Analytic Number Theory

Solution key of Test No 1, 17 February 2016

Solution of Question 1.

Part a) By the hypotheses the sum $F(s_0) = \sum_{n=1}^{\infty} f(n)n^{-s_0}$ is convergent. We use the partial summation formula with the arithmetic function $g(n) = f(n)n^{-s_0}$ and the function $G(x) = x^{s_0-s}$, for $x > 0$. The function $G(x)$ is continuous and has a continuous derivative. By the summation formula, we have

$$\sum_{n=1}^{N} g(n)G(n) = (\sum_{n=1}^{N} g(n))G(N) - \int_1^N (\sum_{n\leq t} g(n))G'(t)\, dt.$$

[*2 points for this part, correct form of the summation by parts, and the conditions on* G.]

Hence,

$$\begin{aligned}\sum_{n=1}^{N} f(n)n^{-s} &= \sum_{n=1}^{N} f(n)n^{-s_0} \cdot n^{s_0-s}\\ &= \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s} - \int_1^N S(x)(s_0-s)x^{s_0-s-1}\, dx\\ &= \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s}(s-s_0) \int_{x=1}^{N} S(x)x^{s_0-s-1}\, dx\end{aligned}$$

[*1 points for the above calculations.*] Taking limit as N tends to ∞,

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n)n^{-s_0} = F(s_0)$$

is finite by the assumption, which implies that

$$\lim_{N\to\infty} \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s} = 0$$

[*1 point for this term.*]

Therefore,

$$F(s) = \lim_{N\to\infty} \sum_{n=1}^{N} f(n)n^{-s} = (s_0-s) \int_{x=1}^{\infty} S(x)x^{s_0-s-1}\, dx.$$

Finally, the infinite integral is finite since $|S(X)|$ is uniformly bounded from above, and $\mathrm{Re}(s) > \mathrm{Re}\, s_0$. [*1 point for the convergence of this term.*]

Part b)

By the hypotheses, $\mathrm{Re}(s_2) \in S_1$, and hence S_1 is not empty. [*1 point.*]

On the other hand, since $F(s_1)$ is divergent, by part a), $F(s)$ must be divergent for every s with $\mathrm{Re}(s) < \mathrm{Re}(s_1)$. This means that S_1 is bounded from below by $\mathrm{Re}(s_1)$. [*1 point.*]

In particular, S_1 has a finite infimum.

Solution of Question 2.

Part a) Choose $\delta > 0$ with $\mathrm{Re}(s) = \alpha + \delta$. Use the partial summation with $f(n) = a_n$ and the C^1 function $F(x) = x^{-s}$, we obtain

$$\sum_{n=1}^{N} a_n n^{-s} = S(N)N^{-s} + \int_1^N S(x)sx^{-s-1}\,dx$$

[*2 point for initiating the correct idea with the correct arithmetic function and* $F(x)$*.*]

By the relation

$$\lim_{X\to\infty} \frac{\log|S(X)|}{\log X} = \alpha$$

there is $N_0 > 1$ such that for all $X \geq N_0$ we have

$$\log|S(X)| \leq (\alpha + \delta/2)\log X.$$

In other words,

$$|S(X)| \leq X^{\alpha+\delta/2}$$

[*2 point for understanding the right way to use the value of the limit.*]

Hence,

$$\lim_{N\to\infty} S(N)N^{-s} \leq \lim_{N\to\infty} N^{\alpha+\delta/2}N^{-(\alpha+\delta)} = \lim_{N\to\infty} N^{-\delta/2} = 0.$$

[*1 point.*]

Similarly,

$$\left|s\int_{N_0}^N S(x)x^{-s-1}\,dx\right| \leq |s|\int_{N_0}^N x^{\alpha+\delta/2}x^{-\alpha-\delta-1}dx \leq |s|\int_{n_0}^N x^{-1-\delta/2}\,dx < \infty.$$

[*1 point.*]

The above bounds prove that $A(s)$ is a convergent series.

Part b) Let us denote the partial sums of the series $A(s)$ with

$$A(N) = \sum_{n=1}^{N} a_n n^{-s}$$

Since $A(s)$ is convergent, there is $M > 0$ such that for all $N \geq 1$ we have $|A_N(s)| \leq M$. Moreover, since the series $A(s)$ converges, we must have $s \geq 0$. These imply that

$$\begin{aligned} |S(N)| = \Big|\sum_{n=1}^{N} a_n \cdot n^{-s} \cdot n^s\Big| &= \Big|\sum_{n=1}^{N}(A(n) - A(n-1)) \cdot n^s\Big| \\ &= \Big|\sum_{n=1}^{N} A(n)n^s - \sum_{n=1}^{N} A(n-1) \cdot n^s\Big| \\ &= \Big|\sum_{n=1}^{N} A(n)n^s - \sum_{n=0}^{N-1} A(n) \cdot (n+1)^s\Big| \\ &= \Big|\sum_{n=1}^{N-1} A(n)(n^s - (n+1)^s) + A(N)N^s\Big| \\ &\leq M \sum_{n=1}^{N-1}((n+1)^s - n^s) + MN^s \\ &\leq 2MN^s \end{aligned}$$

[*3 point for the calculations, and 1 point for the correct constant M.*]

The above equation implies that

$$\log |S(N)| \leq \log 2 + \log M + s \log N.$$

Hence,

$$\alpha = \lim_{N \to \infty} \frac{\log |S(N)|}{\log N} \leq s.$$

[*1 point.*]

Part c)

By Part a, the series $A(\alpha+1)$ is convergent and by Part b for every s with $\operatorname{Re} s < \alpha$, $A(s)$ is divergent. Thus, the series $A(s)$ has a finite abscissa of convergence, which we denote by σ_1.

[*1 point for any argument that shows the abscissa of convergence exists and is finite.*]

By Part a of the question, for every s with $\operatorname{Re}(s) > \alpha$, $A(s)$ is convergent. This implies that $\sigma_1 \leq \alpha$.

On the other hand, by Part b of the question, if $A(s)$ is convergent, then $s \geq \alpha$. This implies that $\sigma_1 \geq \alpha$. Combining the two inequalities, we conclude that $\sigma_1 = \alpha$.

[*1 point.*]

Using Question 2 above, we can answer Problem 8 in Problem Sheet No 2.

Recall Problem 8

Problem 8. Show that $\sigma_1 = 0$ and $\sigma_0 = 1$ for the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$. For each $\alpha \in [0,1]$ construct an example in which $\sigma_1 = \alpha$ and $\sigma_0 = 1$.

Solution of Problem 8. Let $s = \sigma + it$.

We know that

$$\sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

is convergent if and only if $\sigma > 1$. This implies that $\sigma_0 = 1$.

On the other hand, for $\sigma \leq 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}$ is divergent. However, for every $\sigma > 0$, by the alternating series test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}$ is convergent. (need to see that the sequence $1/n^\sigma$ is monotone decreasing!). This implies that $\sigma_1 = 0$. This implies the first part of the problem.

We need to build an example of a Dirichlet series such that $\sigma_1 = \alpha$ and $\sigma_0 = 1$. If $\alpha = 0$, the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ provides the answer to the problem. If $\alpha = 1$, we take the series $\sum_{n=1}^{\infty} 1/n^s$. So, below we assume that $\alpha \in (0,1)$.

Define the function $h(x) = x^\alpha$, for $x > 0$. The function $h(x)$ is strictly increasing and for every integer $n \geq 1$ we have

$$|h(n+1) - h(n)| \leq 1 \cdot \sup_{t \in [n,n+1]} |h'(t)| = \sup_{t \in [n,n+1]} \alpha t^{\alpha-1} \leq 1 \cdot \frac{1}{n} \leq 1.$$

Inductively we define the sequence of numbers $a_n \in \{+1, -1\}$, for $n \geq 1$, such that the partial sums $S(n) = \sum_{n=1}^{N} a_n$ satisfies

$$|S(n) - n^\alpha| = |S(n) - h(n)| \leq 1. \tag{1}$$

We set $a_1 = +1$. It satisfies Equation (1) for $n = 1$.

Assume that a_i are defined for $1 \leq i \leq n$, and $S(i)$ satisfies Equation (1) for $i \leq n$.

Define,

$$a_{n+1} = \begin{cases} +1 & \text{if } S(n) \leq h(n+1) \\ -1 & \text{if } S(n) > h(n+1). \end{cases}$$

We need to show that Equation (1) holds for $n+1$.

When $a_{n+1} = +1$ we have

$$\begin{aligned} & h(n) - 1 \leq S(n) \leq h(n+1) \\ \implies & \quad h(n) \leq S(n+1) \leq h(n+1) + 1 \\ \implies & \quad h(n) - h(n+1) \leq S(n+1) - h(n+1) \leq +1 \\ \implies & \quad |S(n+1) - h(n+1)| \leq +1 \end{aligned}$$

When $a_{n+1} = -1$ we have

$$
\begin{aligned}
& h(n+1) < S(n) \le h(n) + 1 \\
\implies \quad & h(n+1) - 1 \le S(n+1) \le h(n) \\
\implies \quad & -1 \le S(n+1) - h(n+1) \le h(n) - h(n+1) \\
\implies \quad & |S(n+1) - h(n+1)| \le +1
\end{aligned}
$$

This finishes the proof of Equation (1) for $n+1$. By induction, we have the infinite sequence a_n so that the partial sums $S(n)$ satisfies Equation (1) for all n. In particular, we have

$$
\begin{aligned}
\lim_{X\to\infty} \left| \frac{\log S(X)}{\log X} - \alpha \right| &= \lim_{X\to\infty} \left| \frac{\log S(X)}{\log X} - \frac{\log X^\alpha}{\log X} \right| \\
&= \lim_{X\to\infty} \left| \frac{\log S(X) - \log X^\alpha}{\log X} \right| = \lim_{N\to\infty} \left| \frac{\log S(N) - \log N^\alpha}{\log N} \right| \le \lim_{N\to\infty} \frac{1}{\log N} = 0.
\end{aligned}
$$

That is,

$$
\lim_{X\to\infty} \frac{\log S(X)}{\log X} = \alpha. \tag{2}
$$

The Dirichlet series we introduce is

$$
A(s) = \sum_{n=1}^{\infty} a_n n^{-s}.
$$

It is clear that for $A(s)$ we have $\sigma_0 = 1$. By Question 2 above, $\sigma_1 = \alpha$.