \rho)_w w_z w_{\bar{z}} = 0,$$

the heat flow is given by

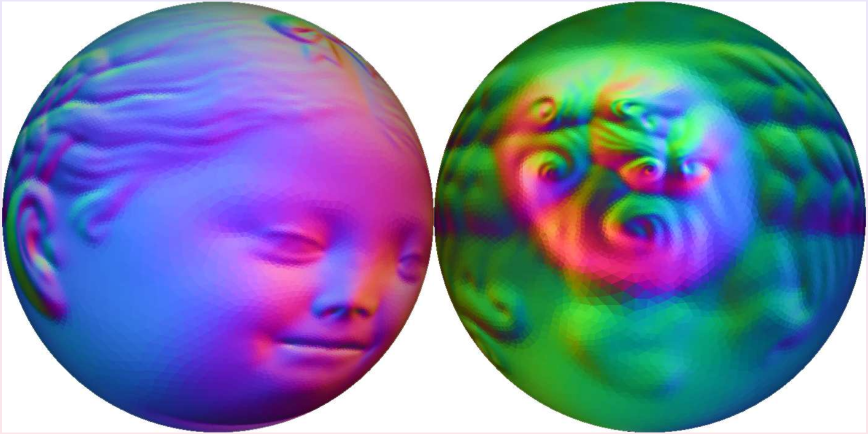
$$\frac{\partial w}{\partial t} = -w_{z\bar{z}} - (\log \rho)_w w_z w_{\bar{z}}.$$

Theorem

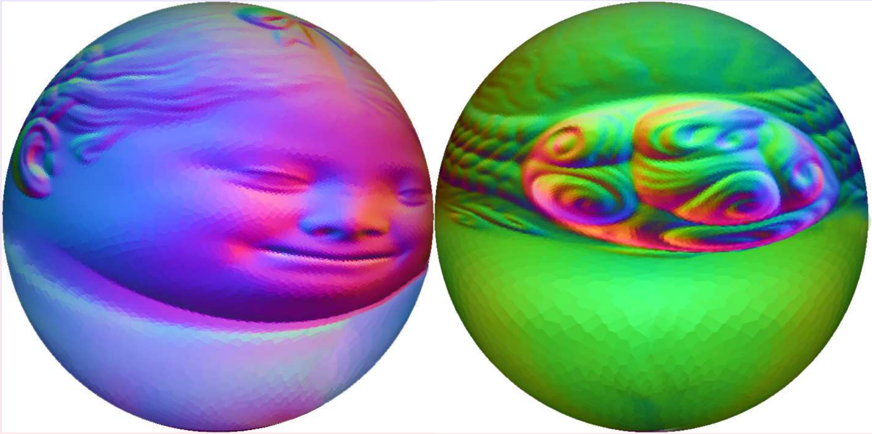
Suppose a degree one harmonic map $w : (M, \sigma|dz|^2) \to (N, \rho|dw|^2)$ is given, the curvature on the target is negative everywhere, then w is a diffeomorphism.

Spherical harmonic map

Spherical harmonic map



Spherical harmonic map



Computational Method - Holomorphic Form

Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H^1(M)$.

Theorem (Hodge Decomposition)

$$\Omega^k(M) = Img d^{k-1} \oplus Img \delta^{k+1} \oplus H^k_\Delta(M).$$

Compute Harmonic 1-forms

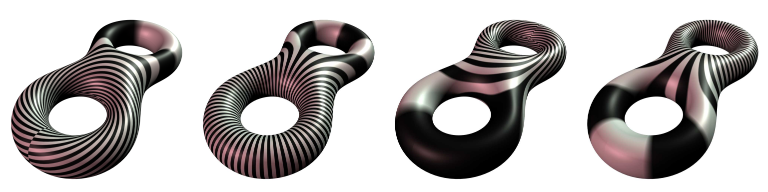
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f \in C^0(M, \mathbb{R})$, such that

$$\delta^1(\omega + df) = 0,$$

then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Compute Harmonic 1-forms

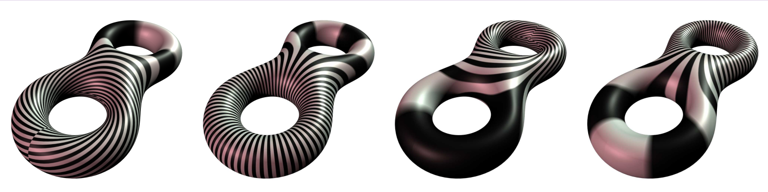
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f : V \to \mathbb{R}$, such that

$$\sum_j w_{ij}(\omega + df)([v_i, v_j]) = 0, \forall v_i \in V.$$

Then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Hodge Star Operator

Let $(S,\mathbf{g})$ be a metric surface, $\{e_1,e_2\}$ be an orthonormal frame field, $\{\frac{\partial}{\partial u},\frac{\partial}{\partial v}\}$ be the base vector fields, $\{du,dv\}$ be the dual differential 1-form fields.

$$^*du = dv, ^*dv = -du.$$

Conjugate harmonic 1-forms $\omega + \sqrt{-1}^*\omega$

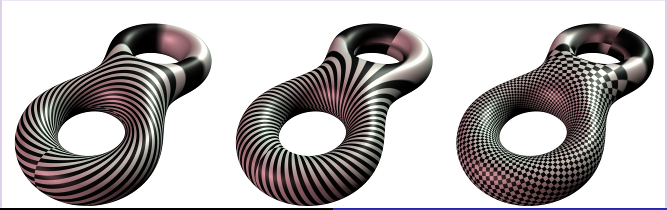
Hodge Star Operator

If ω is a harmonic 1-form, so is ${}^*\omega$. Suppose $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ is the set basis of harmonic 1-forms, then ${}^*\omega = \sum_k \lambda_k \omega_k$. Locally, on each triangle ${}^*(adx + bdy) = ady - bdx$. Solve linear system

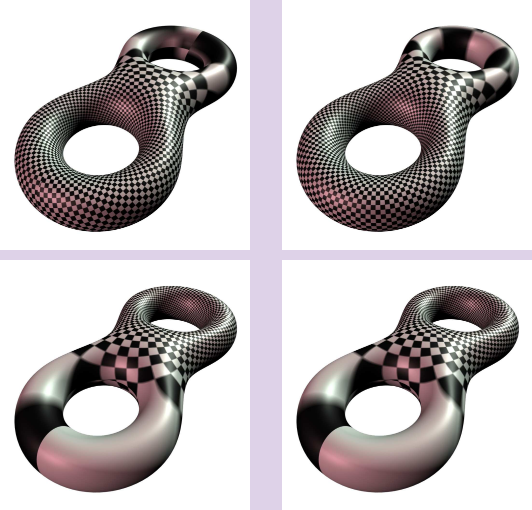
$$\int_M \omega_i \wedge {}^*\omega = \sum_k \lambda_k \int_M \omega_i \wedge \omega_k, i = 1, 2, \cdots, 2g.$$

to solve λ_k's, where ${}^*\omega$ on the left hand side is locally evaluated.

Conjugate harmonic 1-forms $\omega + \sqrt{-1}{}^*\omega$



Holomorphic 1-form Basis



Topological Quadrilateral

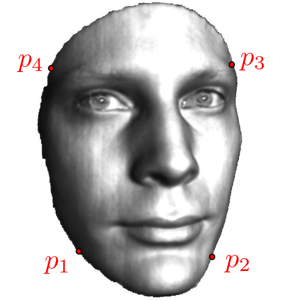



Figure: Topological quadrilateral.

Topological Quadrilateral

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Topological Quadrilateral

Assume the boundary of Q consists of four segments $\partial Q = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, such that

$$\partial \gamma_1 = p_2 - p_1, \partial \gamma_2 = p_3 - p_2, \partial \gamma_3 = p_4 - p_3, \gamma_4 = p_1 - p_4.$$

We compute two harmonic functions $f_1, f_2 \to \mathbb{R}$, such that

$$\left\{ \begin{array}{lcl} \Delta f_1 & = & 0 \\ f_1|_{\gamma_1} & = & 0 \\ f_1|_{\gamma_3} & = & 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_2 \cup \gamma_4} & = & 0 \end{array} \right. \left\{ \begin{array}{lcl} \Delta f_2 & = & 0 \\ f_2|_{\gamma_2} & = & 0 \\ f_2|_{\gamma_4} & = & 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1 \cup \gamma_3} & = & 0 \end{array} \right.$$

Topological Quadrilateral

The df_1 and df_2 are two exact harmonic 1-forms. We need to find a scalar λ, such that ${}^*df_1 = \lambda df_2$, this can be achieved by solving the following equation,

$$\int_S df_1 \wedge {}^*df_2 = \lambda \int_S df_1 \wedge df_2.$$

Then the desired holomorphic 1-form $\omega = df_1 + i\lambda df_2$. The conformal mapping is given by

$$\phi(p) = \int_q^p \omega,$$

where q is the base point, the path from q to p is arbitrarily chosen.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Topological Annulus

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$*, the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.*

Topological Annulus

First, we compute a harmonic function $f : S \to \mathbb{R}$, such that

$$\begin{cases} f|_{\gamma_1} & = & 0 \\ f|_{\gamma_2} & = & 1 \\ \Delta f & = & 0 \end{cases}$$

Then df is an exact harmonic 1-form. Then we compute a harmonic 1-form τ, such that $\int_{\gamma_1} \tau = 1$.

Topological Annulus

Then we compute a constant λ, such that ${}^*df = \lambda\tau$, by solving the following equation,

$$\int_S df \wedge {}^*df = \lambda \int_S df \wedge \tau.$$

Then $\omega = df + i\lambda\tau$ is a holomorphic 1-form. Let $Img(\int_{\gamma_1} \omega) = k$. The conformal mapping is given by

$$\phi(p) = exp^{\frac{2\pi}{k}\int_q^p \omega}.$$

Topological Annulus

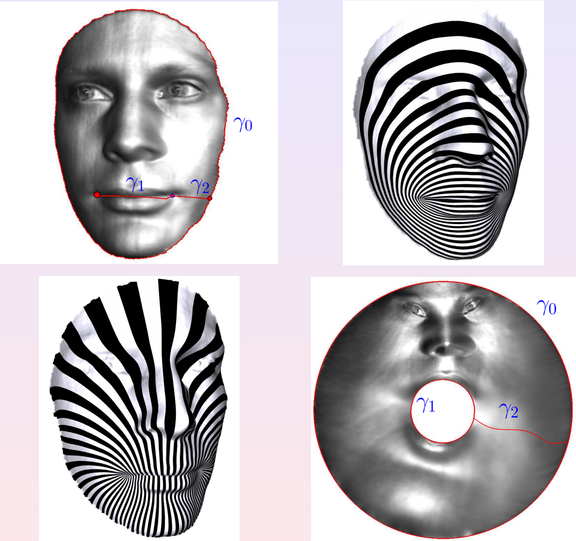


Figure: Topological annulus.

Riemann Mapping

Conformal Module

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Topological Disk

Punch a small hole in the disk, then use the algorithm for topological annulus to compute the conformal mapping. The punched hole will be mapped to the center.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Conformal Slit Mapping

Figure: Harmonic forms and holomorphic forms.

Slit Mapping

Suppose there are $n+1$ boundary components$\{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n\}$. $\{\omega_1, \omega_2, \cdots, \omega_n\}$ are the holomorphic 1-form basis. Choose two boundary components, γ_0, γ_1, solve linear equation $\omega = \sum_{k=1}^{n} \lambda_k \omega_k$,

$$img\left(\int_{\gamma_0} \omega\right) = 2\pi, img\left(\int_{\gamma_1} \omega\right) = -2\pi, img\left(\int_{\gamma_k} \omega\right) = 0, 2 \leq k \leq n.$$

Then the mapping is given by

$$p \rightarrow \exp \int_q^p \omega,$$

where q is the base point on the surface.

Conformal Circular Slit Mapping

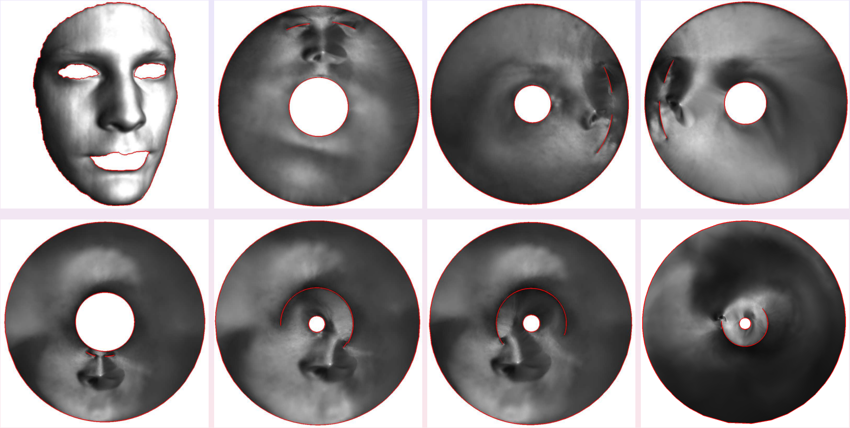


Figure: Conformal circular slit mapping.

Hole Filling

Adding sample points in the center hole, use Delaunay triangulation to fill in with boundary constraints.

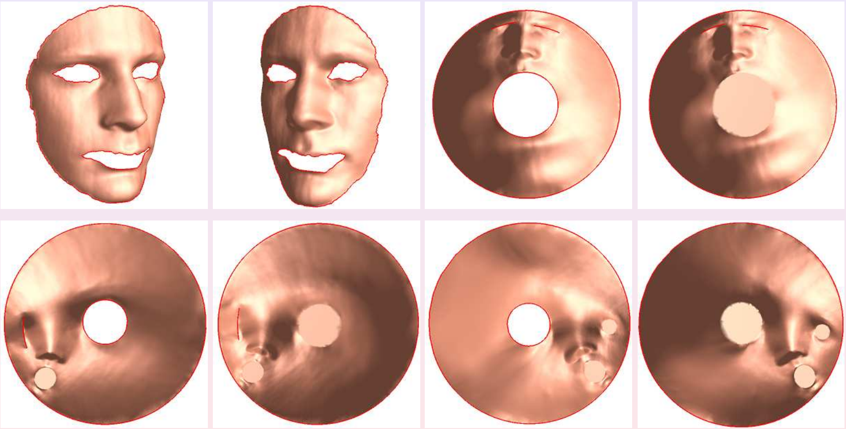


Figure: Fill interior holes.

Koebe's Iteration - I

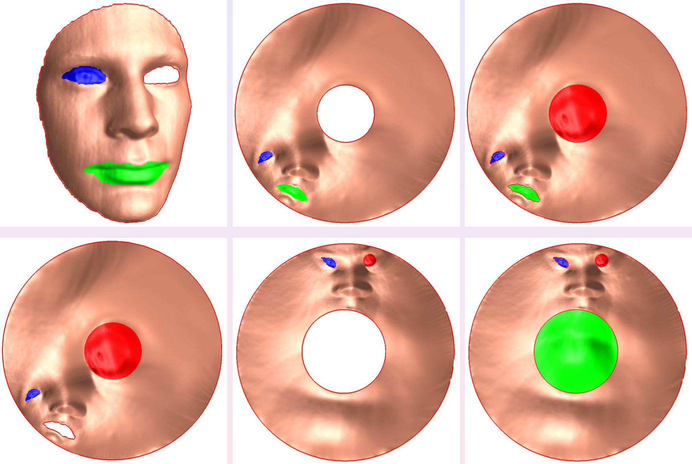


Figure: Koebe's method for computing conformal maps for multiply connected domains.

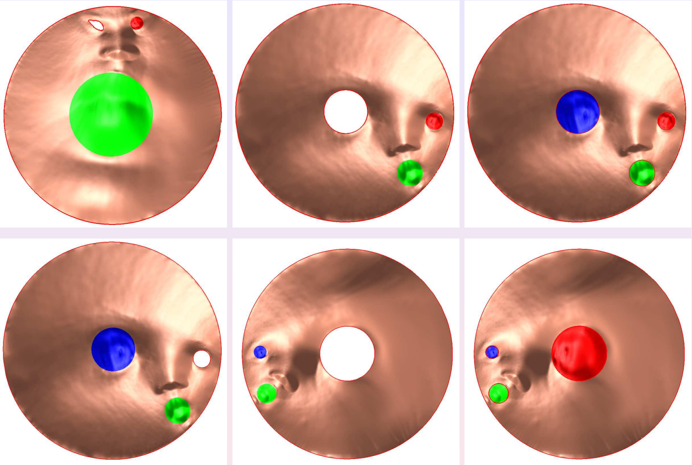


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

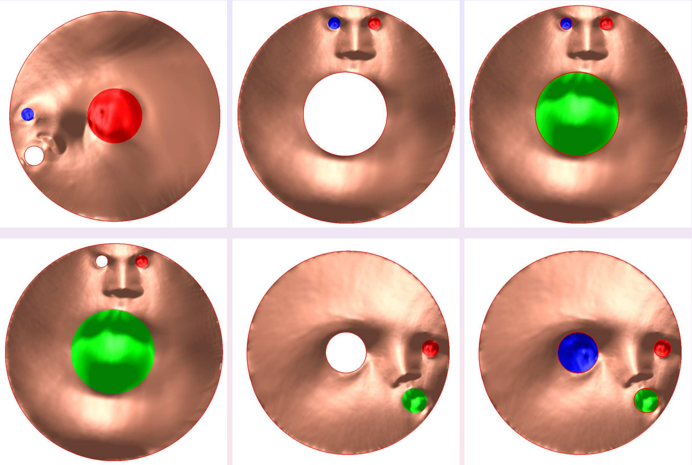


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Theorem (Gu and Luo 2009)

Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

Topological Torus

Topological torus

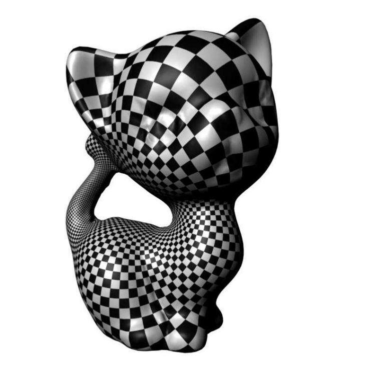
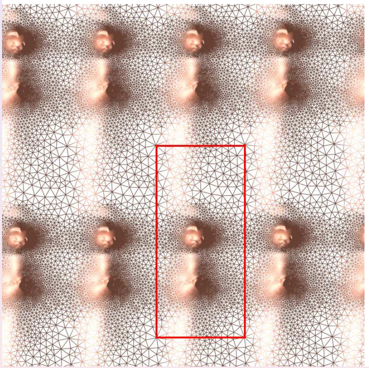



Figure: Genus one closed surface.

Topological Torus

1. We compute a basis for the fundamental group $\pi_1(S)$, $\{\gamma_1, \gamma_2\}$.
2. Compute the holomorophic 1-form basis ω_1, ω_2, such that $\int_{\gamma_i} \omega_j = \delta_{ij}$.
3. Slice the surface along γ_1, γ_2 to get a fundamental domain $\tilde{S}$,
4. The conformal mapping $\phi : \tilde{S} \to \mathbb{C}$ is given by

$$\phi(p) = \int_q^p \omega_1,$$

where q is the base point, the path from q to p in $\tilde{S}$ can be arbitrarily chosen.

Topological Torus

Suppose $a+ib=\int_{\gamma_2}\omega_1$, then $a+ib$ is the conformal module of the torus. The deck transformation group generators are

$$T_1(z)=z+1, T_2(z)=z+a+ib.$$

By using all deck transformations to translate $\phi(\tilde{S})$, we can conformally map the universal covering space of S onto the whole complex plane $\mathbb{C}$, each fundamental domain is a parallelogram.

Conformal Mapping

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two surfaces with Riemannian metrics. A conformal mapping $\phi : S_1 \to S_2$ is a diffeomorphism, such that

$$\phi^* \mathbf{g}_2 = e^{2\lambda} \mathbf{g}_1.$$

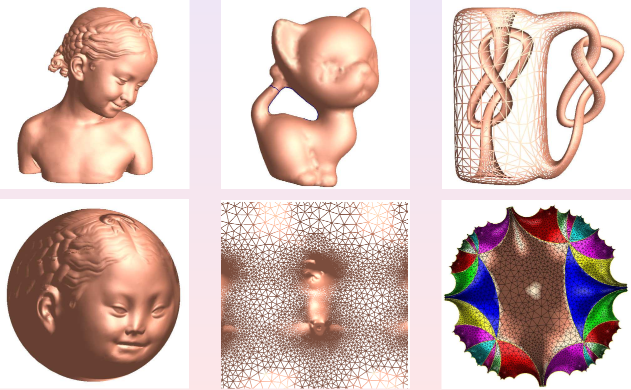
Properties

Conformal mappings preserve infinitesimal circles, and preserve angles.

Uniformization

Theorem (Poincaré Uniformization Theorem)

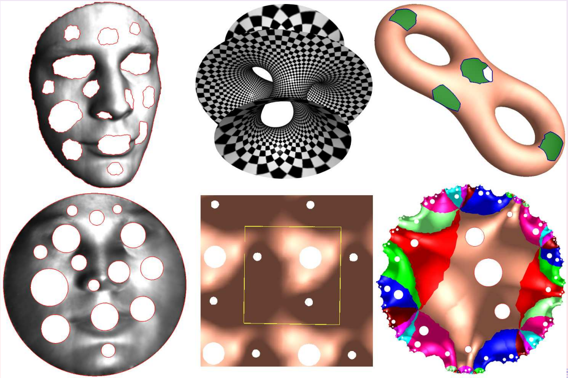
Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ *be a compact 2-dimensional Riemannian manifold with finite number of boundary components. Then there is a metric* $\tilde{\mathbf{g}}$ *conformal to* $\mathbf{g}$ *which has constant Gauss curvature, and constant geodesic curvature.*



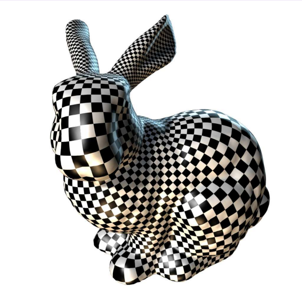
Yamabe Problem

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface M with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

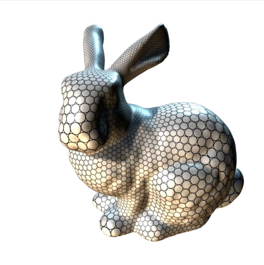
Suppose M is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Surface Ricci Flow

Key Idea

$$K = -\Delta_{\mathbf{g}}\lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = -\Delta_{\mathbf{g}}\frac{d\lambda}{dt}$$

Let $\frac{d\lambda}{dt} = -K$,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

Heat equation!

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

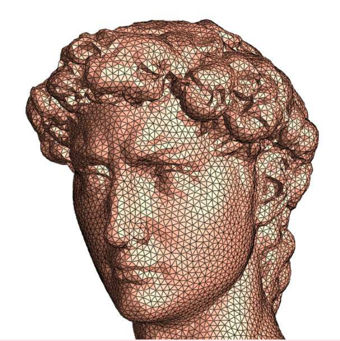
- Ricci flow

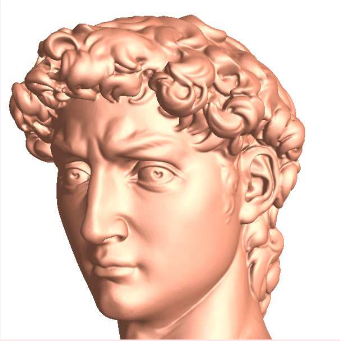
$$\frac{du}{dt} = \bar{K} - K.$$

Discrete Surface Ricci Flow

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





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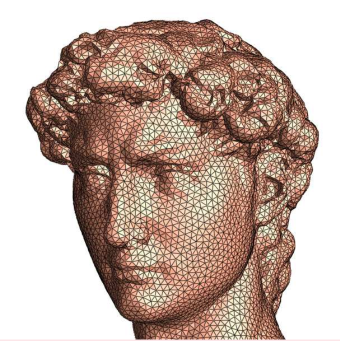

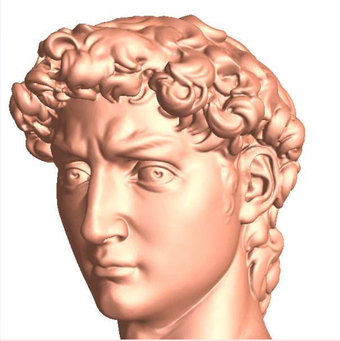

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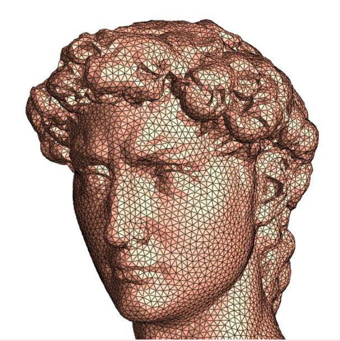
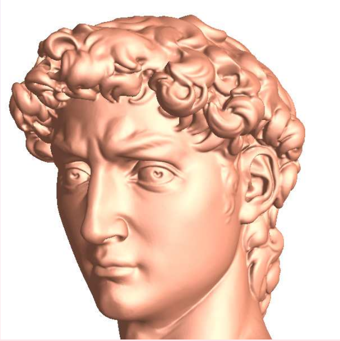




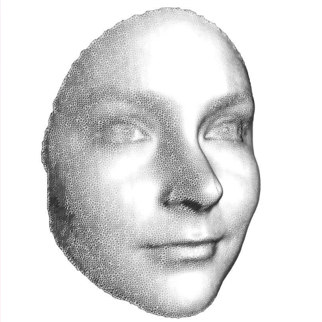
Concepts

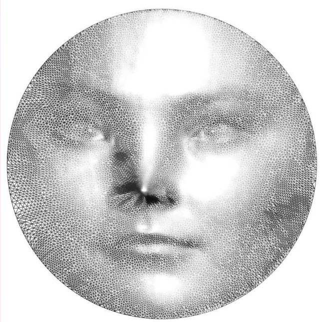
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

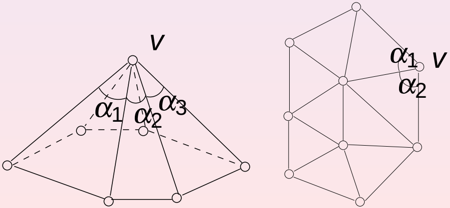
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

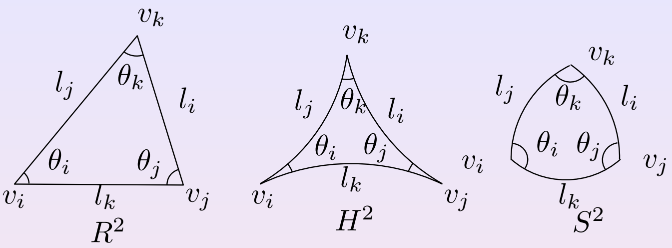
$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



cosine laws

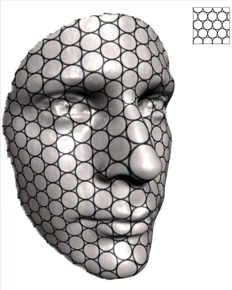
$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j\cosh\theta_k}{\sinh\theta_j\sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (3)$$

Conformal maps Properties

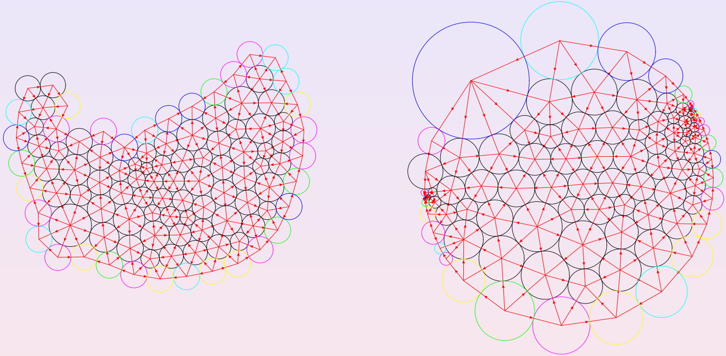
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.



Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Discrete Conformal Metric Deformation vs CP



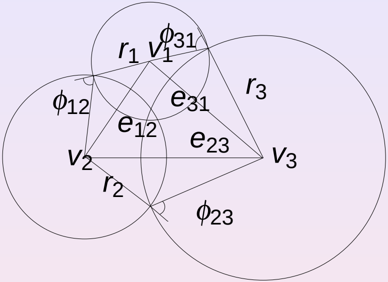
CP Metric

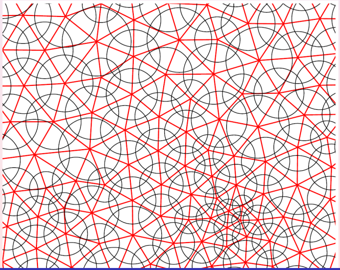
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$





Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

Unified Framework of Discrete Curvature Flow

Analogy

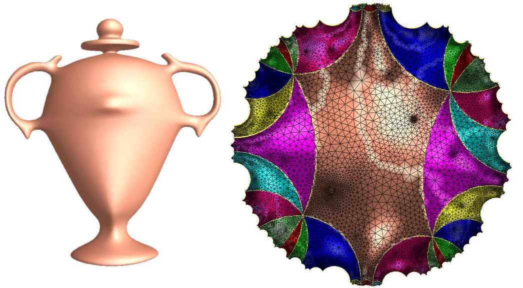
- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

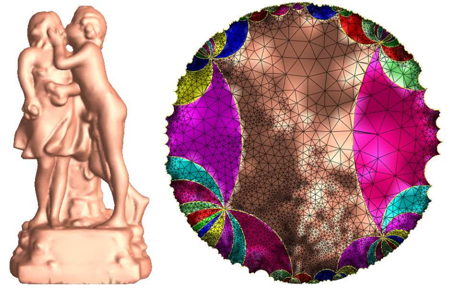
Criteria for Discretization

Key Points

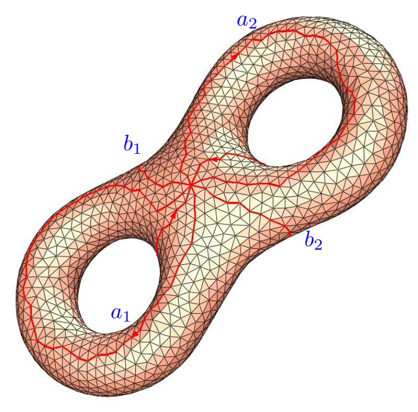
- Convexity of the energy $E(\mathbf{u})$
- Convexity of the metric space (**u**-space)
- Admissible curvature space (**K**-space)
- Preserving or reflecting richer structures
- Conformality

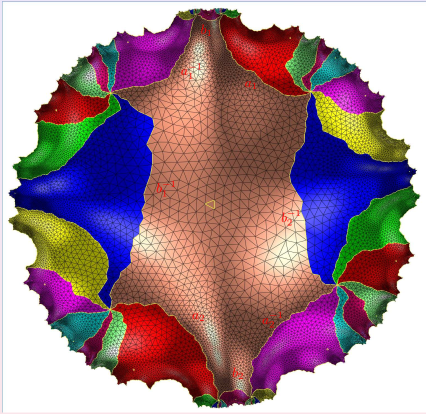
Hyperbolic Ricci Flow





Hyperbolic Yamabe Flow





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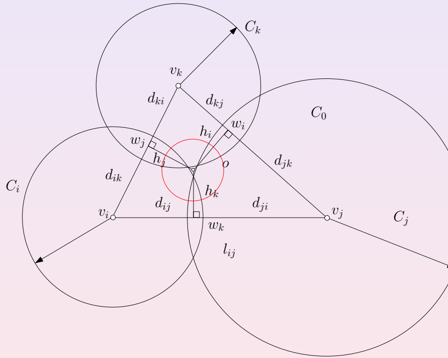
$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\eta_{ij}$$

CP Metric (Σ, Γ, η), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \eta = \{\eta_{ij} < 1 | \forall e_{ij}\}$$



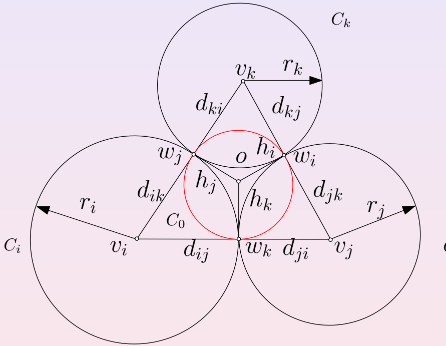
Tangential Circle Packing Metric

Tangential CP Metric

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j,$$

equivalently

$$\eta_{ij} \equiv 1.$$

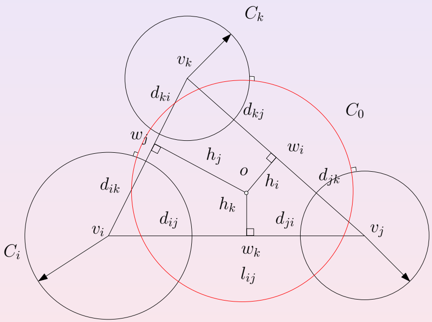


Tangential CP Metric

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

equivalently

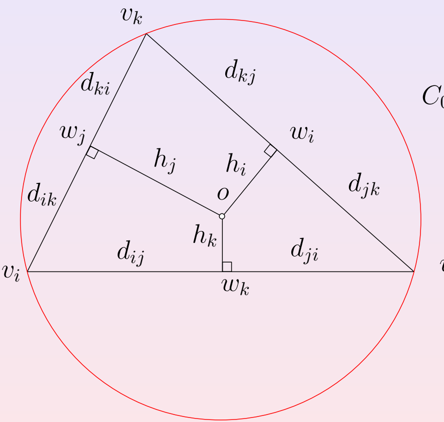
$$\eta_{ij} > 1.$$



Yamabe Flow

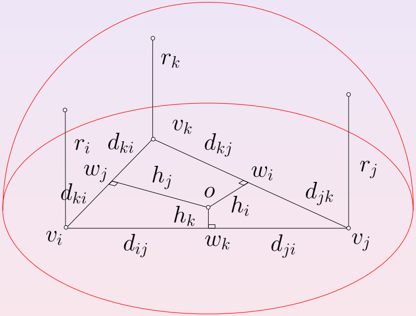
Yamabe Flow

$$l_{ij}^2 = \eta_{ij}\gamma_i\gamma_j,$$



Imaginary Radius Circle Packing Metric

$$l_{ij}^2 = -\gamma_i^2 - \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

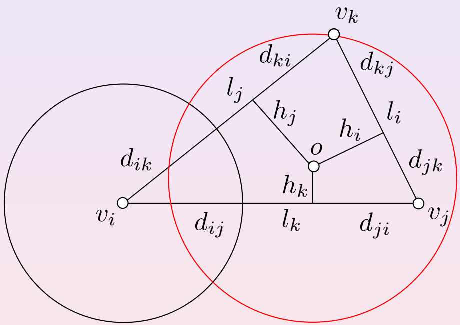


Mixed Type

Mixed Circle Packing

$$l_{ij}^2 = \alpha_i \gamma_i^2 + \alpha_j \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

$$(\alpha_i, \alpha_j, \alpha_k) = (+1, -1, 0)$$



Applications

Medical Imaging Application

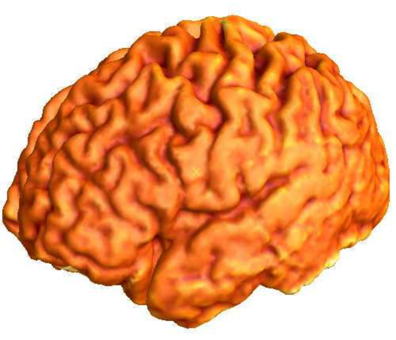
Medical Imaging

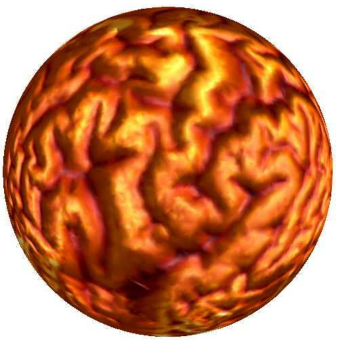
Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.

Brain Cortex Surface

Conformal Brain Mapping for registration, matching, comparison.

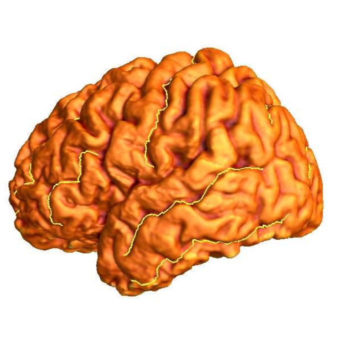


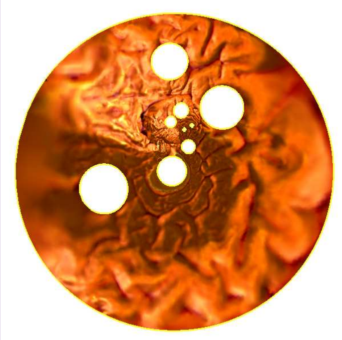


Conformal Brain Mapping

Using conformal module to analyze shape abnormalities.

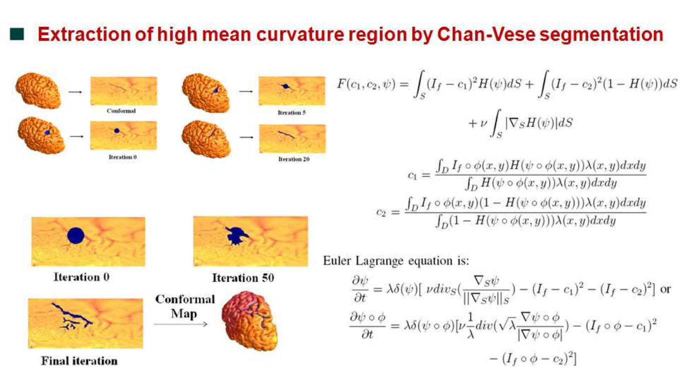
Brain Cortex Surface





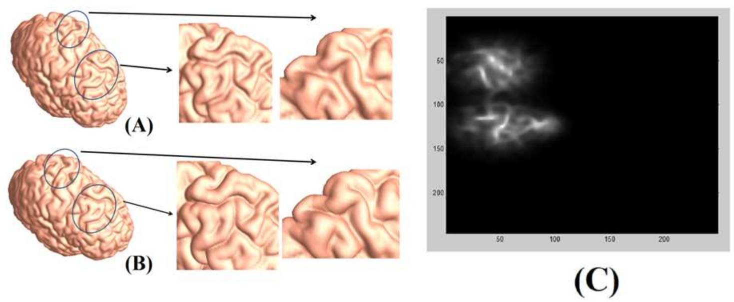
Automatic sulcal landmark Tracking

- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically



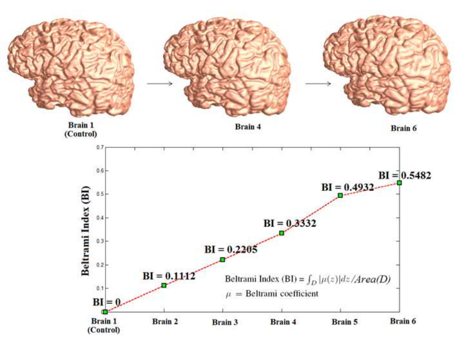
Abnormality detection on brain surfaces

The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.

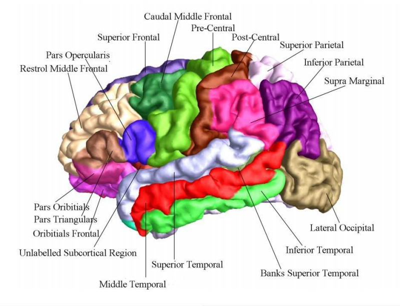


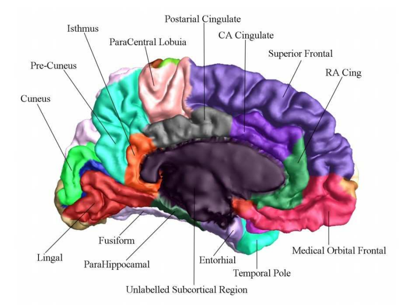
Abnormality detection on brain surfaces

The brain is undergoing gyri thickening (commonly observed in Williams Syndrome) The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.



Alzheimer Study





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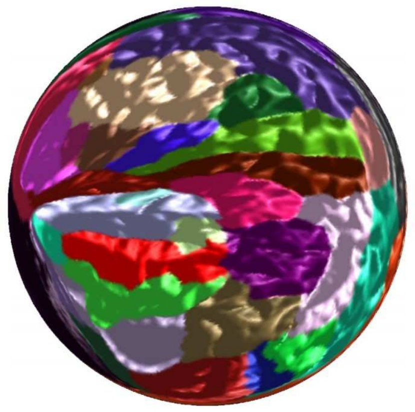
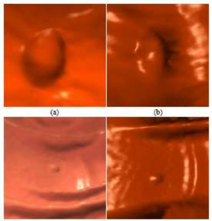
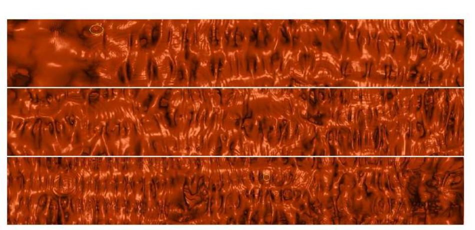


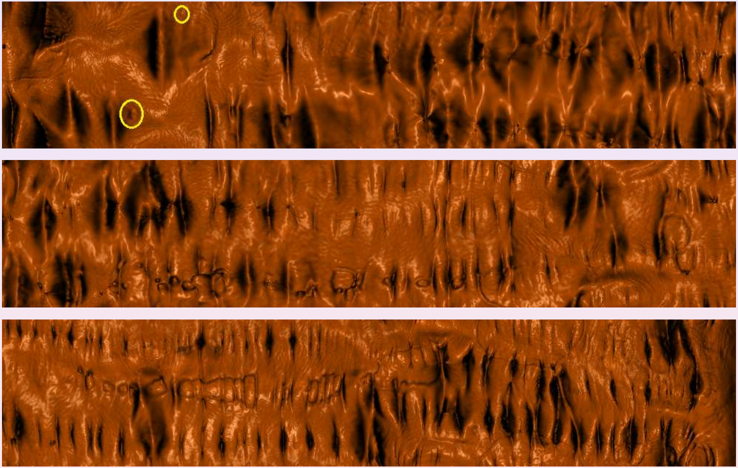
Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.





Colon Flattening

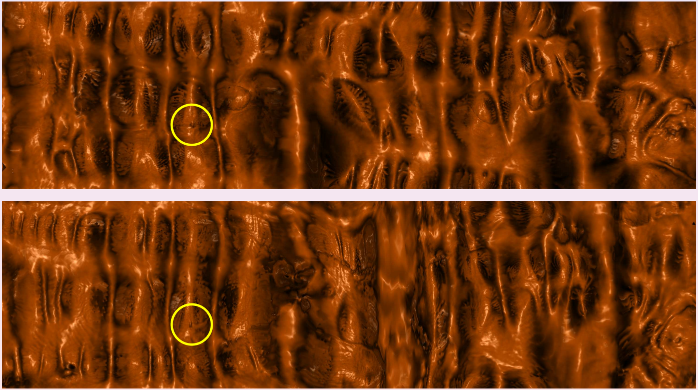


Virtual Colonoscopy

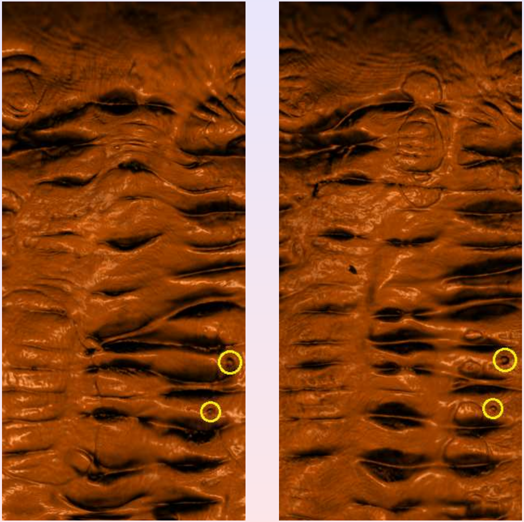
Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.

Virtual Colonoscopy

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.



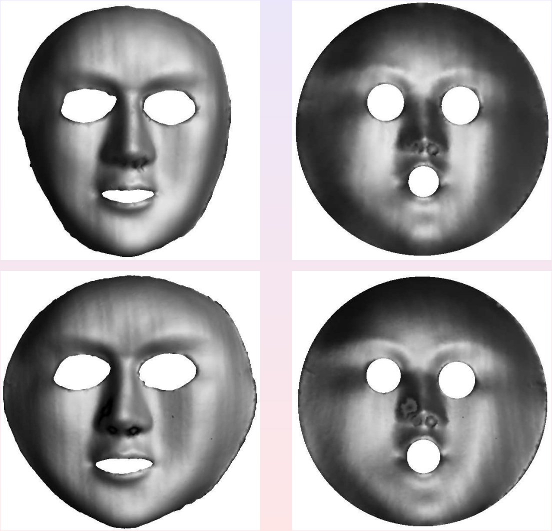
Colon Registration



Vision

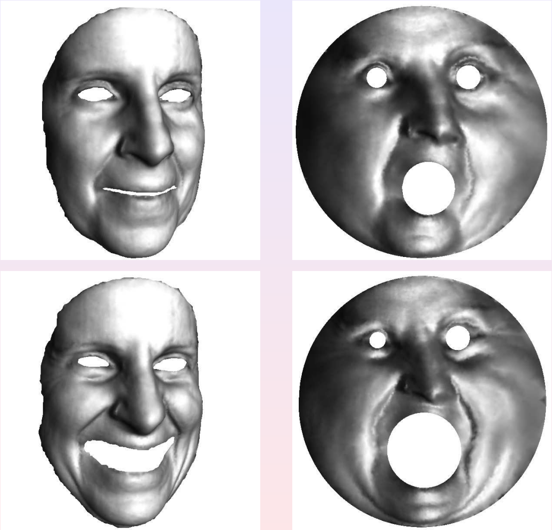
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.

Isometric deformation is conformal. The mask is bent without stretching.



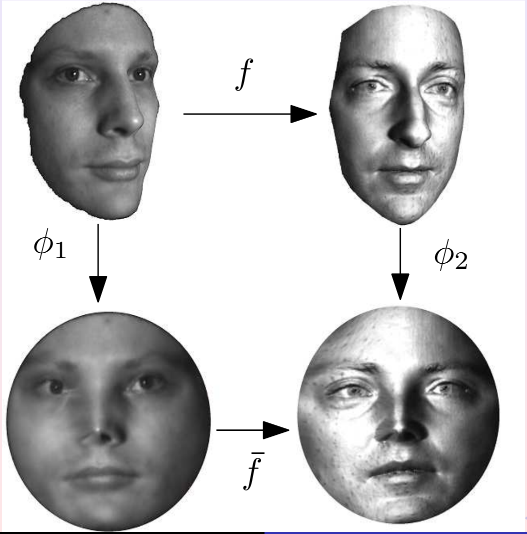
Surface Matching

Facial expression change is not-conformal.

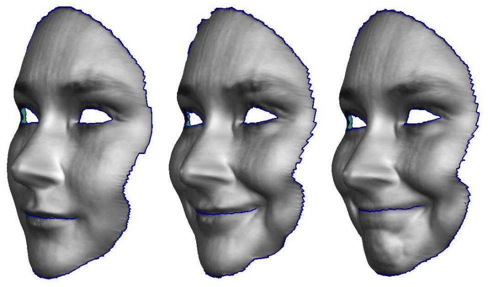


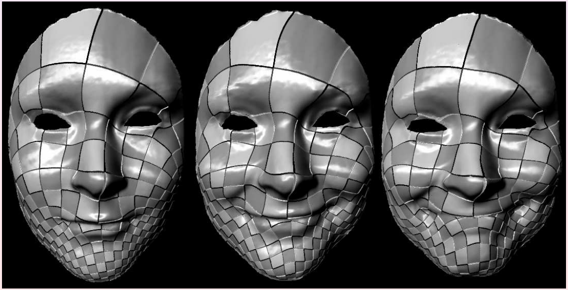
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

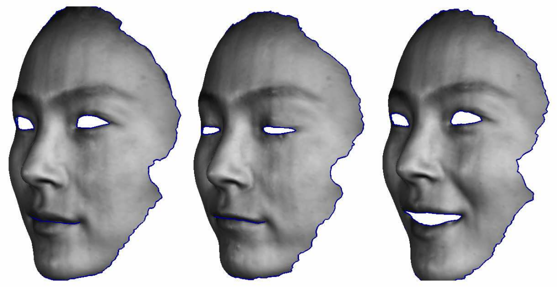


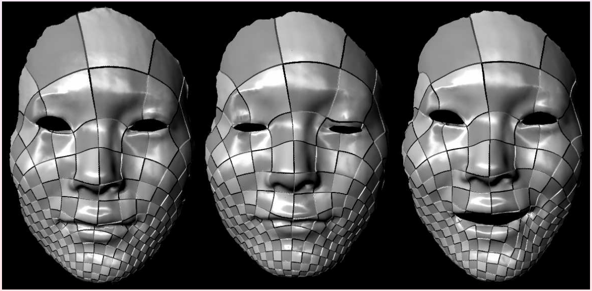
Face Surfaces with Different Expressions are Matched



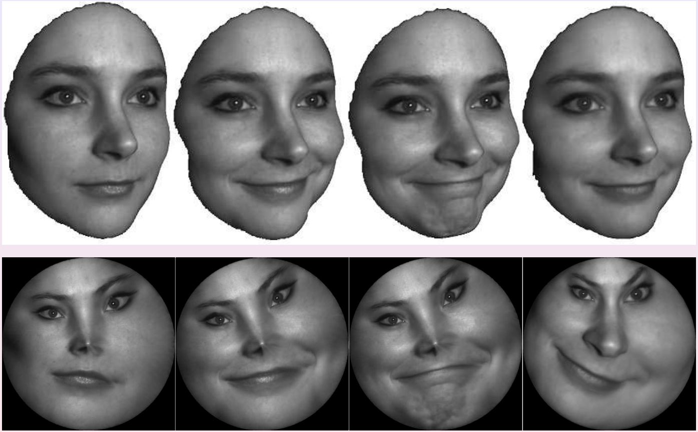


Face Surfaces with Different Expressions are Matched

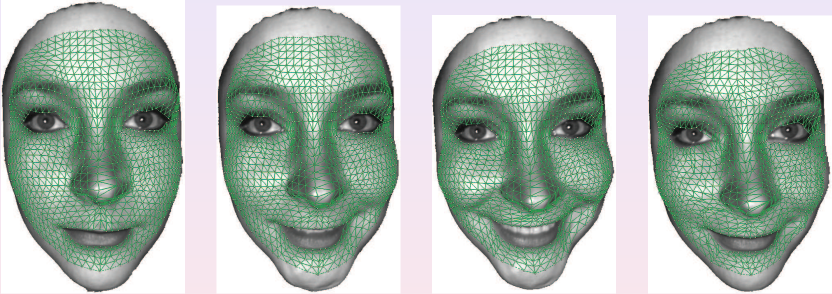




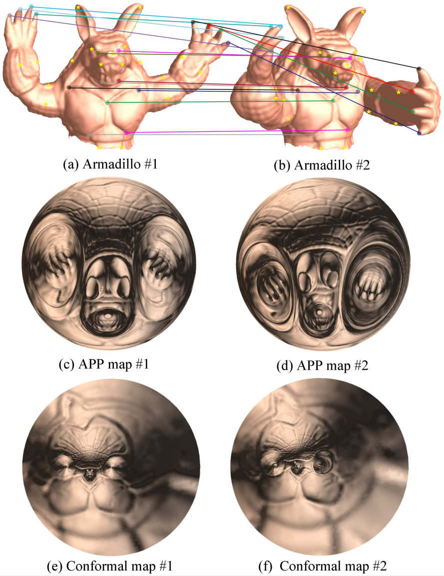
Face Expression Tracking



Face Expression Tracking



Surface Registration



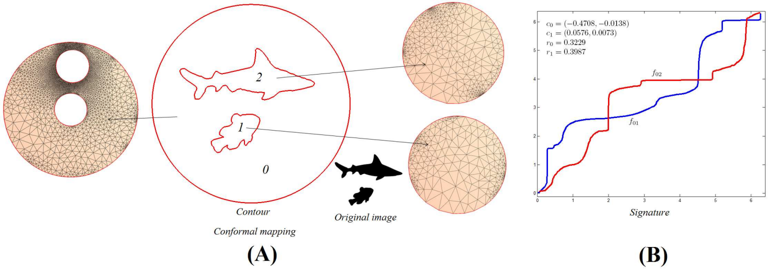
(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

2D Shape Space-Conformal Welding

$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$

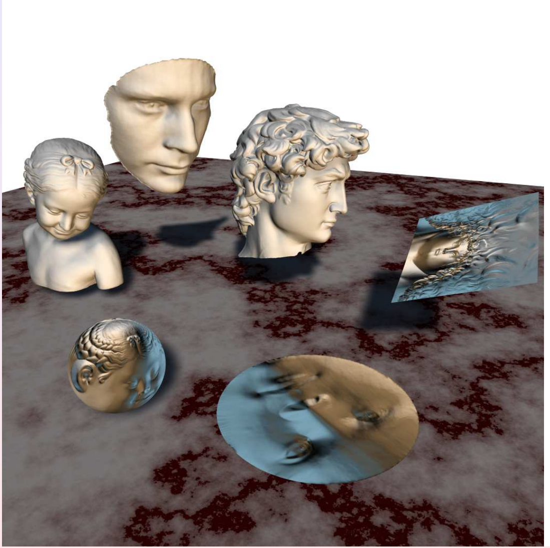


Computer Graphics Application

Graphics

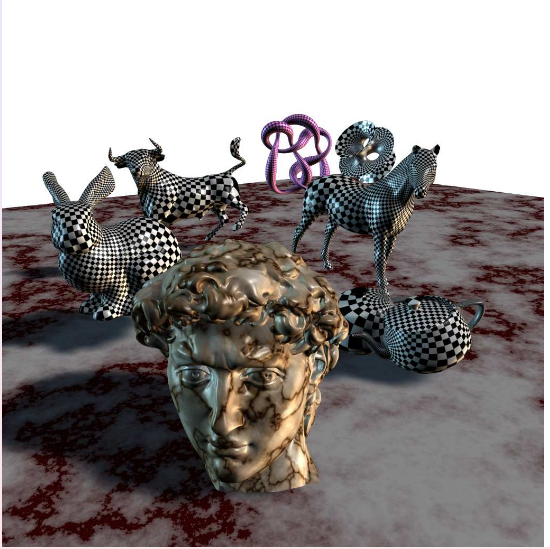
- Surface Parameterization, texture mapping
- Texture synthesis, transfer
- Vector field design
- Shape space and retrieval.

Map the surfaces onto canonical parameter domains

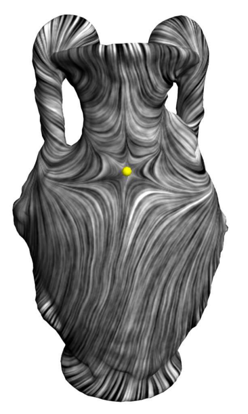


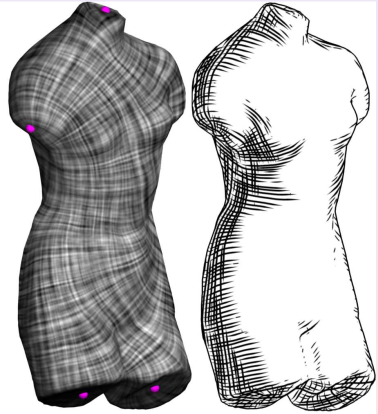
Surface Parameterization

Applied for texture mapping.

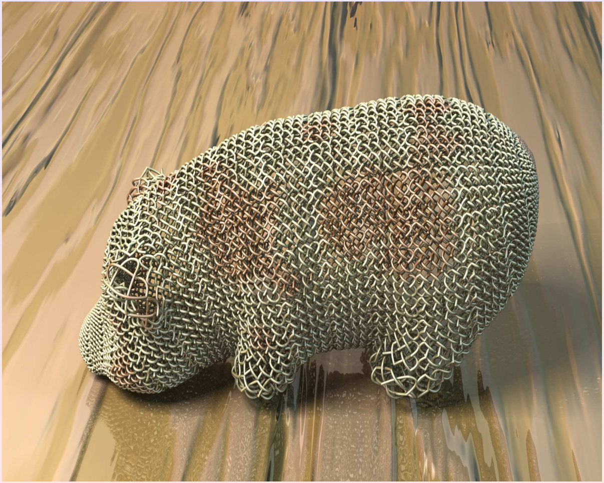


Design vector fields on surfaces with prescribed singularity positions and indices.

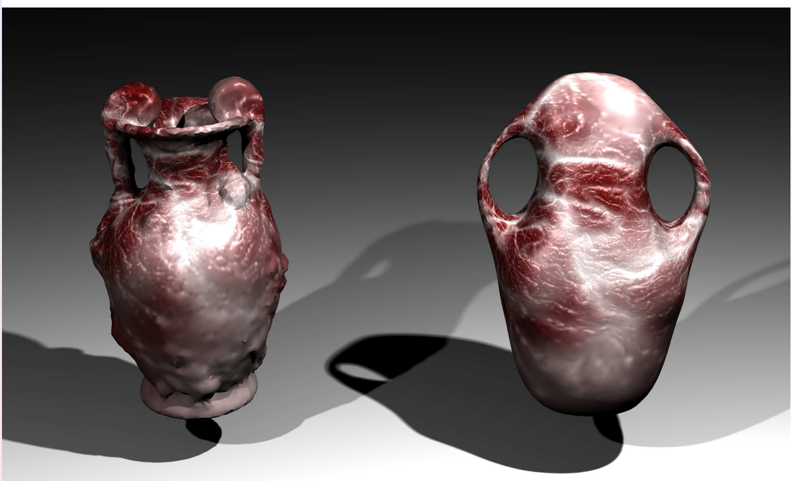




Convert the surface to knot structure using smooth vector fields.

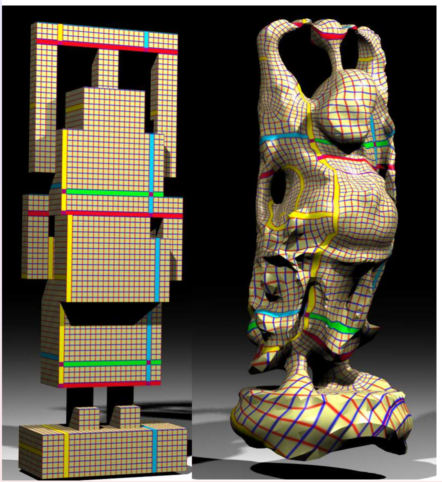


Transfer the texture between high genus surfaces.



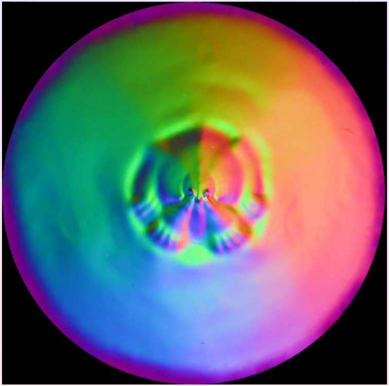
Polycube Map

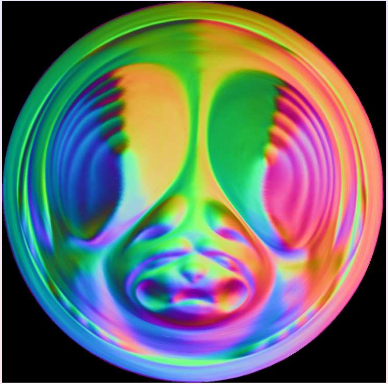
Compute polycube maps for high genus surfaces.



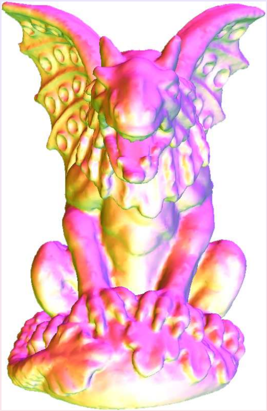
Normal Map

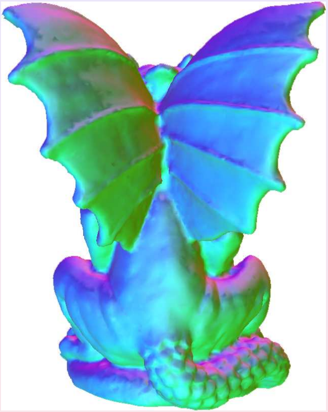
Normal Map



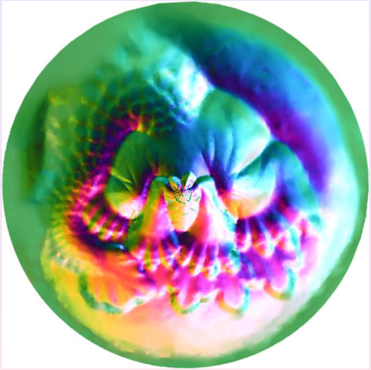


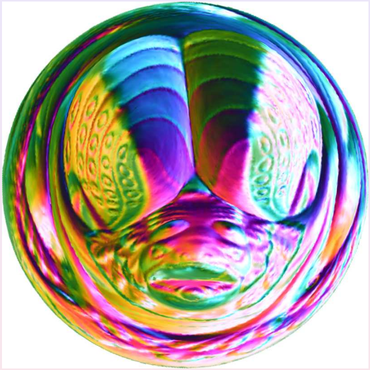
Normal Map





Normal Map





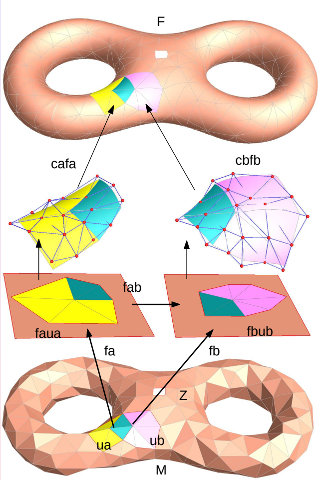
Manifold Spline

- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.

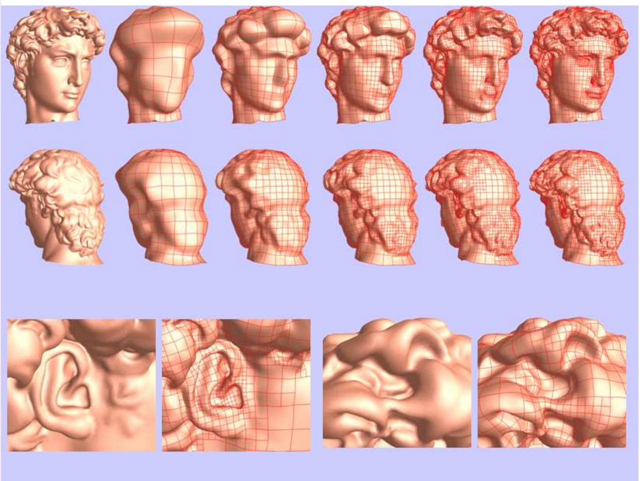
Extraordinary Points

- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.

Manifold Spline

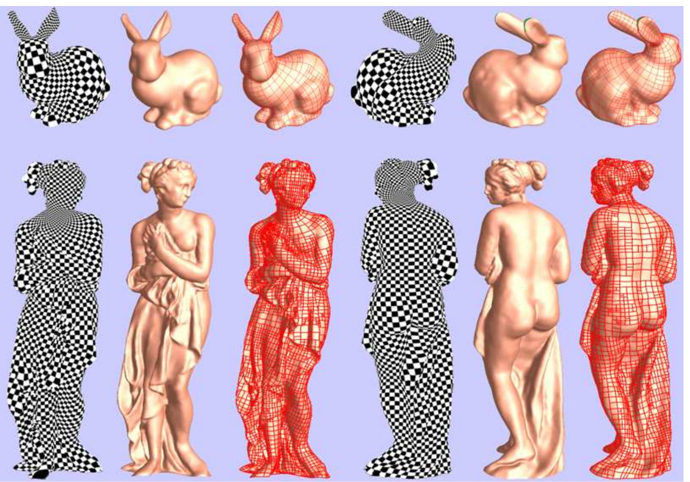


Converting a polygonal mesh to TSplines with multiple resolutions.



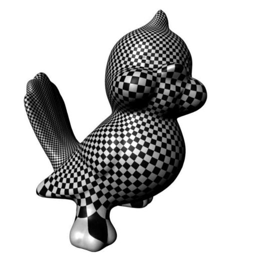
Manifold Spline

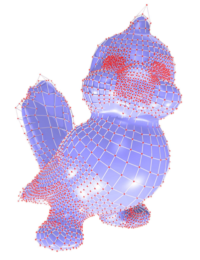
Converting scanned data to spline surfaces.

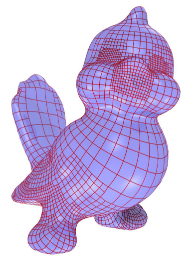


Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

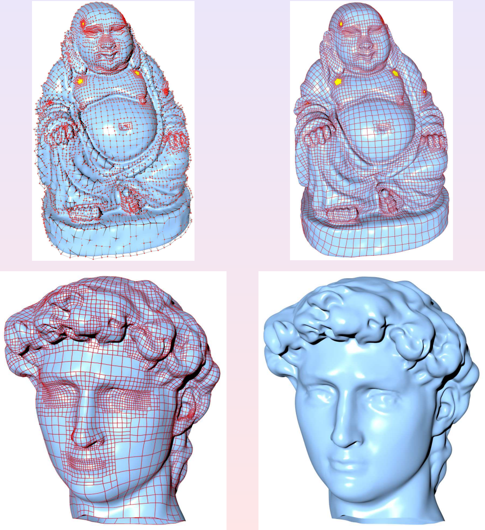






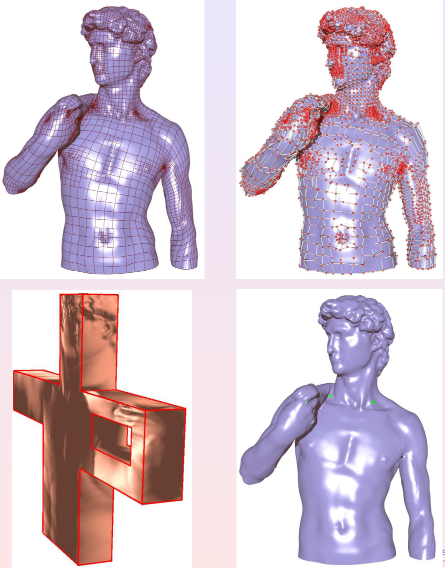
Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

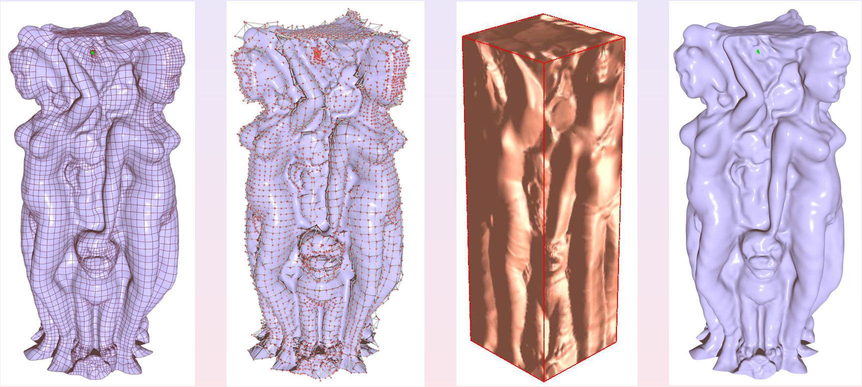


Manifold Spline

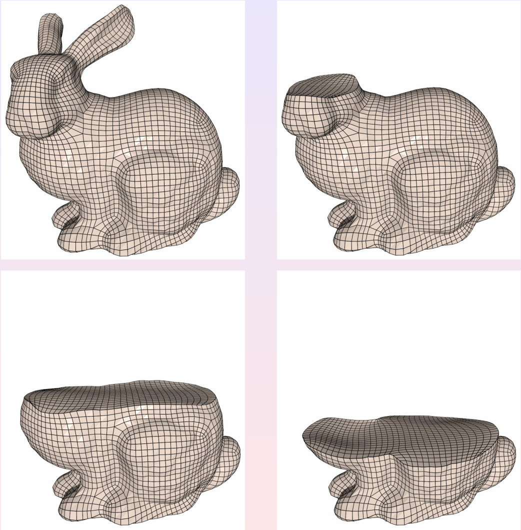
Polygonal mesh to spline, control net and the knot structure.



Manifold Spline



volumetric spline.



Visualization

(a) Front view (b) Angle-preserving (c) Area-preserving (d) Back view

Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x (b) 3x (c) 4x (d) 6x

Importance driven parameterization. The Buddha's head region is magnified by different factors

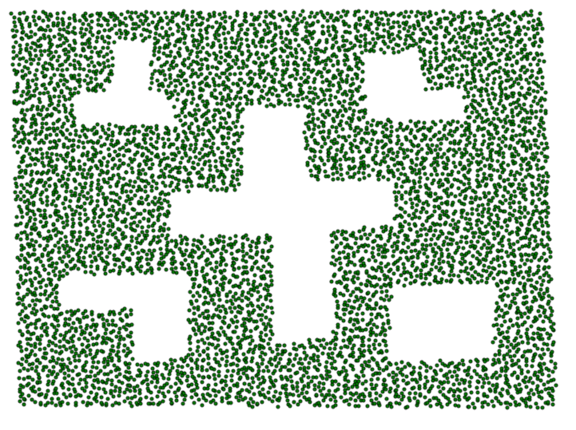
Wireless Sensor Network Application

Wireless Sensor Network

- Detecting global topology.
- Routing protocol.
- Load balancing.
- Isometric embedding.

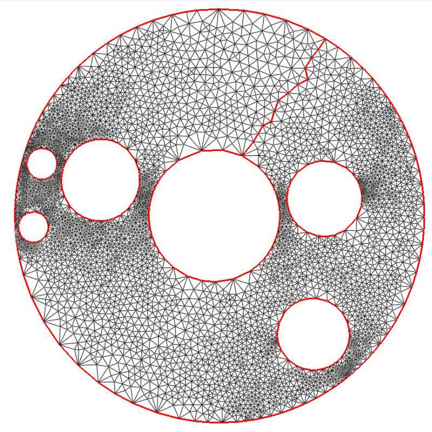
Greedy Routing

Given sensors on the ground, because of the concavity of the boundaries, greedy routing doesn't work.

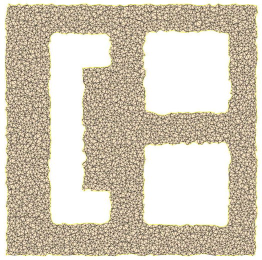
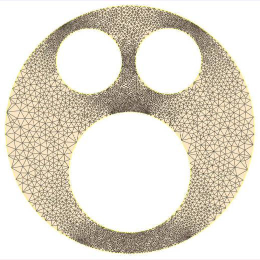
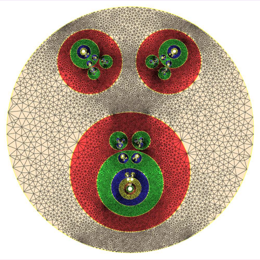


Greedy Routing

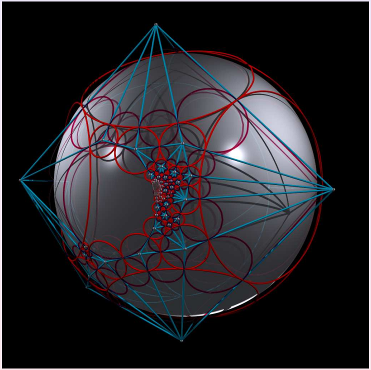
Map the network to a circle domain, all boundaries are circles, greedy routing works.

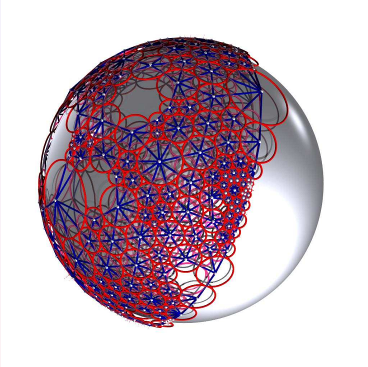


Schoktty Group - Circular Reflection

Optimal Planar Graph Embedding.

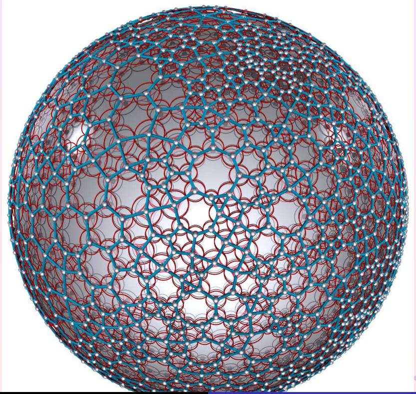




Graph Embedding

Thurston-Andreev Theorem

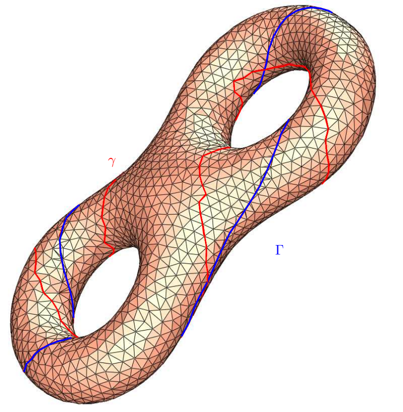
A planar graph can be embedded on the unit sphere, such that the face circles are orthogonal to vertex circles; the circles at the vertices of an edge are tangent to each other. Such kind of embedding differ by a Möbius transformation.

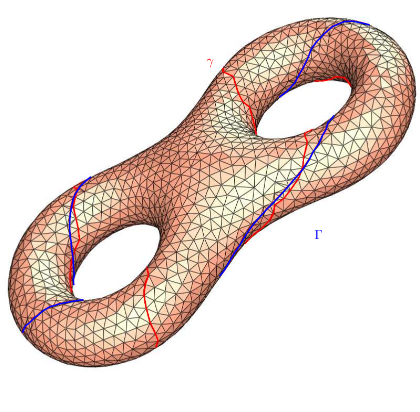


Computational Topology Application

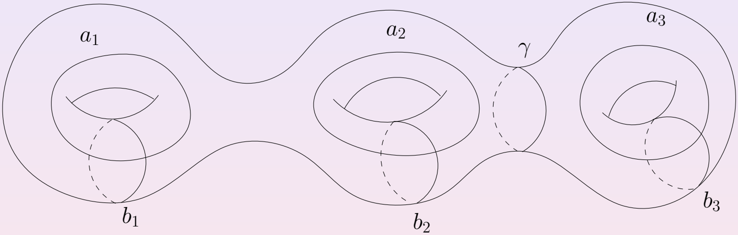
Canonical Homotopy Class Representative

Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.



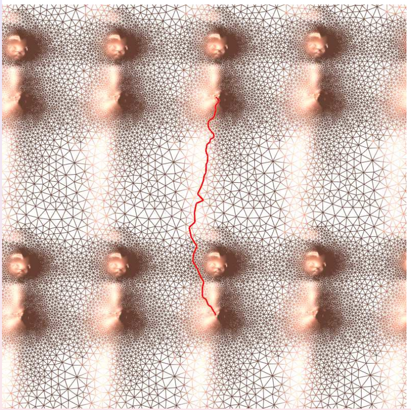


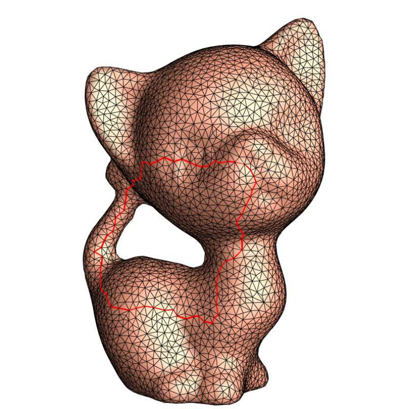
Shortest word Problem (NP Hard):



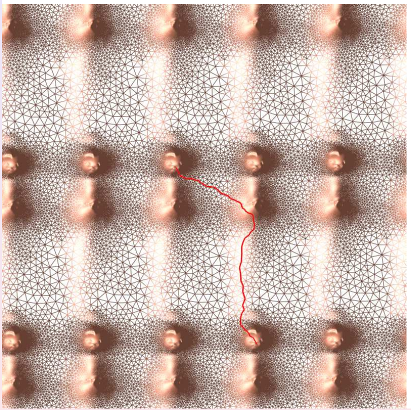
$$\gamma = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = (a_3 b_3 a_3^{-1} b_3^{-1})^{-1}$$

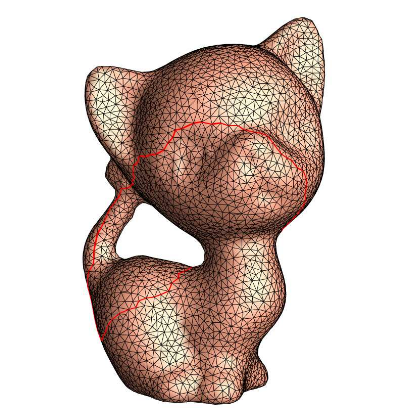
Loop Lifting



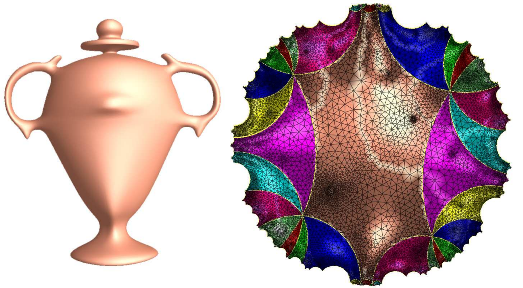


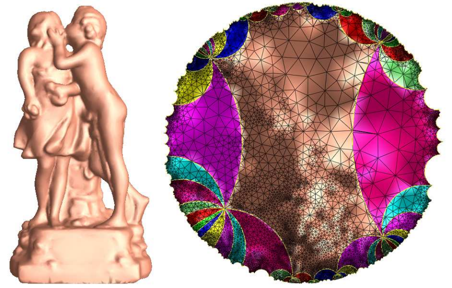
Loop Lifting



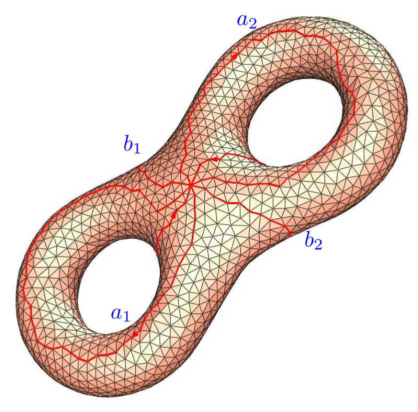


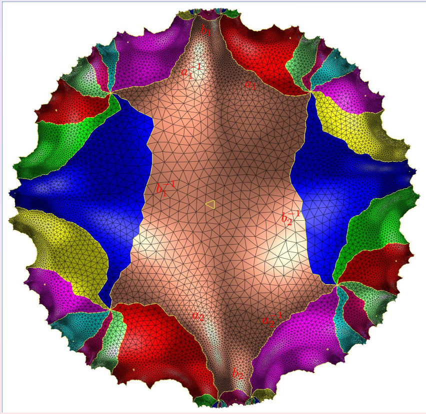
Hyperbolic Ricci Flow



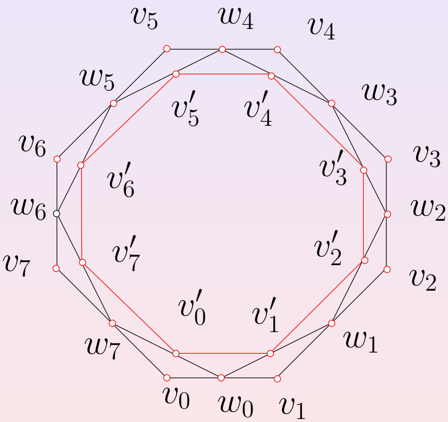


Lifting a loop from base surface to the universal covering space.

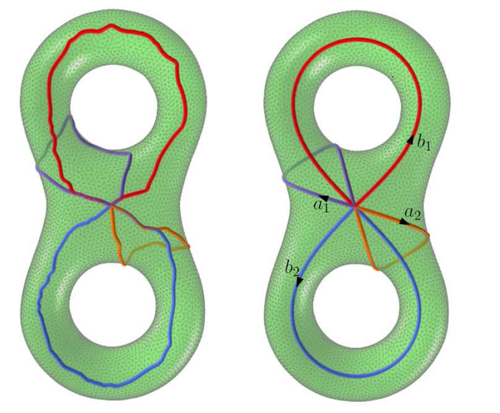




Birkoff curve shortening deforms a loop to a geodesic.



Birkoff curve shortening deforms a loop to a geodesic.



Computing Shortest Word

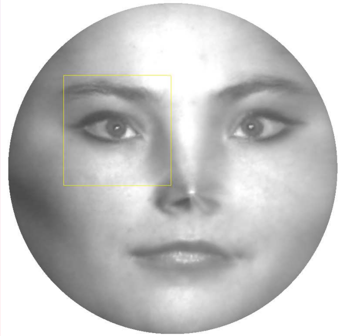
1. Compute the uniformization metric using Ricci flow.
2. Compute the geodesic loop by Birkoff curve shortening.
3. Lift the geodesic loop to the universal covering space.
4. Trace the lifted loop to compute the word.

Meshing

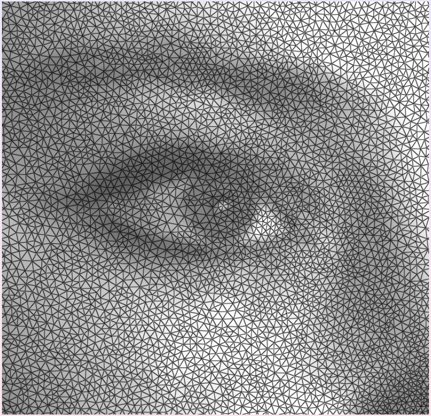
Theorem

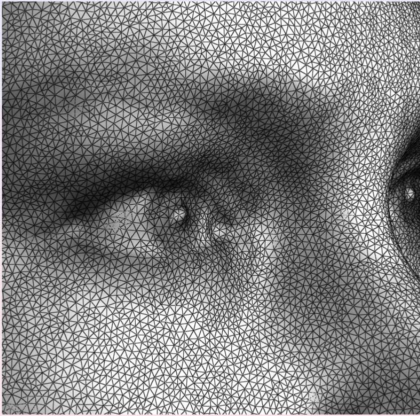
Suppose S is a surface with a Riemannian metric. Then there exist meshing method which ensures the convergence of curvatures.

Key idea: Delaunay triangulations on uniformization domains. Angles are bounded, areas are bounded.



Meshing





Theorem

Let M be a compact Riemannian surface embedded in $\mathbb{E}^3$ with the induced Euclidean metric, T the triangulation generated by Delaunay refinement on conformal uniformization domain, with circumradius bound ε. If B is the relative interior of a union of triangles of T, then

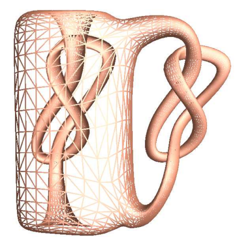
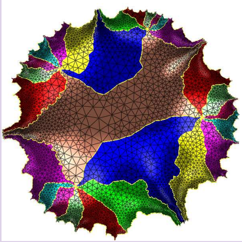
$$\begin{array}{rcl} |\phi_T^G(B) - \phi_M^G(\pi(B))| & \leq & K\varepsilon \\ |\phi_T^H(B) - \phi_M^H(\pi(B))| & \leq & K\varepsilon \end{array}$$

where $\pi : T \to M$ is the closest point projection, ϕ^H, ϕ^G are the mean and Gaussian curvature measures, where

$$K = O(area(B)) + O(length(\partial B)).$$

Thanks

For more information, please email to gu@cs.sunysb.edu.

Thank you!