Analytic Number Theory

Test No 1, 17 February 2016

Full name: College ID No:

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Question 1. Let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series such that there are complex numbers s_1 and s_2 with $F(s_1)$ divergent and $F(s_2)$ convergent.

a) Suppose that $F(s)$ converges at some $s = s_0$. Show that $F(s)$ converges whenever $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, and that

$$F(s) = (s - s_0) \int_1^{\infty} S(x)x^{s_0-s-1}\, dx$$

where

$$S(x) = \sum_{n \leq x} f(n)n^{-s_0}.$$

[5 points for Part a]

b) Conclude that the set

$$S_1 = \{\mathrm{Re}(s) \mid s \in \mathbb{C} \text{ and } F(s) \text{ is a convergent series}\},$$

has a finite infimum.

[2 points for Part b]

Question 2. Let a_n, $n \geq 1$, be a sequence in $\{+1, -1\}$, and $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series, $s \in \mathbb{C}$. Assume that

$$\lim_{X\to\infty} \frac{\log S(X)}{\log X} = \alpha \in (0, 1),$$

where $S(X) = \sum_{n \leq X} a_n$.

a) Prove that for every s with $\mathrm{Re}\, s > \alpha$, the Dirichlet series $A(s)$ is convergent. [6 points for Part a]

b) Prove that if $A(s)$ is convergent for some s, then $\mathrm{Re}\, s \geq \alpha$. [5 points for Part b]

[Hint for part b: Use the relation $|S(N)| = |\sum_{n=1}^{N} a_n n^{-s} n^s|$ and conclude that $|S(N)| \leq MN^s$, for some real constant M independent of N.]

c) Conclude that the abscissa of convergence of $A(s)$ is equal to α. [2 points for part c]