Analytic Number Theory

Solutions

Solution to Problem 1. Recall that for $\operatorname{Re} s > \operatorname{Re} s_0$ we have

$$F(s) = (s - s_0) \int_1^\infty S(x) x^{s_0 - s - 1}\, dx$$

where $S(x) = \sum_{n \le x} f(n) n^{-s_0}$. We proved in Problem 7 that the integral is finite. Now, fix $\delta > 0$ and assume that $\operatorname{Re} s \geq \operatorname{Re} s_0 + \delta$. We have

$$\Big| \int_1^\infty S(x) x^{s_0-s-1}\, dx - \int_1^X S(x) x^{s_0-s-1}\, dx \Big| \leq \int_X^\infty \big| S(x) x^{s_0-s-1} \big|\, dx \leq M \int_X^\infty x^{-\delta-1}\, dx \leq \frac{1}{\delta} \frac{1}{X^\delta}$$

where $M = \sup\{|S(x)| \mid x \geq 0\}$ is finite, and the upper bound $\frac{1}{\delta X^\delta}$ tends to 0 as X tends to ∞.

Let us define

$$F_N(s) = (s - s_0) \int_1^N S(x) x^{s_0-s-1}\, dx$$

Each map F_n is holomorphic on the region $\operatorname{Re} s > \operatorname{Re} s_0$. Moreover, for every $\delta > 0$, by the above equation, the sequence of maps $F_n(s)$ is uniformly convergent on the region $\operatorname{Re} s \geq \operatorname{Re} s_0 + \delta$. By Lemma 3.2, this implies that F_s is holomorphic on the region $\operatorname{Re} s \geq \operatorname{Re} s_0 + \delta$. As $\delta > 0$ was arbitrary, we conclude that $F(s)$ is holomorphic on the region $\operatorname{Re} s > \operatorname{Re} s_0$.

Solution to Problem 2. a) First we verify whether f is multiplicative. This can be easily done by considering the three cases of the pairs (m, n) are (odd, odd), (odd, even), (even, odd).

We note that $\sum_{n=1}^\infty |f(n)| n^{-s}$ is convergent if and only if $\operatorname{Re} s > 1$. This implies that AAC of this Dirichlet series σ_0 is equal to $+1$. Then, by Theorem 3.4, for every s with $\operatorname{Re} s > \sigma_0$, we have

$$\sum_{n=1}^\infty f(n) n^{-s} = \prod_{p \text{ prime}} \{ \sum_{e=0}^\infty f(p^e) p^{-es} \} = \big(1 - p^{-s} - p^{-2s} - p^{-3s} \dots \big) \prod_{2<p \text{ prime}} \{ \sum_{e=0}^\infty f(p^e) p^{-es} \}$$
$$= \big(2 - \frac{1}{1-2^{-s}}\big) \prod_{2<p \text{ prime}} \big(\frac{1}{1-p^{-s}}\big)$$

b) We can verify whether f is multiplicative by considering the following four cases. Let $(m, n) = 1$ for some $m, n \in \mathbb{N}$.

1) If at least one of n and m is even, then $f(mn) = f(m)f(n) = 0$.

2) If $m \equiv 1 \pmod 4$, and $n \equiv 1 \pmod 4$, then $f(mn) = (-1)^{(mn-1)/2} = 1$ and $f(m)f(n) = (-1)^{(m-1)/2} \cdot (-1)^{(n-1)/2} = 1 \cdot 1 = 1$.

3) If $m \equiv 3 \pmod 4$, and $n \equiv 1 \pmod 4$, then $f(mn) = (-1)^{(mn-1)/2} = -1$ and $f(m)f(n) = (-1)^{(m-1)/2} \cdot (-1)^{(n-1)/2} = (-1) \cdot 1 = -1$.

4) If $m \equiv 3 \pmod 4$, and $n \equiv 3 \pmod 4$, then $f(mn) = (-1)^{(mn-1)/2} = 1$ and $f(m)f(n) = (-1)^{(m-1)/2} \cdot (-1)^{(n-1)/2} = (-1) \cdot (-1) = 1$.

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p \text{ prime}} \{\sum_{e=0}^{\infty} f(p^e)p^{-es}\}$$

For $p = 2$,

$$\sum_{e=0}^{\infty} f(p^e)p^{-es} = f(1) = 1,$$

for primes p of the form $4k+3$, we have

$$\sum_{e=0}^{\infty} f(p^e)p^{-es} = 1 - p^{-s} + p^{-2s} - p^{-3s} + p^{-4s} - \cdots = \frac{1}{1+p^{-s}}$$

and for primes p of the form $4k+1$ we have

$$\sum_{e=0}^{\infty} f(p^e)p^{-es} = 1 + p^{-s} + p^{-2s} + p^{-3s} + p^{-4s} + \cdots = \frac{1}{1-p^{-s}}.$$

Thus,

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \Big(\prod_{p\equiv 3} \frac{1}{1+p^{-s}}\Big)\Big(\prod_{p\equiv 1} (\frac{1}{1-p^{-s}})\Big).$$

Solution to Problem 3.

$$\zeta^2(s) = \sum_{n=1}^{\infty} (u*u)(n)n^{-s} = \sum_{n=1}^{\infty} \big(\sum_{ab=n} (u(a)u(b))\big)n^{-s} = \sum_{n=1}^{\infty} d(n)n^{-s}.$$

However the above relation may be also obtained from the Euler product formula as in

$$\zeta^2(s) = \prod_p \frac{1}{(1-p^{-s})^2} = \prod_p \frac{1}{1-2p^{-s}+p^{-2s}}$$

and on the other hand, since $d(n)$ is a multiplicative function,

$$\sum_{n=1}^{\infty} d(n)n^{-s} = \prod_p \Big(\sum_{e=0}^{\infty} d(p^e)p^{-es}\Big) = \prod_p \Big(\sum_{e=0}^{\infty} (e+1)p^{-es}\Big)$$

So for the relation to hold it is enough to prove that for every prime p

$$\frac{1}{1-2p^{-s}+p^{-2s}} = \sum_{e=0}^{\infty} (e+1)p^{-es}.$$

This can be verified by

$$\begin{aligned}
&(1-2p^{-s}+p^{-2s}) \cdot \sum_{e=0}^{\infty} (e+1)p^{-es} \\
&= \sum_{e=0}^{\infty} (e+1)p^{-es} - \sum_{e=0}^{\infty} 2(e+1)p^{-(e+1)s} + \sum_{e=0}^{\infty} (e+1)p^{-(e+2)s} \\
&= \big(1 + 2p^{-s} + \sum_{e=2}^{\infty} (e+1)p^{-es}\big) - (2p^{-s} + \sum_{e=2}^{\infty} 2(e)p^{-es}) + \sum_{e=2}^{\infty} (e-1)p^{-es} \\
&= 1 + \sum_{e=2}^{\infty} (e+1-2e+e-1)p^{-es} = 1.
\end{aligned}$$

One can use this approach for the other relations.

$$\frac{\zeta^3(s)}{\zeta(2s)} = \prod_p (\frac{1}{(1-p^{-s})^3}) \prod_p (1-p^{-2s}) = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^3} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2} = \prod_p \frac{1+p^{-s}}{1-2p^{-s}+p^{-2s}}$$

On the other hand, since $d(n^2)$ is a multiplicative functions, we must have

$$\sum_{n=1}^{\infty} d(n^2) n^{-s} = \prod_p \Big(\sum_{e=0}^{\infty} d(p^{2e}) p^{-es} \Big) = \prod_p \Big(\sum_{e=0}^{\infty} (2e+1) p^{-es} \Big).$$

Thus, it is enough to show that

$$\frac{1+p^{-s}}{1-2p^{-s}+p^{-2s}} = \sum_{e=0}^{\infty} (2e+1) p^{-es}.$$

This can be verified as follows:

$$\begin{aligned}
(1-2p^{-s}+p^{-2s}) \cdot & \sum_{e=0}^{\infty} (2e+1) p^{-es} \\
&= \sum_{e=0}^{\infty} (2e+1) p^{-es} - \sum_{e=0}^{\infty} 2(2e+1) p^{-(e+1)s} + \sum_{e=0}^{\infty} (2e+1) p^{-(e+2)s} \\
&= \big(1 + 3p^{-s} + \sum_{e=2}^{\infty} (2e+1) p^{-es}\big) - (2p^{-s} + \sum_{e=2}^{\infty} 2(2e-1) p^{-es}) + \sum_{e=2}^{\infty} (e-3) p^{-es} \\
&= 1 + p^{-s} + \sum_{e=2}^{\infty} (2e+1-4e+2+2e-3) p^{-es} = 1 + p^{-s}.
\end{aligned}$$

Similarly,

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p (\frac{1}{(1-p^{-s})^4}) \prod_p (1-p^{-2s}) = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^4} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^3}.$$

On the other hand, since $d^2(n)$ is a multiplicative functions, we must have

$$\sum_{n=1}^{\infty} d(n^2) n^{-s} = \prod_p \Big(\sum_{e=0}^{\infty} d^2(p^{e}) p^{-es} \Big) = \prod_p \Big(\sum_{e=0}^{\infty} (e+1)^2 p^{-es} \Big).$$

Thus, it is enough to show that

$$\frac{1+p^{-s}}{(1-p^{-s})^3} = \sum_{e=0}^{\infty} (e+1)^2 p^{-es}.$$

This can be verified as in the above case.

Solution to Problem 4.

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma(n)n^{-s} &= \sum_{n=1}^{\infty}\Big(\sum_{ab=n}u_1(a)u(b)\Big)n^{-s}\\ &= \sum_{n=1}^{\infty}(u_1 * u)(n)n^{-s}\\ &= \Big(\sum_{n=1}^{\infty}u_1(n)n^{-s}\Big)\Big(\sum_{n=1}^{\infty}u(n)n^{-s}\Big)\\ &= \Big(\sum_{n=1}^{\infty}n\cdot n^{-s}\Big)\zeta(s)\\ &= \zeta(s-1)\zeta(s).\end{aligned}$$

Recall that in Theorem 2.16 we proved that $\phi * u = u_1$. Thus,

$$\Big(\sum_{n=1}^{\infty}\phi(n)n^{-s}\Big)\Big(\sum_{n=1}^{\infty}u(n)n^{-s}\Big) = \sum_{n=1}^{\infty}n\cdot n^{-s}.$$

This implies that,

$$\sum_{n=1}^{\infty}\phi(n)n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

For the last relation we use a different approach. On one hand,

$$\sum_{n=1}^{\infty}|\mu(n)|n^{-s} = \prod_p(\sum_{e=0}^{\infty}|\mu(p^e)|p^{-es}) = \prod_p(1+p^{-s}).$$

We also know that

$$\frac{1}{\zeta(s)} = \prod_p(1-p^{-s}).$$

Multiplying the two formulas together we see that

$$\frac{1}{\zeta(s)}\cdot(\sum_{n=1}^{\infty}|\mu(n)|n^{-s}) = \prod_p(1-p^{-2s}) = 1/\zeta(2s),$$

which implies that

$$\sum_{n=1}^{\infty}|\mu(n)|n^{-s} = \frac{\zeta(s)}{\zeta(2s)}.$$

Solution to Problem 5. We use the relation

$$\sum_{n=1}^{\infty}|\mu(n)|n^{-s} = \frac{\zeta(s)}{\zeta(2s)}$$

prove in the previous part. By Theorem 4.1, we have

$$\frac{1}{|\zeta(s)|} = \Big|\sum_{n=1}^{\infty}\mu(n)n^{-s}\Big| \le \sum_{n=1}^{\infty}|\mu(n)|n^{-\operatorname{Re}s} = \frac{\zeta(\sigma)}{\zeta(2\sigma)}$$

Solution to Problem 6. By Theorem 2.10, $\sum_{n\le X}\log n = X\log X + O(X)$, and by Lemma 3.6 for every $n\in\mathbb{N}$ we have $(\Lambda * u)(n) = \log n$. Thus,

$$\sum_{m\le X}\Lambda(m)[\frac{X}{m}] = \sum_{n\le X}\sum_{a|n}\Lambda(a) = \sum_{n\le X}\sum_{ab=n}\Lambda(a)u(b) = \sum_{n\le X}\log n = X\log X + O(X).$$

To obtain the first equality we have counted how many times $\Lambda(a)$ appears in the second double sum. That is how many $n\le X$ are there with $a|n$. The answer is $[X/a]$.

By the above equation

$$\sum_{m\le X}\Lambda(m)\frac{X}{m} \ge \sum_{m\le X}\Lambda(m)[\frac{X}{m}] = X\log X + O(X).$$

This implies that

$$\sum_{m\le X}\frac{\Lambda(m)}{m} \ge \log X + O(1).$$

Fix an arbitrary $\delta\in(0,1)$. Then, for every $\theta\ge\delta^{-1}$ we have $[\theta]\ge(\theta-1)\ge(1-\delta)\theta$. This implies that

$$\sum_{m\le\delta X}\Lambda(m)\frac{X}{m} \le (1-\delta)^{-1}\sum_{m\le\delta X}\Lambda(m)[\frac{X}{m}] = (1-\delta)^{-1}(X\log X + O(X)).$$

Dividing the last inequality by X we obtain

$$\sum_{m\le\delta X}\frac{\Lambda(m)}{m} \le (1-\delta)^{-1}(\log X + O(1))$$

for every $\delta\in(0,1)$. Combining the two inequalities we have

$$\sum_{m\le Y}\frac{\Lambda(m)}{m} = \log Y + O(1).$$

In particular,

$$\sum_{m\le Y}\frac{\Lambda(m)}{m} \sim \log Y.$$

Solution to Problem 7. For every s with $\operatorname{Re} s>1$, by the Euler product formula we have

$$\log\zeta(s) = -\sum_p \log(1-p^{-s}).$$

Differentiating both sides with respect to s results in

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p\frac{\log p\cdot p^{-s}}{1-p^{-s}} = -\sum_p\log p\sum_{m=0}^{\infty}\frac{1}{p^{(m+1)s}}.$$

Hence,

$$3\frac{\zeta'(\sigma)}{\zeta(\sigma)} + 4\operatorname{Re}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \operatorname{Re}\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} = -\sum_p\frac{\log p}{p^{m\sigma}}\sum_{m=1}^{\infty}\big(3+4\cos(mt\log p)+\cos(2mt\log p)\big)$$

As in the proof of Lemma 3.6, for every $\theta\in\mathbb{R}$, $3+4\cos\theta+\cos(2\theta)\ge 0$. This finishes the proof of the inequality.

Solution to Problem 8. It is enough to have $A_1 > A_0$, and all $A_i \geq 0$. With $N = 2$ we need to have

$$A_0 + A_1 \cos\theta + A_2 \cos(2\theta) \geq 0.$$

replace $\cos\theta = x$ and find the minimum of the function on the interval $[-1, 1]$.