Analytic Number Theory

Solution key of Test No 1, 17 February 2016

Solution of Question 1. First we verify whether f is multiplicative by considering the following four cases. Let $(m,n)=1$ for some $m,n\in\mathbb{N}$.

1) If at least one of n and m is even, then $f(mn)=f(m)f(n)=0$.

2) If $m\equiv 1 \pmod 4$, and $n\equiv 1 \pmod 4$, then $f(mn)=(-1)^{(mn-1)/2}=1$ and $f(m)f(n)=(-1)^{(m-1)/2}\cdot(-1)^{(n-1)/2}=1\cdot 1=1$.

3) If $m\equiv 3 \pmod 4$, and $n\equiv 1 \pmod 4$, then $f(mn)=(-1)^{(mn-1)/2}=-1$ and $f(m)f(n)=(-1)^{(m-1)/2}\cdot(-1)^{(n-1)/2}=(-1)\cdot 1=-1$.

4) If $m\equiv 3 \pmod 4$, and $n\equiv 3 \pmod 4$, then $f(mn)=(-1)^{(mn-1)/2}=1$ and $f(m)f(n)=(-1)^{(m-1)/2}\cdot(-1)^{(n-1)/2}=(-1)\cdot(-1)=1$.

The series

$$\sum_{n=1}^{\infty}|f(n)||n^{-s}|=\sum_{k=0}^{\infty}1\cdot|(2k+1)^{-s}|=\sum_{k=0}^{\infty}(2k+1)^{-\operatorname{Re}(s)},$$

which converges if and only if $\operatorname{Re}(s)>1$.

By a theorem in the lectures, for every s with $\operatorname{Re}(s)>1$ we have

$$\sum_{n=1}^{\infty}f(n)n^{-s}=\prod_{p\text{ prime}}\Big\{\sum_{e=0}^{\infty}f(p^e)p^{-es}\Big\}$$

For $p=2$,

$$\sum_{e=0}^{\infty}f(p^e)p^{-es}=f(1)=1,$$

for primes p of the form $4k+3$, we have

$$\sum_{e=0}^{\infty}f(p^e)p^{-es}=1-p^{-s}+p^{-2s}-p^{-3s}+p^{-4s}-\cdots=\frac{1}{1+p^{-s}}$$

and for primes p of the form $4k+1$ we have

$$\sum_{e=0}^{\infty}f(p^e)p^{-es}=1+p^{-s}+p^{-2s}+p^{-3s}-p^{-4s}+\cdots=\frac{1}{1-p^{-s}}.$$

Thus,

$$\sum_{n=1}^{\infty}f(n)n^{-s}=\Big(\prod_{p\equiv 3}\frac{1}{1+p^{-s}}\Big)\Big(\prod_{p\equiv 1}\frac{1}{1-p^{-s}}\Big).$$

Solution of Question 2. Using the Euler's product formula

$$\zeta(s)=\prod_{p}\Big(\frac{1}{1-p^{-s}}\Big),$$

for $\mathrm{Re}(s) > 1$, we have

$$\frac{\zeta^3(s)}{\zeta(2s)} = \prod_p (\frac{1}{(1-p^{-s})^3}) \prod_p (1-p^{-2s}) = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^3} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2} = \prod_p \frac{1+p^{-s}}{1-2p^{-s}+p^{-2s}}$$

On the other hand, since $d(n) \le n$, we have

$$\sum_{n=1}^{\infty} |d(n^2)||n^{-s}| \le \sum_{n=1}^{\infty} |n^{-s+2}| = \sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)+2}.$$

The above series is convergent for every s with $\mathrm{Re}(s) > 3$. This implies that the abscissa of absolute convergent of the Dirichlet series is less than or equal to 3.

Since $d(n^2)$ is a multiplicative function, by a theorem in the lectures, for every s with $\mathrm{Re}(s) > 3$ we must have

$$\sum_{n=1}^{\infty} d(n^2) n^{-s} = \prod_p \Big(\sum_{e=0}^{\infty} d(p^{2e}) p^{-es} \Big) = \prod_p \Big(\sum_{e=0}^{\infty} (2e+1) p^{-es} \Big).$$

Thus, to prove the identity in the question, it is enough to show that

$$\frac{1+p^{-s}}{1-2p^{-s}+p^{-2s}} = \sum_{e=0}^{\infty} (2e+1) p^{-es}.$$

This can be verified as follows:

$$\begin{aligned}
(1-2p^{-s}+p^{-2s}) \cdot & \sum_{e=0}^{\infty} (2e+1) p^{-es} \\
&= \sum_{e=0}^{\infty} (2e+1) p^{-es} - \sum_{e=0}^{\infty} 2(2e+1) p^{-(e+1)s} + \sum_{e=0}^{\infty} (2e+1) p^{-(e+2)s} \\
&= \big(1 + 3p^{-s} + \sum_{e=2}^{\infty} (2e+1) p^{-es}\big) - (2p^{-s} + \sum_{e=2}^{\infty} 2(2e-1) p^{-es}) + \sum_{e=2}^{\infty} (e-3) p^{-es} \\
&= 1 + p^{-s} + \sum_{e=2}^{\infty} (2e+1-4e+2+2e-3) p^{-es} = 1 + p^{-s}.
\end{aligned}$$