Fourier Series Methods for Numerical Conformal Mapping of Smooth Domains

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Conformal Geometry in Mapping, Imaging, and Sensing

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Outline

1. Introduction
   - Some background
   - Numerical preview and gallery
2. Fourier series methods
   - Fornberg's method for the disk (1980)
     - Analyticity conditions
     - Linearization
     - Discretization by N-pt. trig. interp.
   - Fornberg-like method for the annulus (1998)
   - Multiply connected Fornberg (bounded case, 2009)
3. Remarks and extra details

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Collaborators

Colleagues: Alan Elcrat (WSU) and John Pfaltzgraff (UNC Chapel Hill)

MS/PhD students: Mark Horn, Noureddine Benchama, Lianju (Julian) Wang, and Everett Kropf

Conformal map $w = f(z)$ from disk to target domain

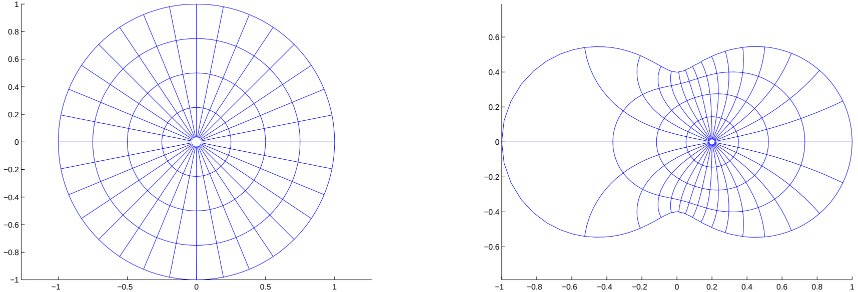

Figure: Fornberg (Fourier series) map from unit disk to interior of an inverted ellipse using 64 Fourier points. $f'(z) \neq 0$, so locally $f(a+h) \approx f(a) + f'(a)h$ and f maps a small circle near $z = a$ to a circle near $f(a)$ magnified by $|f'(a)|$ and rotated by $\arg f'(a)$. Therefore curves intersecting at angle θ at a will be mapped to curves intersecting at angle θ at $f(a)$ and the map is *angle-preserving* or *conformal*. Existence and uniquesness given by Riemann Mapping Theorem with $f(0)$ and $f(1)$ fixed.

Interior mult. conn. case–Kropf's MS thesis (2009)

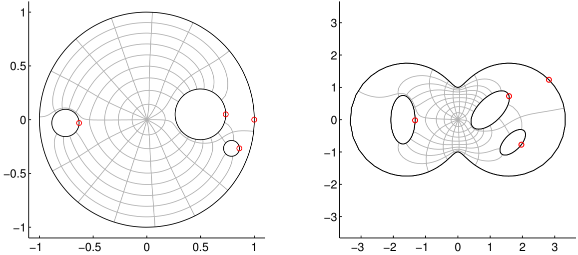

Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.

Boundary correspondence

The boundary Γ of Ω is parametrized by S (e.g., arclength or polar angle), $\Gamma : \gamma(S), 0 \leq S \leq L, \gamma(0) = \gamma(L)$. If $S = S(\theta)$ or its inverse $\theta(S) = \arg f^{-1}(\gamma(S))$ is known, then the map is known for $z \in D$ or $w \in \Omega$ by the Cauchy Integral Formula,

$$w = f(z) = \frac{1}{2\pi i} \int_C \frac{\gamma(S(\theta))}{\zeta - z} d\zeta(\theta)$$

or

$$z = f^{-1}(w) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{i\theta(S)}}{\gamma(S) - w} d\gamma(S).$$

Two classes of "traditional" methods

1. Find $S = S(\theta)$ such that $f(e^{i\theta}) = \gamma(S(\theta))$. We will discuss this case. These methods solve a nonlinear integral equation for $S(\theta)$ by linearly convergent methods of successive approximation (Picard-like iteration) such as Theodorsen's method, or quadratically convergent Newton-like methods such as Fornberg's or Wegmann's methods. Cost: $O(N \log N)$ with FFTs.
2. Find $\theta = \theta(S)$ such that $f^{-1}(\gamma(S)) = e^{i\theta(S)}$. These methods solve linear integral equations arising from potential theory for $\theta(S)$ or $\theta'(S)$. Cost: $O(N^2)$ operation counts, but can handle more highly distorted regions.

MANY other methods exist, as we see at this meeting, based on ideas from computational geometry, circle packing, Riemann-Hilbert problems, orthogonalization, compositions of explicit maps (Grassmann, Marshall),...

A few general references

[1.] T. A. Driscoll and L. N. Trefethen, *Schwarz-Christoffel mapping*, Cambridge U. Press, 2002.
[2.] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, 1964.
[3.] X. D. Gu and S.-T. Yau, *Computational Conformal Geometry*, International Press, 2008.
[4.] P. Henrici, *Applied and Computational Complex Analysis, Vol. 3*, Wiley, 1986.
[5.] K. Stephenson, *Introduction to Circle Packing*, Cambridge, 2005.
[6.] R. Wegmann, *Methods for Numerical Conformal Mapping*, survey article in Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, 2005, pp. 351–477.

Key idea for this talk: Taylor/Laurent series = Fourier series

For $|z| < |\zeta| = 1, \zeta = e^{i\theta}, d\zeta = ie^{i\theta} d\theta$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma(S(\theta))}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(S(\theta)) \left(1 + \frac{z}{\zeta} + \left(\frac{z}{\zeta}\right)^2 + \cdots \right) \frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \gamma(S(\theta))(1 + ze^{-i\theta} + z^2 e^{-2i\theta} + \cdots) d\theta \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \gamma(S(\theta)) e^{-ik\theta} d\theta \right) z^k \\ &= \sum_{k=0}^{\infty} a_k z^k, \end{aligned}$$

Taylor coeff. = Fourier coeff. $a_k := \frac{1}{2\pi} \int_0^{2\pi} \gamma(S(\theta)) e^{-ik\theta} d\theta$.

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3 Remarks and extra details

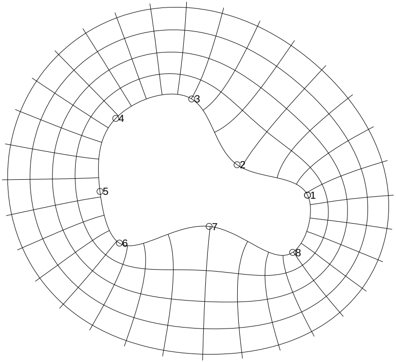


Figure: Fornberg map from exterior of unt disk to exterior of spline

Simply-connected case: crowding=large distortions=Ill-conditioning

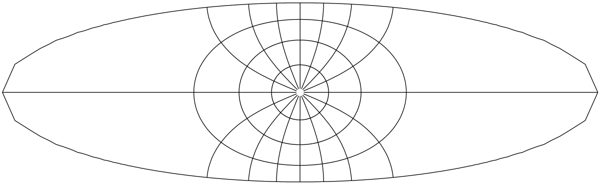


Figure: Fornberg (Fourier series) map from unit disk to interior of ellipse using 1024 Fourier points.

Map from annulus–D. and Pfaltzgraff (1998)

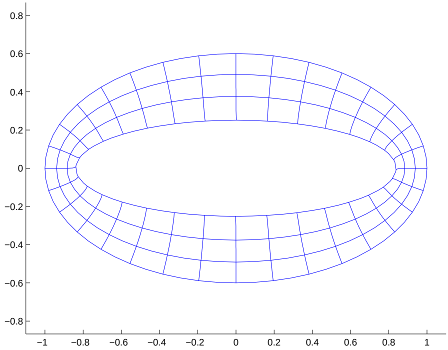


Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Exterior mult. conn. case–Benchama's PhD thesis (2003)

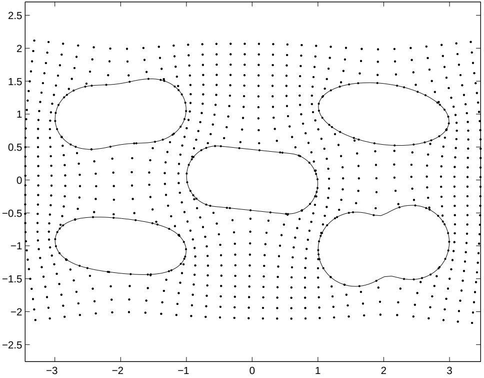
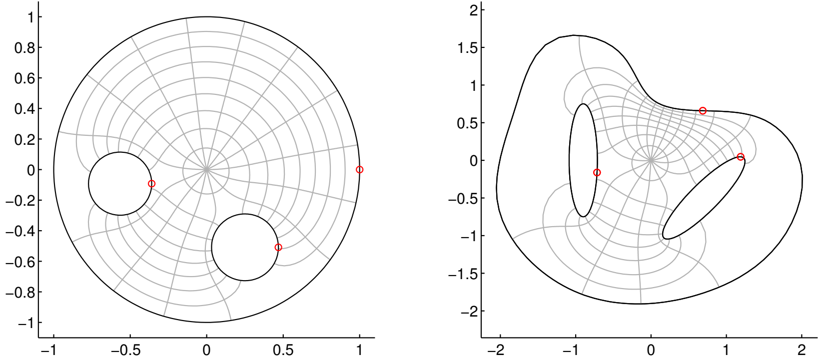


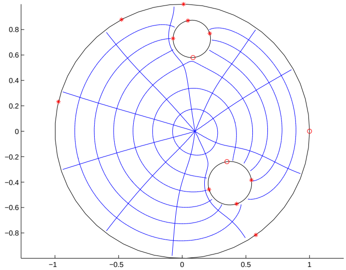
Figure: Fornberg map to the exterior of five curves.

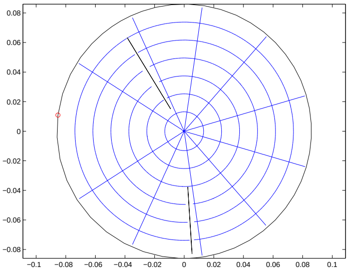
Interior mult. conn. case–Kropf's MS thesis (2009)

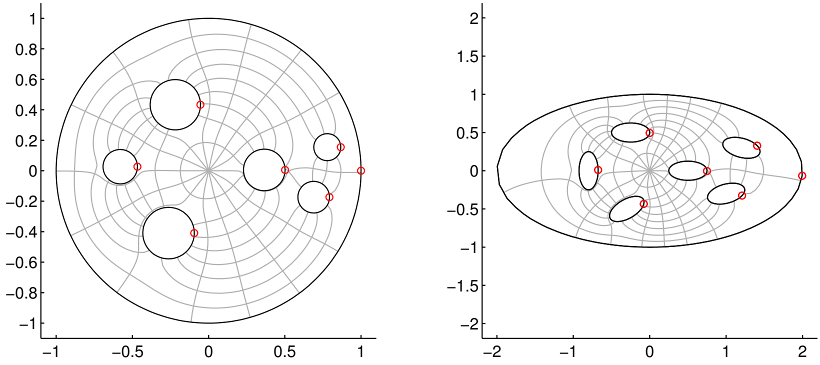


- A target region (on the right) with an outer spline boundary which is parametrized by arclength.

Radial slit map from Kropf's PhD thesis (2012)

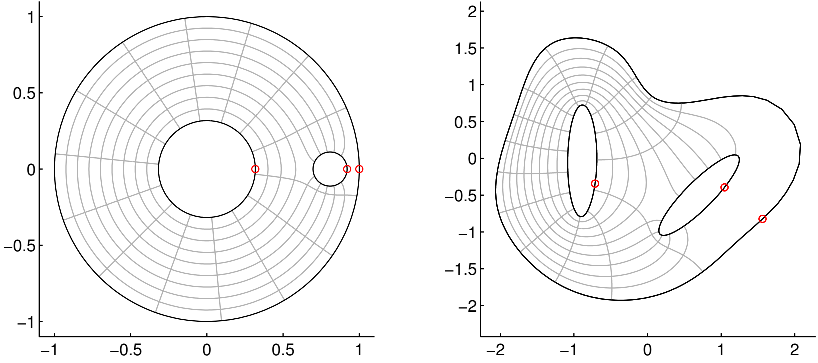






- A target region with $m = 7$.

Numerical Example



- Annulus with circular holes as a computational domain.

Outline

Conformal map $w = f(z)$ from disk to target domain

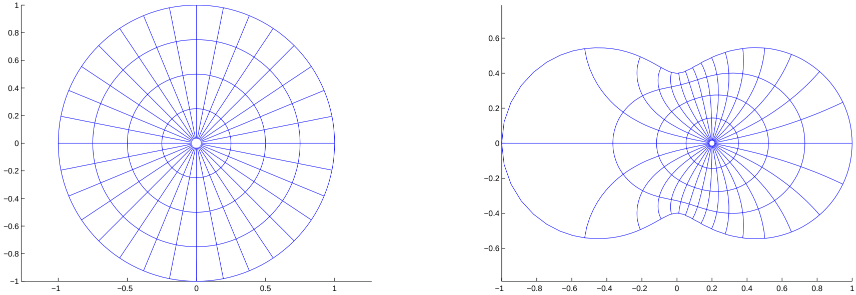


Figure: Fornberg (Fourier series) map from unit disk to interior of an inverted ellipse using 64 Fourier points. Normalization fixes three real parameters: $f(0)$ fixed and $f(1)$ fixed.

Some useful linear operators

For $h = h(\theta)$, 2π-periodic, $h(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$

$$
\begin{aligned}
Jh(\theta) &:= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta = c_0 \\
P_+h(\theta) &:= \sum_{k=1}^{\infty} c_k e^{ik\theta} \\
P_-h(\theta) &:= \sum_{k=-\infty}^{0} c_k e^{ik\theta}
\end{aligned}
$$

Note that $P_\pm^2 = P_\pm$ are *projection operators* onto subspaces of $L^2[0, 2\pi]$ whose nonpositive/positive indexed Fourier coefficients 0. Also note

$$
\begin{aligned}
P_+h &= \frac{1}{2}(I + iK - J)h, \\
P_-h &= \frac{1}{2}(I - iK + J)h.
\end{aligned}
$$

(Infinite) matrix form $Fh := \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} =: \underline{c}$ and $Kh = F^{-1}\hat{K}Fh$

$$= F^{-1} \begin{bmatrix} \ddots & & & & & & \\ & i & & & & & \\ & & i & & & & \\ & & & 0 & & & \\ & & & & -i & & \\ & & & & & -i & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = F^{-1} \begin{bmatrix} \vdots \\ ic_{-2} \\ ic_{-1} \\ 0 \\ -ic_1 \\ -ic_2 \\ \vdots \end{bmatrix},$$

Condition for analytic extension of boundary values

Theorem

A function $h \in Lip(\Gamma)$ can be continued analytically into D^+ (i.e., $f(t) = h(t), t \in \Gamma$) if and only if

$$f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{h(t)}{t - z} dt = 0, \quad z \in D^-,$$

or, equivalently, if

$$\frac{1}{2\pi i} \int_\Gamma t^n h(t) dt = 0, \quad n = 0, 1, 2, \ldots.$$

Proof.

Cauchy Integral Theorem and Sokhotskyi jump relations, $f^+ - f^- = h$; see, e.g., Henrici, ACCA, v. 3, Muskhelishvili, SIE.

Condition for unit D=disk

Theorem

A function $f \in \mathrm{Lip}(C)$ on the boundary C of the unit disk extends to an analytic function in the interior of the disk with $f(0) = 0$ if and only if

$$P_{-} f(e^{i\theta}) = 0. \tag{1}$$

That is, negative indexed coefficients are 0.

Linearization

Given the kth Newton iterate $S = S^k(\theta)$, find correction $U^k(\theta)$, real, such that

$$f(e^{i\theta}) = \gamma(S^k(\theta) + U^k(\theta)) \approx \xi(\theta) + e^{i\beta(\theta)}U(\theta)$$

extends analytically to the interior of the unit disk with $f(0) = 0$, where $\xi(\theta) = \gamma(S^{(k)}(\theta)), \beta(\theta) = \arg\gamma'(S^{(k)}(\theta))$, and $U(\theta) := |\gamma'(S^{(k)}(\theta)|U^{(k)}(\theta)$ extends analytically to the interior of the unit disk with $f(0) = 0$. The analyticity condition

$$2P\_f = (I - iK + J)f = 0$$

implies that

$$(I - iK + J)e^{i\beta(\theta)}U(\theta) = -2P_{-}\xi(\theta).$$

U real gives

$$(I + R)U = r$$

where $R = \mathrm{Re}(e^{-i\beta}(J - iK)e^{i\beta})$ and $r = -\mathrm{Re}(e^{-i\beta}(I - iK + J)\xi)$.

R is a compact operator (Widlund, Wegmann)

$$RU(\theta) := \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin\left(\beta(\phi) - \beta(\theta) + \frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)} U(\phi)\, d\phi,$$

and for γ sufficiently smooth R^{in} is a symmetric, compact operator on L^2.

Discretization by N-pt. trig. interp.

With $E = \mathrm{diag}_j(e^{i\beta(\theta_j)})$, $j = 0, 1, \cdots, N-1$, discretization gives

$$A\underline{U} = (I_N + R_N)\underline{U} = \underline{r}.$$

where the matrix

$$I_N + R_N = \frac{2}{N}\mathrm{Re}(E^H F^H P_N F E)$$

(with $P_N := \mathrm{diag}[1, 0, \ldots, 0, 1, \ldots, 1]$) is symmetric and pos.(semi)def. with eigenvalues well-grouped around 1 and conjugate gradient converges superlinearly.

Matrix-vector multiplications costs $O(N \log N)$ with FFT.

The *Newton update* is given by

$$\underline{S}^{(k+1)} = \underline{S}^{(k)} + \underline{U}^{(k)},$$

with $U_0 = 0$ set to fix a boundary point

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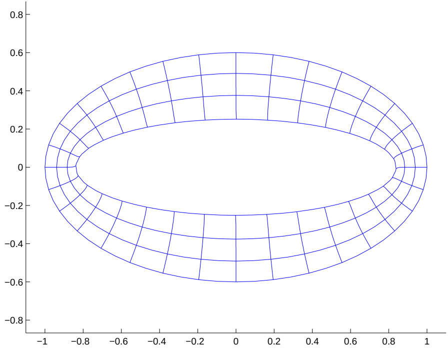


Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Analyticty conditions

Let C_1 and C_2 denote the outer and inner boundaries, respectively, of the annulus $\rho < |z| < 1$, and put $C = C_1 - C_2$.

Theorem

A function $h \in Lip(C)$ extends analytically to the annulus $\rho < |z| < 1$ if and only if

$$\int_{C_1} h(z) z^k dz = \int_{C_2} h(z) z^k dz, \quad k \in \mathbf{Z}.$$

If we let

$$h(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \qquad h(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}$$

then the above condition becomes $\rho^k a_k = b_k, \quad k \in \mathbf{Z}$ or (to prevent overflow)

$$\rho^k a_k = b_k, a_{-k} = \rho^k b_{-k}, k = 0, 1, 2, \ldots.$$

Mapping problem

Target region Ω bounded by two smooth curves $\Gamma_1 : \gamma_1(S_1)$ and $\Gamma_2 : \gamma_2(S_2)$.

Problem: Find the *boundary correspondences* $S_1(\theta)$ and $S_2(\theta)$ and the *conformal modulus* ρ such that $f(z)$ is analytic in the annulus $\rho < |z| < 1$ and $f(e^{i\theta}) = \gamma_1(S_1(\theta))$ and $f(\rho e^{i\theta}) = \gamma_2(S_2(\theta))$.

Linearization for Newton-like method

At each Newton step we want to compute corrections $U_1(\theta),\ U_2(\theta)$, and $\delta\rho$ to $S_1(\theta),\ S_2(\theta)$, and ρ. With S_j arclength, $\beta_j(\theta) := \arg \gamma_j'(S_j(\theta)),\ \xi_j(\theta) := \gamma_j(S_j(\theta)),\ j = 1,2,\ \zeta(\theta) := f'(\rho e^{i\theta})e^{i\theta} = -ie^{i\beta_2(\theta)}dS_2(\theta)/d\theta/\rho$, as in [LM] we linearize about S_1, S_2, and ρ,

$$\begin{aligned} \gamma_j(S_j(\theta) + U_j(\theta)) &\approx \gamma_j(S_j(\theta)) + \gamma_j'(S_j(\theta))U_j(\theta)),\ j = 1,2, \\ f((\rho + \delta\rho)e^{i\theta}) &\approx f(\rho e^{i\theta}) + f'(\rho e^{i\theta})\delta\rho e^{i\theta} \end{aligned}$$

giving

$$\begin{aligned} f(e^{i\theta}) &\approx \xi_1(\theta) + e^{i\beta_1(\theta)}U_1(\theta) \\ f(\rho e^{i\theta}) &\approx \xi_2(\theta) + e^{i\beta_2(\theta)}U_2(\theta) - \zeta(\theta)\delta\rho. \end{aligned}$$

We find $U_1, U_2, \delta\rho$ to force these BVs to satisfy the analyticity conditions for the annulus.

Linear system

Letting a_k and b_k now denote the N discrete Fourier coefficients and using the $N-$periodicity $a_{k+N} = a_k$, we have with $N = 2M$

$$\underline{a} = (a_0, a_1, \ldots, a_M, a_{M+1}, \ldots, a_{N-1})^T = (a_0, a_1, \ldots, a_M, a_{-M+1}, \ldots, a_{-1})^T$$

$\underline{b}$ is defined similarly. Next define the $N \times N$ matrices $P_1 = \mathrm{diag}(1, \rho, \ldots, \rho^{M-1}, 1, \ldots, 1)$ and $P_2 = -\mathrm{diag}(1, \ldots, 1, 1, \rho^{M-1}, \ldots, \rho)$. If we set $a_M = b_M$ as in [Fo2, eq. 6], we write the discrete form of our analyticity conditions as

(29) $$P_1\underline{a} + P_2\underline{b} = 0.$$

Linear system

With $E_j := \mathrm{diag}_{l=0,\ldots,N-1}(e^{i\beta_j(\theta_l)}),\ j=1,2$, our discrete linearizations become

(30) $$N\underline{a} = F\underline{\xi}_1 + FE_1\underline{U}_1$$

(31) $$N\underline{b} = F\underline{\xi}_2 + FE_2\underline{U}_2 - F\underline{\zeta}\delta\rho.$$

Substituting these linearizations into the discrete analyticity conditions gives our linear system for $\underline{U}_1,\ \underline{U}_2$, and $\delta\rho$,

$$(C\ \underline{w})\underline{U} = P_1FE_1\underline{U}_1 + P_2FE_2\underline{U}_2 - P_2F\underline{\zeta}\delta\rho = -P_1F\underline{\xi}_1 - P_2F\underline{\xi}_2 =: \underline{c}.$$

where $C = (P_1FE_1\ \ P_2FE_2)$ is a complex $N \times 2N$ matrix, $\underline{w} = -P_2F\underline{\zeta}$ is a complex N-vector, and

$$\underline{U} = \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \delta\rho \end{bmatrix}.$$

We have a system of N complex equations in $2N+1$ real unknowns, $\underline{U}$. To satisfy the normalization $f(1) = \gamma_1(0)$, we add the equation $\underline{q}^T \underline{U} = U_0 = \delta := 0$, where $\underline{q}^T = (1, 0, \ldots, 0)^T$ is a $2N+1$-vector. We write

$$D = \begin{bmatrix} \dfrac{C}{\sqrt{N}} & \dfrac{\underline{w}}{\underline{q}^T/2} \end{bmatrix}, \ \underline{g} := \begin{bmatrix} \dfrac{\underline{c}}{\delta} \end{bmatrix}.$$

and our system now becomes

$$D\underline{U} = \underline{g},$$

a system of N complex equations and 1 real equation for the $2N+1$ real unknowns, $\underline{U}$. Using the normal equations and $\underline{U}$ real, we have

$$A\underline{U} = \frac{2}{N}\mathrm{Re}(D^H D)\underline{U} = \underline{r} := \frac{2}{N}\mathrm{Re}(D^H \underline{g}).$$

As in the simply connected case, we solve the system by the conjugate gradient method using FFTs.

The matrix A is a discretization of the identity plus a compact operator as in the disk case. We have the following $2N+1 \times 2N+1$-matrix

$$A = \frac{2}{N}\mathrm{Re}(D^H D) = \begin{bmatrix} A_{11} & A_{12} & \underline{w}_1 \\ A_{12}^T & A_{22} & \underline{w}_2 \\ \underline{w}_1^H & \underline{w}_2^H & 2\underline{w}^H\underline{w}/N \end{bmatrix} + \frac{1}{2}\underline{qq}^T$$

where $A_{ij} = \frac{2}{N}\mathrm{Re}(E_i^H F^H P_i P_j F E_j)$ and $\underline{w}_i = \frac{2}{N}\mathrm{Re}(E_i^H F^H P_i \underline{w}),\ i,j = 1,2.$ Now it is easy to see that A_{11} is a (low rank perturbation of) the discretization of

$$2\mathrm{Re}(e^{-i\beta_1}(P_- + l_1 *)e^{i\beta_1}) = I + R_1 + C_1$$

with $N-$point trigonometric interpolation where
$R_1 = \mathrm{Re}(e^{-i\beta_1}(J - iK)e^{i\beta_1}$ is compact, $*$ is convolution,
$l_1(\theta) = \rho^2 e^{i\theta}/(1-\rho^2 e^{i\theta}) = \sum_{k=1}^{\infty} \rho^{2k} e^{ik\theta}$, and
$C_1 = 2\mathrm{Re}(e^{-i\beta_1} l_1 * (e^{i\beta_1}))$ is the product of bounded operators and a convolution and is, hence, compact.

Newton update

$$\underline{S}_1^{(k+1)} = \underline{S}_1^{(k)} + \underline{U}_1^{(k)}$$
$$\underline{S}_2^{(k+1)} = \underline{S}_2^{(k)} + \underline{U}_2^{(k)}$$
$$\rho^{(k+1)} = \rho^{(k)} + \delta\rho^{(k)}.$$

Outline

Interior mult. conn. case–Kropf's MS thesis (2009)

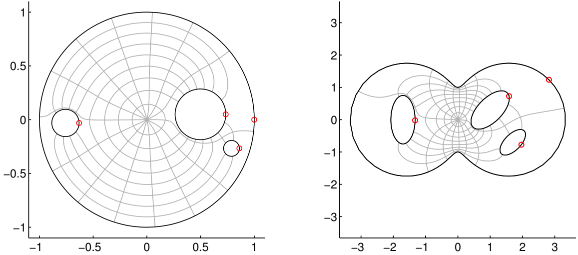
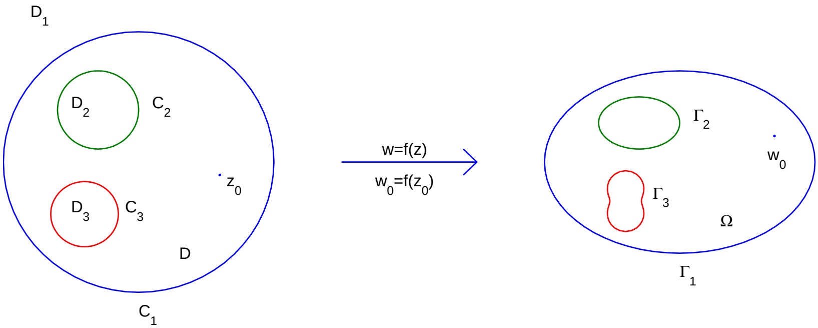


Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.

Computational Goal



- The goal is to compute the conformal map $f : D \to \Omega$.
- To do this we must calculate
  1. the centers c_ν and radii ρ_ν of the circles C_ν, $2 \le \nu \le m$, and
  2. the boundary correspondences $S_\nu(\theta)$, where $0 \le \theta \le 2\pi$,

  such that $f(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta))$, $1 \le \nu \le m$.

Form of the Map

Theorem

The conformal map described above has the series representation

$$f(z) = \sum_{j=0}^{\infty} a_{1,j} z^j + \sum_{\nu=2}^{m} \sum_{j=1}^{\infty} a_{\nu,-j} \left(\frac{\rho_\nu}{z - c_\nu} \right)^j,$$

where for $1 \leq \nu \leq m$ *and* $j > 0$ *the Fourier coefficients* $a_{\nu,j}$ *are defined*

$$a_{\nu,j} := \frac{1}{2\pi} \int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta}) e^{-ij\theta} \, d\theta.$$

Analytic Continuation

Theorem

Let C be a positively oriented, Lipschitz continuous curve with D the region bounded by C and D^- the compliment of $D \cup C$. A function $f \in \mathrm{Lip}(C)$ can be continued analytically into D if and only if

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0, \quad \forall z \in D^-.$$

Now applied to multiply connected circle domain D.

Analyticity Conditions

Theorem

A function $f \in \mathrm{Lip}(C)$ extends analytically into D if and only if for all $k \geq 0$

$$a_{1,-(k+1)} - \sum_{\nu=2}^{m} \sum_{j=0}^{k} \binom{k}{j} \rho_\nu^{j+1} c_\nu^{k-j} a_{\nu,-(j+1)} = 0$$

and

$$\sum_{j=0}^{\infty} B_{k+1,j} \rho_\ell^k c_\ell^j a_{1,k+j} - a_{\ell,k} - \sum_{\substack{\nu=2 \\ \nu \neq \ell}}^{m} \sum_{j=0}^{\infty} \frac{\rho_\ell^k}{(c_\nu - c_\ell)^{k+1}} B_{k+1,j} \frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} a_{\nu,-(j+1)} = 0.$$

Note on Analyticity Conditions

For the analyticity conditions we need to define some binomial coefficients.

Definition

For $k > 0$ and $x, y \in \mathbb{C}$,

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j \quad \text{where} \quad \binom{k}{j} := \frac{k!}{j!(k-j)!}.$$

Definition

For $k > 0$ and $|z| < 1$,

$$\frac{1}{(1-z)^k} = \sum_{j=0}^{\infty} B_{k,j} z^j \quad \text{where} \quad B_{k,j} := \frac{k(k+1)\cdots(k+j-1)}{j!}.$$

Note on Proof of Analyticity Conditions

The proof involves

1. applying the above analytic continuation Theorem for an arbitrary point z in each $D_1, \ldots, D_m$,
2. expanding the function in the appropriate Laurent series, and
3. setting the resulting series equal to 0.

Proof of Analyticity Conditions

(Outside C_1)

Proof.

For z in D_1 we have $|z| > 1$ and $|\zeta|/|z| < 1$ for ζ on any $C_1, \ldots, C_m$, thus

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = -\frac{1}{2\pi i}\int_C f(\zeta)\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\zeta}{z}\right)^k\,d\zeta$$

$$= -\sum_{k=0}^{\infty} z^{-k-1}\frac{1}{2\pi i}\int_C f(\zeta)\zeta^k\,d\zeta = 0.$$

The last integral on the RHS must be zero for all $k \geq 0$.

Proof of Analyticity Conditions

(Outside C_1)

Proof.

$$0 = \frac{1}{2\pi i}\int_C f(\zeta)\zeta^k\,d\zeta = \frac{1}{2\pi i}\int_{C_1} f(\zeta)\zeta^k\,d\zeta - \sum_{\nu=2}^{m}\frac{1}{2\pi i}\int_{C_\nu} f(\zeta)\zeta^k\,d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{i(k+1)\theta}\,d\theta \qquad (\text{Note}: \quad \zeta^k = (c_\nu + \rho_\nu e^{i\theta})^k$$

$$- \sum_{\nu=2}^{m}\sum_{j=0}^{k}\binom{k}{j}\rho_\nu^{j+1}c_\nu^{k-j}\frac{1}{2\pi}\int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta})e^{i(j+1)\theta}\,d\theta$$

$$= a_{1,-(k+1)} - \sum_{\nu=2}^{m}\sum_{j=0}^{k}\binom{k}{j}\rho_\nu^{j+1}c_\nu^{k-j}a_{\nu,-(j+1)}.$$

Map Normalization

- The map is normalized by specifying three real conditions:
  - $f(1) = \gamma_1(0)$ and
  - $$w_0 = f(z_0) = \sum_{k=0}^{\infty} a_{1,k} z_0^k + \sum_{\nu=2}^{m} \sum_{k=1}^{\infty} a_{\nu,-k} \left(\frac{\rho_\nu}{z_0 - c_\nu} \right)^k .$$

Linearization

We now write $f(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta))$ as a linear problem.

- For an initial guess $S_\nu(\theta)$ and 2π periodic correction $U_\nu(\theta)$,

$$\gamma_\nu(S_\nu(\theta) + U_\nu(\theta)) \approx \gamma_\nu(S_\nu(\theta)) + \gamma_\nu'(S_\nu(\theta))U_\nu(\theta).$$

- For an initial guess of c_ν and ρ_ν with corrections δc_ν and $\delta\rho_\nu$,

$$(f + \delta f)(c_\nu + \delta c_\nu + (\rho_\nu + \delta\rho_\nu)e^{i\theta}) \approx (f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) + f'(c_\nu + \rho_\nu e^{i\theta})(\delta c_\nu + \delta\rho_\nu e^{i\theta}).$$

- Setting the RHS of these approximations equal gives

$$(f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta)) + \gamma_\nu'(S_\nu(\theta))U_\nu(\theta) - f'(c_\nu + \rho_\nu e^{i\theta})(\delta c_\nu + \delta\rho_\nu e^{i\theta}).$$

Linearization

More concisely

- For convenience define
  - $\xi_\nu(\theta) := \gamma_\nu(S_\nu(\theta))$,
  - $\eta_\nu(\theta) := \gamma'_\nu(S_\nu(\theta))$, and
  - $\zeta_\nu(\theta) := -f'(c_\nu + \rho_\nu e^{i\theta})e^{i\theta} = i\rho_\nu^{-1}\eta_\nu S'_\nu(\theta)$.
- The linearization conditions can then be written
  - $(f + \delta f)(e^{i\theta}) = \xi_1(\theta) + \eta_1(\theta)U_1(\theta)$
  - $(f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) = \xi_\nu(\theta) + \eta_\nu(\theta)U_\nu(\theta) + \zeta_\nu(\theta)(\delta\rho_\nu + \delta c_\nu e^{-i\theta})$

  for the updates around C_1 and around C_ν, $2 \leq \nu \leq m$, respectively.

Newton Updates

- After the linear system has been solved, the updates are applied at each step (n) as follows:
  - $S_\nu^{(n)}(\theta) = S_\nu^{(n-1)}(\theta) + U_\nu^{(n-1)}(\theta)$

  for $1 \leq \nu \leq m$, and
  - $c_\nu^{(n)} = c_\nu^{(n-1)} + \delta c_\nu^{(n-1)}$
  - $\rho_\nu^{(n)} = \rho_\nu^{(n-1)} + \delta \rho_\nu^{(n-1)}$

  for $2 \leq \nu \leq m$.

Discrete analyticity conditions

- $$a_{1,-(k+1)} - \sum_{\nu=2}^{m} \sum_{j=0}^{k} \binom{k}{j} \rho_\nu^{j+1} c_\nu^{k-j} a_{\nu,-(j+1)} = 0,$$

- $$\sum_{j=0}^{M-1} B_{k+1,j} \rho_\ell^k c_\ell^j a_{1,k+j} - a_{\ell,k}$$
$$- \sum_{\substack{\nu=2 \\ \nu \neq \ell}}^{m} \sum_{j=0}^{M-1} \frac{\rho_\ell^k}{(c_\nu - c_\ell)^{k+1}} B_{k+1,j} \frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} a_{\nu,-(j+1)} = 0,$$

- $$\sum_{j=0}^{M-1} a_{1,j} z_0^j + \sum_{\nu=2}^{m} \sum_{j=1}^{M} a_{\nu,-j} \left(\frac{\rho_\nu}{z_0 - c_\nu} \right)^j = w_0.$$

Matrix Form

of the Analyticity and Normalization Conditions

- The discrete system of equations can be written

$$P\underline{a} = P_1\underline{a}_1 + \cdots + P_m\underline{a}_m = [P_1 \quad \cdots \quad P_m] \begin{bmatrix} \underline{a}_1 \\ \vdots \\ \underline{a}_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_0 \end{bmatrix} := \underline{r}.$$

Discrete Linearization Conditions

- We need to define the vectors and vector functions
  - $\underline{\theta} := \frac{2\pi}{N}(0, 1, \ldots, N-1)^T$,
  - $\underline{\xi}_\nu := \xi_\nu(\underline{\theta})$,
  - and similarly for $\underline{\eta}_\nu$, $\underline{\zeta}_\nu$, and $\underline{U}_\nu$.
- If F is the discrete Fourier transform matrix, $E_\nu := \mathrm{diag}(\underline{\eta}_\nu)$, $\underline{q} := e^{-i\underline{\theta}}$, and $*$ is the Hadamard product, then the linearization conditions become
  - $N\underline{a}_1 = F\underline{\xi}_1 + FE_1\underline{U}_1$ and
  - $N\underline{a}_\nu = F\underline{\xi}_\nu + FE_\nu\underline{U}_\nu + \delta\rho_\nu F\underline{\zeta}_\nu + \delta c_\nu F(\underline{q} * \underline{\zeta}_\nu)$.

New Linear System

- For ease of exposition, assume $m = 3$ for the rest of this section.
- Combining the discrete system of equations for the analyticity and normalization conditions with the discretized linear conditions gives

$$
\begin{aligned}
&P_1 F E_1 \underline{U}_1 \\
&\quad + P_2(F E_2 \underline{U}_2 + \delta\rho_2 F \underline{\zeta}_2 + (\mathrm{Re}\,\delta c_2 + i\mathrm{Im}\,\delta c_2) F(\underline{q} * \underline{\zeta}_2)) \\
&\quad + P_3(F E_2 \underline{U}_3 + \delta\rho_3 F \underline{\zeta}_3 + (\mathrm{Re}\,\delta c_3 + i\mathrm{Im}\,\delta c_3) F(\underline{q} * \underline{\zeta}_3)) \\
&\qquad = N\underline{r} - P_1 F \underline{\xi}_1 - P_2 F \underline{\xi}_2 - P_3 F \underline{\xi}_3 := \underline{\tilde{g}}.
\end{aligned}
$$

More Convenience Notation

- Let $\underline{w}_\nu := P_\nu F \underline{\zeta}_\nu$,
- $\underline{wq}_\nu := P_\nu F(\underline{q} * \underline{\zeta}_\nu)$,
- $W := \begin{bmatrix} \underline{w}_2 & \underline{w}_3 & \underline{wq}_2 & i\underline{wq}_2 & \underline{wq}_3 & i\underline{wq}_3 \end{bmatrix}$,
- and of course $P := \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$.
- Also define the real vector $\underline{U} :=$

$$\begin{bmatrix} \underline{U}_1^T & \underline{U}_2^T & \underline{U}_3^T & \delta\rho_2 & \delta\rho_3 & \mathrm{Re}\,\delta c_2 & \mathrm{Im}\,\delta c_2 & \mathrm{Re}\,\delta c_3 & \mathrm{Im}\,\delta c_3 \end{bmatrix}^T.$$

The Matrix $\tilde{D}$

- Combining all of this we now have

$$\tilde{D}\underline{U} := \begin{bmatrix} P_1 & P_2 & P_3 & W \end{bmatrix} \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \underline{U} = \underline{\tilde{g}}.$$

The Matrix D

Through normalization

- We add a row to this system to force $U_1(0) = 0$ at every iteration.
- This satisfies the normalization condition $f(1) = \gamma_1(0)$.
- To do this define the vector $\underline{v}^T := (1, 0, \ldots, 0)$, and then

$$D := \begin{bmatrix} \tilde{D} \\ \frac{\sqrt{N}}{2} \underline{v}^T \end{bmatrix} \quad \text{and} \quad \underline{g} := \begin{bmatrix} \tilde{g} \\ 0 \end{bmatrix}.$$

The Matrix A

- Taking the "normal equations" and using the fact $\underline{U}$ is real,

$$A\underline{U} := \frac{2}{N}\mathrm{Re}\,(D^H D)\underline{U} = \frac{2}{N}\mathrm{Re}\,(D^H \underline{g}) := \underline{b}.$$

- This system can now be solved efficiently using the conjugate gradient method.

The Matrix A Decomposed

- Define
  - $A_{kj} := (2/N)\mathrm{Re}\,(E_k^H F^H P_k^H P_j F E_j)$ and
  - $X_k := (2/N)\mathrm{Re}\,(E_k^H F^H P_k^H W)$.
- Then A can be written

$$A = \frac{2}{N}\mathrm{Re}\,(D^H D) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & X_1 \\ A_{21} & A_{22} & A_{23} & X_2 \\ A_{31} & A_{32} & A_{33} & X_3 \\ X_1^T & X_2^T & X_3^T & W^H W \end{bmatrix} + \frac{1}{2}\underline{v}\,\underline{v}^T,$$

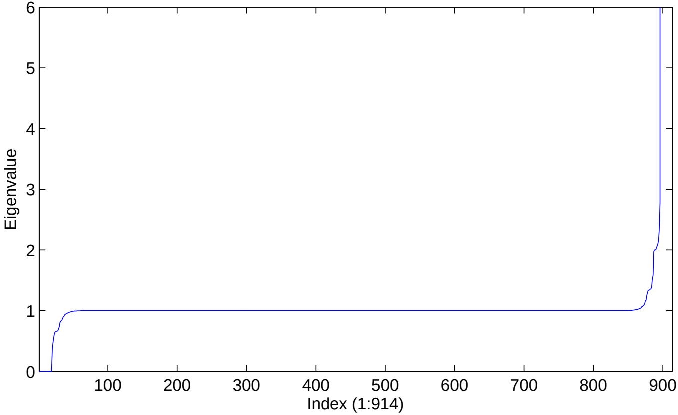
Eigenvalues of A

- To understand the eigenvalues of A it suffices to examine the submatrix

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

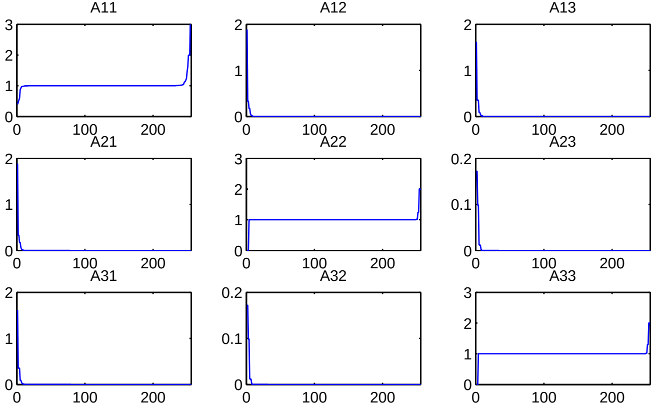
- For the eigenvalues:
  - The diagonal entries can be shown to be discretizations of the identity plus a compact operator, and
  - the off-diagonal entries can be shown to be discretizations of a compact operator.
- In effect $\hat{A}$ is a low-rank perturbation of the identity, and the eigenvalues cluster around 1.
- This is the property which makes the conjugate gradient method an efficient solver to use for this problem.

Eigenvalues of A Cluster Around 1



- This map had $m = 7$ and $N = 128$.

Eigenvalues of $\hat{A}$



- This map had connectivity $m = 3$ with $N = 256$.

Remarks and future work

- The extensions of Fornberg's original method are essentially complete. $I + \textit{compact}$ inner systems carry over.
- (The ellipse method was not presented here.)
- The MATLAB codes need to be refined and integrated.
- Further comparisons with Wegmann's methods needs to be done