Analytic Number Theory

Solutions

Solution to Problem 1. We have

$$\beta = \begin{cases} 1/2 & \text{if } |t| \le e^2, \\ 1 - 1/\log|t| & \text{if } |t| \ge e^2. \end{cases}$$

If $|t| \le e^2$, we have $x^{-\beta} = x^{-1/2} = O(x^{-1})$ iff $x^{1/2} = O(1)$ on the interval $1 \le x \le |t|$. The function $x^{1/2} \le e$ on the interval $[1, |t|] \subseteq [1, e^2]$.

If $|t| \ge e^2$, we have $x^{-\beta} = x^{1/\log|t| - 1} = O(x^{-1})$ iff $x^{1/\log|t|} = O(1)$ iff $e^{\log x/\log|t|} = O(1)$, all on the interval $1 \le x \le |t|$. The function $\log x/\log|t| \le 1$ on the interval $[1, |t|]$.

For $\sigma \ge \alpha$,

$$\sum_{n=1}^{M} n^{-\sigma} \le \sum_{n=1}^{M} n^{-\alpha} = 1 + \int_1^{|t|} x^{-\alpha}\,dx = 1 + \frac{1}{1-\alpha} x^{1-\alpha}\Big|_1^{|t|}$$

$$= 1 - \frac{1}{1-\alpha} + \frac{1}{1-\alpha}|t|^{1-\alpha} = 1 + 100|t|^{1-\alpha} = O(|t|^{1-\alpha}).$$

For $\sigma \ge \beta$, we have

$$\sum_{n=1}^{M} n^{-\sigma} \le \sum_{n=1}^{M} n^{-\beta} \le \sum_{n=1}^{M} n^{-1} \le 1 + \int_1^{|t|} x^{-1}\,dx = O(\log|t|).$$

Similarly, for $\sigma \ge \alpha$,

$$\sum_{n=M}^{\infty} n^{-\sigma-1} \le \sum_{n=M}^{\infty} n^{-\alpha-1} \le M^{-\alpha-1} + \int_M^{\infty} x^{-\alpha-1}\,dx \le M^{-\alpha-1} + \frac{1}{-\alpha} x^{-\alpha}\Big|_M^{\infty}$$

$$\le M^{-\alpha-1} + \frac{1}{\alpha} M^{-\alpha} = O(M^{-\alpha}) = O(|t|^{-\alpha}).$$

If σ is a constant $\ge \beta$, for every t (since β depends on t), we must have $\sigma \ge 1$. Then, $\sum_{n=M}^{\infty} n^{-\sigma-1} \le \sum_{n=M}^{\infty} n^{-2} = O(1/|t|)$. However, we can assume that σ depends on t as well and proceed as above to obtain

$$\sum_{n=M}^{\infty} n^{-\sigma-1} \le \sum_{n=M}^{\infty} n^{-\beta(n)-1} \le M^{-\beta(M)-1} + \int_M^{\infty} x^{-\beta(x)-1}\,dx$$

$$= \frac{e}{M^2} + e \cdot \int_M^{\infty} x^{-2}\,dx \le \frac{e}{M^2} + \frac{e}{M} = O(M^{-1}) = O(|t|^{-1}).$$

By the proof of Theorem 4.3, for $1 \le \mathrm{Re}(s) \le 2$ we have

$$|\zeta(s)| \le \frac{1}{|s-1|} + \sum_{n \le |t|} n^{-\sigma} + |s| \sum_{n \ge |t|} n^{-\sigma-1}.$$

Then, for $\sigma = \mathrm{Re}\, s \ge \alpha$, using the above inequalities, we conclude that

$$|\zeta(s)| \le \frac{1}{|\sigma+it-1|} + O(|t|^{1-\alpha}) + |\sigma+it| O(|t|^{-\alpha}) = O(1) + O(|t|^{1-\alpha}) + O(|t|^{-\alpha+1}) = O(|t|^{-\alpha+1}).$$

If $\mathrm{Re}\, s \ge \beta(t)$, then we obtain

$$|\zeta(s)| \le \frac{1}{|\sigma+it-1|} + O(\log|t|) + |\sigma+it| O(|t|^{-1}) = O(1) + O(\log|t|) + O(1) = O(\log|t|).$$

Solution to Problem 2. Let Γ be the circle of radius $1/(4\log|t|)$ about $s = \sigma + it$. For $|t| \ge 3$ we have

$$\frac{1}{4\log|t|} \le \frac{1}{4}.$$

Let $w = x + iy \in \Gamma$. We have, $x \ge 3/4 - 1/4 = 1/2$, and $|y| \ge |t| - 1/4 \ge 3 - 1/4 \ge 2$.

Moreover, for $|t| \ge e^2$, we have

$$x \ge \mathrm{Re}\, s - \frac{1}{4\log|t|} \ge 1 - \frac{3}{4\log|t|}.$$

Similarly,

$$|y - t| \le \frac{1}{4\log|t|},$$

which implies

$$\log|y| \le \log(|t| + \frac{1}{4\log|t|}).$$

Now there is $t_0 > 3$ such that for all $|t| \ge t_0$ we have

$$1 - \frac{3}{4\log|t|} \ge 1 - \frac{1}{\log(t + \frac{1}{4\log|t|})}.$$

Combining the above bounds, we conclude that for $|t| \ge t_0$ we have $x \ge 1 - \frac{1}{\log y}$. (In other words, Γ lies to the right-hand side of the curve $(\beta(t), t)$.)

For $2 \le |t| \le t_0$ and $0 \le \mathrm{Re}\, s \le 2$, the function $\zeta'(s)$ is holomorphic. In particular, $|\zeta'(s)|$ is bounded from above on this compact region. That is $|\zeta'(s)| = O(1)$. It remains to prove the bound in the Question for $|t| \ge t_0$.

By the inequality in Question 1 (for $\sigma \geq \beta$ applied at the point $x+iy$) we have $|\zeta(w)| \leq O(\log|y|)$. However, since $|y-t| \leq 1/(4\log|t|)$, we have $\log|y| = O(\log|t|)$.

By the Cauchy integral formula, we have

$$|\zeta'(s)| \leq \Big|\frac{1}{2\pi i}\int_\Gamma \frac{\zeta(w)}{(w-s)^2}\,dw\Big| \leq \frac{1}{2\pi}\int_\Gamma \frac{|\zeta(w)|}{|w-s|^2}|dw|$$
$$\leq \frac{1}{2\pi}O(\log|t|)\cdot 16\log^2|t|\cdot 2\pi\frac{1}{4\log t} = O(\log^2|t|).$$

Solution to Problem 3. Fix $x \geq 2$ and define

$$A = \{(p,e) \mid p \in \mathbb{N}, e \in \mathbb{N}, p^e \leq x\}.$$

For each $n \in \mathbb{N}$ we let $A_n = \{(p,n) \mid p \in \mathbb{N}, p^n \leq x\}$. Then, we have $A = \cup_{n\geq 1} A_n$. We note that $A_n = \emptyset$, for $n > \log x/\log 2$. Moreover, for every $n \geq 2$, $\#A_n \leq \#A_2 \leq x^{1/2}$. These imply that

$$\psi(x) - \theta(x) = \sum_A \log p - \sum_{A_1}\log p = \sum_{A_2}\log p + \sum_{A_3}\log p + \sum_{A_4}\log p + \cdots$$
$$\leq \frac{\log x}{\log 2}\sum_{A_2}\log p \leq \frac{\log x}{\log 2}\sum_{p^2\leq x}\log p \leq \frac{\log x}{\log 2}\sum_{m\leq x^{1/2}}\log m$$
$$\leq \log x \cdot (x^{1/2}\log(x^{1/2}) + O(x^{1/2})) = O(x^{1/2}\log^2 x).$$

Above we have used that $\sum_{n\leq N}\log n = N\log N + O(N)$.

Using the partial summation formula with the functions $f(n)$ and $F(n)$ we obtain

$$\sum_{p\leq x} 1 = \sum_{n\leq x} f(n)F(n) = S(x)F(x) - \int_1^x S(x)F'(x)dx,$$

where

$$S(x) = \sum_{n\leq x} f(n) = \sum_{p\leq x}\log p = \theta(x).$$

Thus,

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \theta(x)\frac{1}{x\log x}\,dx,$$

since $S(x) = 0$ for $x < 2$.

If $\psi(x) = x + O(E(x))$ then we have

$$\theta(x) = \psi(x) - O(x^{1/2}\log^2 x) = x + O(E(x)) - O(x^{1/2}\log^2 x) = x + O(E(x)).$$

Therefore,

$$\begin{aligned}
\pi(x) &= \frac{x + O(E(x))}{\log x} + \int_2^x \frac{t + O(E(t))}{t\log^2 t}\,dt \\
&= \frac{x}{\log x} + \frac{O(E(t))}{\log x} + \int_2^x \frac{1}{\log^2 t}\,dt + \int_2^x \frac{O(E(t))}{t\log^2 t}\,dt \\
&= \frac{x}{\log x} + \frac{O(E(x))}{\log x} + \Big(\mathrm{Li}(x) - C - \frac{x}{\log x}\Big) + \int_2^x \frac{O(E(t))}{t\log^2 t}\,dt \\
&\le \mathrm{Li}(x) + O(E(x)) + \int_2^x t^{-1/2}\,dt \\
&= \mathrm{Li}(x) + O(E(x)) + O(x^{1/2}) = \mathrm{Li}(x) + O(E(x)).
\end{aligned}$$

Solution to Problem 4. Let $\beta = 1 + \delta$ and choose $\delta = \frac{F(2x)^{1/2}}{2x}$. Then,

$$\psi_1(\beta x) - \psi_1(x) = \int_x^{\beta x} \psi(t)\,dt \ge \psi(x)\frac{F(2x)^{1/2}}{2}.$$

Thus,

$$\begin{aligned}
\psi(x)\frac{F(2x)^{1/2}}{2} &\le \psi_1(\beta x) - \psi_1(x) = \psi_1(x + \frac{F(2x)^{1/2}}{2}) - \psi_1(x) \\
&= \frac{1}{2}(x + \frac{F(2x)^{1/2}}{2})^2 + O(F(x + \frac{F(2x)^{1/2}}{2})) - (\frac{1}{2}x^2 + O(F(x))) \\
&= \frac{1}{2}x^2 + \frac{1}{2}xF(2x)^{1/2} + \frac{1}{4}F(2x) + O(F(x + \frac{1}{2}F(2x)^{1/2})) - \frac{1}{2}x^2 - O(F(x)) \\
&= \frac{1}{2}xF(2x)^{1/2} + \frac{1}{4}F(2x) + O(F(x + \frac{1}{2}F(2x)^{1/2})) - O(F(x))
\end{aligned}$$

Since $F(x)$ in increasing and non-negative, we have $F(x) \le F(x + \frac{1}{2}F(2x)^{1/2})$. Thus, $O(F(x + \frac{1}{2}F(2x)^{1/2})) - O(F(x)) = O(F(x + \frac{1}{2}F(2x)^{1/2}))$. Also, since $F(x) \le x^2$ we have $\frac{1}{2}F(2x)^{1/2} \le x$. Thus, $O(F(x + \frac{1}{2}F(2x)^{1/2})) = O(F(2x))$. Therefore, by the above inequalities, we have

$$\psi(x)\frac{F(2x)^{1/2}}{2} \le \frac{1}{2}xF(2x)^{1/2} + \frac{1}{4}F(2x) + O(F(2x)),$$

which diving through by $\frac{F(2x)^{1/2}}{2}$ implies that

$$\psi(x) \le x + O(F(2x)^{1/2}).$$

The lower bound on $\psi(x)$ is obtained in a similar fashion.

Solution to Problem 5. In Problems Sheet 3 we saw that formally,

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} d^2(n) n^{-s}.$$

By Theorem 4.2, the left-hand side of the above equation is defined for $\operatorname{Re} s > 0$, except for the singularities at $s = 1$ for $\zeta(s)$ and $s = 1/2$ for $\zeta(2s)$. (Indeed, By Corollary 6.7, $\zeta(s)$ extends over $\mathbb{C}$ but we don't need that here.) Below we show that the right-hand side of the above equation is defined for $\operatorname{Re} s > 1$.

By Theorem 2.9, for every $\epsilon > 0$ there is a constant c_ϵ such that $d(n) \le c_\epsilon n^\epsilon$. This implies that for every s with $\operatorname{Re} s > 1 + 2\epsilon$ the series

$$\sum_{n=1}^{\infty} d^2(n)|n^{-s}| \le c_\epsilon \sum_{n=1}^{\infty} n^{2\epsilon - \sigma}$$

is finite. Since ϵ was arbitrary, the right-hand side of the equation is absolutely convergent for $\operatorname{Re} s > 1$.

As $c > 1$, by the above equation on the vertical line $c + i\mathbb{R}$, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta^4(s)}{\zeta(2s)} \frac{x^{s+1}}{s(s+1)}\, ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\sum_{n=1}^{\infty} d^2(n) n^{-s}\Big) \frac{x^{s+1}}{s(s+1)}\, ds$$

Now, to switch the places of the sum and the integral, we need to verify that

$$\sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} |d^2(n) n^{-s} \frac{x^{s+1}}{s(s+1)}|\, |ds| \le \sum_{n=1}^{\infty} d^2(n) n^{-c} \int_{c-i\infty}^{c+i\infty} |\frac{x^{s+1}}{s(s+1)}|\, |ds| < \infty.$$

However, the integral $\int_{c-i\infty}^{c+i\infty} |\frac{x^{s+1}}{s(s+1)}|\, |ds|$ is finite and independent of n. By the above argument, the series is convergent for $c > 1$.

Now, using the values of the integrals in Lemma 5.5,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\sum_{n=1}^{\infty} d^2(n) n^{-s}\Big) \frac{x^{s+1}}{s(s+1)}\, ds &= \sum_{n=1}^{\infty} d^2(n) x \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}\, ds \\ &= \sum_{n \le x} d^2(n) x (1 - n/x) \\ &= \sum_{n \le x} d^2(n)(x - n).\end{aligned}$$

The function $x^{s+1}/(s(s+1))$ has poles of order 1 at $s=0$ and $s=-1$. The function $\zeta(2s)$ has no zero for $\operatorname{Re} s \geq 1/2$. The function $\zeta^4(s)$ has a pole of order 4 at $s=1$. Let $R \in \mathbb{R}$ denote the residue of $\zeta^4(s)$ at $s=1$. By the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^{s+1}}{s(s+1)}\,ds &= \frac{1}{2\pi i}\int_{7/8-i\infty}^{7/8+i\infty}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^{s+1}}{s(s+1)}\,ds + \operatorname{Res}(\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^{s+1}}{s(s+1)}; s=1)\\ &= \frac{1}{2\pi i}\int_{7/8-i\infty}^{7/8+i\infty}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^{s+1}}{s(s+1)}\,ds + \frac{R}{\pi^2/6}\frac{x^2}{2}\end{aligned}$$

In the last equality of the above equation we have used $\zeta(2)=\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

For $\operatorname{Re} s > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\mu(n)n^{-s},$$

So on the line $7/8 + i\mathbb{R}$, we have

$$\Big|\frac{1}{\zeta(2s)}\Big| \leq \sum_{n=1}^{\infty}|\mu(n)n^{-2s}| \leq \sum_{n=1}^{\infty} n^{-7/4} < \infty.$$

Let C_1 be an upper bound for $|1/\zeta(2s)|$ on the line $7/8+i\mathbb{R}$.

We have

$$\begin{aligned}\Big|\frac{1}{2\pi i}\int_{7/8-i\infty}^{7/8+i\infty}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^{s+1}}{s(s+1)}\,ds\Big| &\leq \frac{C_1}{2\pi}\int_{7/8-i\infty}^{7/8+i\infty}|\zeta^4(s)|\Big|\frac{x^{s+1}}{s(s+1)}\Big|\,|ds|\\ &\leq \frac{C_1 x^{15/8}}{2\pi}\int_{7/8-i\infty}^{7/8+i\infty}|\zeta^4(s)|\Big|\frac{1}{s(s+1)}\Big|\,|ds|\\ &\leq \frac{C_1 x^{15/8}}{2\pi}\Big(\int_{7/8-2i}^{7/8+2i} + 2\int_{7/8+2i}^{7/8+i\infty}\Big)|\zeta^4(s)|\Big|\frac{1}{s(s+1)}\Big|\,|ds|\\ &\leq C_2x^{15/8} + C_3x^{15/8}\int_{7/8+2i}^{7/8+i\infty}|\zeta^4(s)|\frac{1}{t^2}\,dt\\ &\leq C_2x^{15/8} + C_3x^{15/8}\int_{7/8+2i}^{7/8+i\infty}|\zeta^4(s)|\frac{1}{t^2}\,dt\end{aligned}$$

For some constants C_2 and C_3 independent of x.

For $\alpha = 7/8 \in [1/100, 99/100]$, the inequality in Question 1 gives us $|\zeta(7/8+it)| \leq |t|^{1/8}$. Then, there is a constant C_4 such that

$$\int_{7/8+2i}^{7/8+i\infty}|\zeta^4(s)|\frac{1}{t^2}\,dt \leq \int_2^{\infty}(t^{1/8})^4\frac{1}{t^2} \leq \int_2^{\infty} t^{-3/2} \leq C_4.$$

Combining the above bounds we have

$$\sum_{n\leq x} d^2(n)(x-n) = \frac{3R}{\pi}x^2 + O(x^{15/8}).$$

This is stronger than the estimate in the question. One can find the exact value of R by identifying the coefficient of $1/(s-1)$ in the expansion of $\zeta^4(s)$, but we are not concerned with this value here.

Solution to Problem 6. For $x \geq 1$, let us define $A(x) = \sum_{n\leq x} d^2(n)$, and $H(x) = \int_1^x A(t)dt$. By the estimate in Question 5,

$$H(x) = \int_1^x A(t)\,dt = \sum_{n\leq x} d^2(n)(x-n) = x^2 P(\log x) + O(x^{15/8}),$$

where P is a cubic polynomial, say, $P(x) = a_0 + a_1 \log x + a_2 x^2 + a_3 x^3$.

Then,

$$H(x) = a_3 x^2 \log^3 x + O(x^2 \log^2 x).$$

This implies that

$$H(x) = a_3 x^2 \log^3 x + o(x^2 \log^3 x)$$

Since $d(n) \geq 0$, for $n \geq 1$, the function $A(x)$ is increasing. Given $\alpha < 1 < \beta$, we can apply the argument in the proof of Lemma 5.3 to conclude that

$$\begin{aligned} \frac{A(x)}{x\log^3 x} &\leq \frac{H(\beta x) - H(x)}{(\beta-1)x^2\log^3 x} \\ &= \frac{a_3\beta^2 x^2 \log^3 \beta + a_3\beta^2 x^2 \log^3 x + o(x^2\log^2 x) - a_3 x^2 \log^3 x - o(x^2 \log^2 x)}{(\beta-1)x^2\log^3 x} \end{aligned}$$

Hence,

$$\limsup_{x\to\infty} \frac{A(x)}{x\log^3 x} \leq \frac{a_3(\beta^2-1)}{(\beta-1)} = a_3(\beta+1).$$

Since $\beta > 1$ was arbitrary we must have

$$\limsup_{x\to\infty} \frac{A(x)}{x\log^3 x} \leq 2a_3.$$

In a similar fashion one can show that

$$\liminf_{x\to\infty} \frac{A(x)}{x\log^3 x} \geq \frac{a_3(1-\alpha^2)}{(1-\alpha)} = a_3(1+\alpha)$$

which produces,

$$\liminf_{x\to\infty} \frac{A(x)}{x\log^3 x} \geq 2a_3.$$

Combining the two bounds we conclude that the following limit exists and

$$\lim_{x\to\infty} \frac{A(x)}{x\log^3 x} = 2a_3.$$

That is,

$$A(x) \sim 2a_3 x \log^3 x.$$