Analytic Number Theory

Solutions

Solution to Problem 1. We have

$$\frac{d(n)}{n^{1/4}} = \prod_i \frac{1+e_i}{p^{e_i/4}},$$

where $n = \prod p_i^{e_i}$.

By the proof of Theorem 2.9

$$\frac{1+e_i}{p^{e_i/4}} \leq 1 \text{ for } p_i \geq 16$$

and

$$\frac{1+e_i}{p^{e_i/4}} \leq \frac{4}{\log 2} \text{ for } p_i \leq 16$$

Thus, for all $n \in \mathbb{N}$,

$$d(n) \leq \frac{4^6}{(\log 2)^6} \cdot n^{1/4}.$$

We need to find n_0 such that for all $n \geq n_0$ we have

$$\frac{4^6}{(\log 2)^6} < n^{1/4}$$

which is guaranteed by $n > 1.9 \times 10^{18} \geq (4/\log 2)^{24}$.

For primes $p \leq 16$ we may have a better bound on $(1+e_i)/p_i^{e_i/4}$. One needs to find the maximum values of the functions $g_p(x) = \frac{1+x}{p^{x/4}}$ for $p = 2, 3, 5, 7, 11, 13$. By taking derivative we see that $g_p'(\frac{4}{\log p} - 1) = 0$, while for $x > \frac{4}{\log p} - 1$, $g_p(x)$ is decreasing. By some elementary calculations we can obtain a better bound.

Solution to Problem 2. We use the partial summation formula with $f(n) = d(n)$ and $F(x) = 1/x$. By Theorem 2.10 we have $S(X) = \sum_{1 \leq n \leq X} d(n) = X \log X + O(X)$. Hence,

$$\begin{aligned}
\sum_{1 \leq n \leq X} \frac{d(n)}{n} &= \sum_{1 \leq n \leq X} f(n)F(n) \\
&= \big(X \log X + O(X)\big)\frac{1}{X} - \int_1^X \big(x \log x + O(x)\big)\frac{-1}{x^2}\, dx \\
&= \log X + O(1) + \int_1^X \frac{1}{x} \log x \, dx + O\Big(\int_1^X \frac{1}{x}\, dx\Big) \\
&= \log X + O(1) + \frac{1}{2}\log^2 X + O(\log X) \\
&= \frac{1}{2}\log^2 X + O(\log X) + O(1).
\end{aligned}$$

In the above equation we have used the integration by parts with $f(t) = \log t$ and $g(t) = \log t$. This implies the desired asymptotic relation.

Solution to Problem 3. Using the partial summation with $f(n) = 1$, $F(x) = 1/x$ we have

$$s(X) = \sum_{1\le n\le X} 1 = [X],$$

and hence

$$\begin{aligned}
\sum_{1\le n\le X} \frac{1}{n} &= \sum_{1\le n\le X} f(n)F(n) \\
&= [X]\frac{1}{X} - \int_1^X [x]\frac{-1}{x^2}\,dx \\
&= \frac{[X]-X+X}{X} + \int_1^X \frac{[x]-x}{x^2}\,dx + \int_1^X \frac{1}{x}\,dx \\
&= 1 + O(\frac{1}{X}) + \int_1^X \frac{[x]-x}{x^2}\,dx + \log X \\
&= \gamma + O(\frac{1}{X}) + \log X.
\end{aligned}$$

Solution to Problem 4. We use Lemma 2.11 with the increasing function $f(x) = x$ to obtain

$$\int_1^{[x]} x\,dx \le \sum_{n\le x} n \le [x] + \int_1^{[x]} x\,dx,$$

which reduces to

$$\frac{[x]^2}{2} \le \sum_{n\le x} n \le [x] + \frac{[x]^2}{2}$$

Hence,

$$\begin{aligned}
\sum_{n\le x} n - \frac{1}{2}x^2 &\le [x] + \frac{[x]^2 - x^2}{2} + \frac{x^2}{2} - \frac{1}{2}x^2 \\
&= [x] + \frac{([x]-x)([x]+x)}{2} \\
&\le O(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{n\le x} n - \frac{1}{2}x^2 &\ge \frac{[x]^2 - x^2}{2} + \frac{x^2}{2} - \frac{1}{2}x^2 \\
&= \frac{([x]-x)([x]+x)}{2} \\
&\ge -x.
\end{aligned}$$

The above two inequalities imply that $\sum_{n\le x} n = x^2/2 + O(x)$.

Moving to the next stage, we have

$$\begin{aligned}
\sum_{n\le x}\sigma(n) &= \sum_{n\le x}\sum_{v|n} v \\
&= \sum_{u\le x}\sum_{v\le x/u} v \\
&= \sum_{u\le x}\Big(\frac{(x/u)^2}{2}+O(x/u)\Big) \\
&= \frac{1}{2}x^2\sum_{u\le x}u^{-2}+x(\sum_{u\le x}1/u) \\
&= \frac{1}{2}x^2\sum_{u\le x}u^{-2}+O(x(1+\log x)).
\end{aligned}$$

$$\sum_{u>x}\frac{1}{u^2}\le\sum_{u>x}\frac{1}{u^2-u}=\sum_{u>x}\Big(\frac{1}{u-1}-\frac{1}{u}\Big)\le\frac{1}{[x]}\le\frac{2}{x}.$$

Also, recall that $\sum_{n=1}^{\infty}1/n^2=\pi^2/6$. Combining these with the above equation we obtain

$$\begin{aligned}
\sum_{n\le x}\sigma(n)-\frac{\pi^2}{12}x^2 &= \frac{1}{2}x^2\sum_{u\le x}u^{-2}+O(x(1+\log x))-\frac{\pi^2}{12}x^2 \\
&= \frac{1}{2}x^2(\frac{\pi^2}{6}-\sum_{u>x}\frac{1}{u^2})+O(x(1+\log x))-\frac{\pi^2}{12}x^2 \\
&= \frac{1}{2}x^2(-\sum_{u>x}\frac{1}{u^2})+O(x(1+\log x)) \\
&= O(x)+O(x(1+\log x))=O(x(1+\log x)).
\end{aligned}$$

Solution to Problem 5.

$$\begin{aligned}
\sum_{n\le X}d(n) &= \sum_{n\le X}\sum_{v|n}1 \\
&= \sum_{u,v\ge 1,uv\le X}1 \\
&= \sum_{u\le\sqrt{X}}\sum_{v\le X/u}1+\sum_{v\le\sqrt{X}}\sum_{u\le X/v}1-\sum_{u\le\sqrt{X}}\sum_{v\le\sqrt{X}}1
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{u\le\sqrt{X}}\sum_{v\le X/u}1 &= \sum_{u\le\sqrt{X}}[\frac{X}{u}]\\
&= X\sum_{u\le\sqrt{X}}\frac{1}{u}+O(\sqrt{X})\\
&= X\big(\log\sqrt{X}+\gamma+O(1/\sqrt{X})\big)+O(\sqrt{X})\\
&= \frac{1}{2}X\log X+X\gamma+O(\sqrt{X})
\end{aligned}$$

The second sum is equal to the above one. For the third sum we have

$$\begin{aligned}
\sum_{u\le\sqrt{X}}\sum_{v\le\sqrt{X}}1 &= \sum_{u\le\sqrt{X}}[\sqrt{X}]\\
&= \sqrt{X}\sqrt{X}+O(\sqrt{X})=X+O(\sqrt{X}).
\end{aligned}$$

Combining the above equations we obtain the desired asymptotic formula.

Solution to Problem 6. Let

$$S_0=\{\mathrm{Re}(s): s\in\mathbb{C},\sum_{n=1}^{\infty}|f(n)n^{-s}| \text{ is convergent}\},$$

and

$$S_1=\{\mathrm{Re}(s): s\in\mathbb{C},\sum_{n=1}^{\infty}f(n)n^{-s} \text{ is convergent}\}.$$

By the definitions $\inf S_0=\sigma_0$ and $\inf S_1=\sigma_1$.

Let s be a complex number such that $\sum_{n=1}^{\infty}f(n)n^{-s}$ is convergent. Then, by the convergence criteria the terms of the series must tend to zero. In particular, there is n_0 such that for all $n\ge n_0$ we have $|f(n)n^{-s}|\le 1$.

Assume s' be a complex number with $\mathrm{Re}(s')>\mathrm{Re}(s)+1$. Define $\eta=s'-s$ so that $\mathrm{Re}(\eta)>1$. Then,

$$\sum_{n=n_0}^{\infty}\Big|\frac{f(n)}{n^{s'}}\Big|=\sum_{n=n_0}^{\infty}\Big|\frac{f(n)}{n^{s}}\Big|\Big|\frac{1}{n^{\eta}}\Big|\le\sum_{n=n_0}^{\infty}\frac{1}{n^{\mathrm{Re}(\eta)}}<\infty.$$

This implies that s' belongs to S_0, that is, $\sigma_0\le\mathrm{Re}(s)+1$.

By the above argument, $\sigma_0-1-\delta\notin S_1$, for every $\delta>0$. That is, S_1 is bounded from below. Moreover, S_1 contains S_0 and is not empty. These imply that S_1 has an infimum. Finally, we have

$$\sigma_0=\inf_{S_1}\sigma_0\le\inf_{S_1}\mathrm{Re}(s)+1\le\sigma_1+1.$$

This finishes the proof of the statement.

The other inequality follows from $S_0\subseteq S_1$, that is, $\inf S_1\le\inf S_0$.

Solution to Problem 7. The sum $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ is convergent at $s = s_0$.

Consider the arithmetic function $g(n) = f(n)n^{-s_0}$ ans denote its partial sum with

$$S(x) = \sum_{n \leq x} f(n)n^{-s_0}.$$

Fix a complex constant s with $\operatorname{Re} s > \operatorname{Re} s_0$. Let $G(x) = x^{s_0 - s}$, for $x > 0$. Using the partial summation formula we have

$$\begin{aligned}
\sum_{n=1}^{N} f(n)n^{-s} &= \sum_{n=1}^{N} f(n)n^{-s_0} \cdot n^{s_0-s} \\
&= \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s} - \int_{x=1}^{N} S(x)(s_0 - s)x^{s_0-s}\frac{1}{x}\,dx \\
&= \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s} - (s_0 - s)\int_{x=1}^{N} S(x)x^{s_0-s-1}\,dx
\end{aligned}$$

Now we take limit at N tends to ∞. We have

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n)n^{-s} = F(s),$$

and on the other hand

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n)n^{-s_0} = F(s_0)$$

is finite by the assumption, which implies that

$$\lim_{N\to\infty} \Big(\sum_{n=1}^{N} f(n)n^{-s_0}\Big) \cdot N^{s_0-s} = 0$$

Thus,

$$F(s) = (s_0 - s)\int_{x=1}^{\infty} S(x)x^{s_0-s-1}\,dx$$

The infinite integral is finite since $|S(X)|$ is uniformly bounded from above, and $\operatorname{Re}(s) > \operatorname{Re} s_0$.

Let $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma_1$. By the definition of σ_1, there is s_0 with $\sigma_1 < \operatorname{Re} s_0 < \operatorname{Re}(s)$ such that $\sum_{n=1}^{\infty} f(n)n^{-s_0}$ is convergent. In particular the partial sums of this series are uniformly bounded from above in absolute value, and tend to $F(s_0)$ Also note that $\operatorname{Re} s_0 - s - 1 < 1$. Therefore, by the above formula

$$\int_{x=1}^{\infty} S(x)x^{\operatorname{Re}(s_0-s-1)}\,dx < \infty.$$

is well-defined.

Solution to Problem 8. Let $s = \sigma + it$. We know that

$$\sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

is convergent if and only if $\sigma > 1$. This implies that $\sigma_0 = 1$.

On the other hand, for $\sigma \leq 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}$ is divergent. However, for every $\sigma > 0$, by the alternating series test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma}$ is convergent. (need to see that the sequence $1/n^\sigma$ is monotone decreasing!). This implies that $\sigma_1 = 0$.

We need to build an example of a Dirichlet series such that $\sigma_1 = \alpha$ and $\sigma_0 = 1$.

If $\alpha = 0$ then the above example provides the answer to the problem, and if $\alpha = 1$ then we take the series $\sum_{n=1}^{\infty} 1/n^s$. Below we assume that $\alpha \in (0, 1)$.

Define the function $h(x) = x^\alpha$, for $x > 0$. The function $h(x)$ is strictly increasing and for every integer $n \geq 1$ we have

$$|h(n+1) - h(n)| \leq 1 \cdot \sup_{t \in [n,n+1]} |h'(t)| = \sup_{t \in [n,n+1]} \alpha t^{\alpha-1} \leq 1 \cdot 1 = 1.$$

Let a_n, for $n \geq 1$, be a sequence of numbers and define $S(N) = \sum_{n=1}^{N} a_n$. Inductively we define the sequence of numbers $a_n \in \{+1, -1\}$, for $n \geq 1$, such that the corresponding $S(n)$ satisfies $|S(N) - N^\alpha| = |S(N) - h(N)| \leq 1$. Let $a_1 = +1$ which satisfies the inequality for $N+1$. Assume that a_i are defined for $1 \leq i \leq n$, and let

$$a_{n+1} = \begin{cases} +1 & \text{if } S(n) \leq h(n+1) \\ -1 & \text{if } S(n) > h(n+1). \end{cases}$$

When $a_{n+1} = +1$ we have

$$\begin{aligned} & h(n) - 1 \leq S(n) \leq h(n+1) \\ \Longrightarrow \quad & h(n) \leq S(n+1) \leq h(n+1) + 1 \\ \Longrightarrow \quad & h(n) - h(n+1) \leq S(n+1) - h(n+1) \leq +1 \\ \Longrightarrow \quad & |S(n+1) - h(n+1)| \leq +1 \end{aligned}$$

When $a_{n+1} = -1$ we have

$$\begin{aligned} & h(n+1) < S(n) \leq h(n) + 1 \\ \Longrightarrow \quad & h(n+1) - 1 \leq S(n+1) \leq h(n) \\ \Longrightarrow \quad & -1 \leq S(n+1) - h(n+1) \leq h(n) - h(n+1) \\ \Longrightarrow \quad & |S(n+1) - h(n+1)| \leq +1 \end{aligned}$$

For the sequence a_i, for $i \geq 1$ defined above we have

$$\lim_{N\to\infty} \Big| \frac{\log S(N)}{\log N} - \alpha \Big| = \lim_{N\to\infty} \Big| \frac{\log S(N)}{\log N} - \frac{\log N^\alpha}{\log N} \Big| = \lim_{N\to\infty} \Big| \frac{\log S(N) - \log N^\alpha}{\log N} \Big| \leq \lim_{N\to\infty} \frac{1}{\log N} = 0.$$

That is,

$$\lim_{N\to\infty} \frac{\log S(N)}{\log N} = \alpha. \tag{1}$$

The Dirichlet series we introduce is

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Let us denote the partial sums of this series with the notation

$$A_N(s) = \sum_{n=1}^{N} a_n n^{-s}.$$

It is clear that for $A(s)$ we have $\sigma_0 = 1$. We want to show that $\sigma_1 = \alpha$. We will prove this in two steps.

Step 1: $\alpha \leq \sigma_1$.

Let s be a complex number with $\operatorname{Re} s > \sigma_1$. Then, the series

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-1}$$

is convergent. In particular, there is $M > 0$ such that for all $N \geq 1$ we have $|A_N(s)| \leq M$. We have

$$\begin{aligned}
|S(N)| &= \Big|\sum_{n=1}^{N} a_n \cdot n^{-s} \cdot n^s\Big| \\
&= \Big|\sum_{n=1}^{N} (A(n) - A(n-1)) \cdot n^s\Big| \\
&= \Big|\sum_{n=1}^{N} A(n)n^s - \sum_{n=1}^{N} A(n-1) \cdot n^s\Big| \\
&= \Big|\sum_{n=1}^{N} A(n)n^s - \sum_{n=0}^{N-1} A(n) \cdot (n+1)^s\Big| \\
&= \Big|\sum_{n=1}^{N-1} A(n)(n^s - (n+1)^s) + A(N)N^s\Big| \\
&\leq M \sum_{n=1}^{N-1} ((n+1)^s - n^s) + MN^s \\
&\leq 2MN^s
\end{aligned}$$

The above equation implies that

$$\log |S(N)| \leq \log 2 + \log M + s \log N.$$

Hence,

$$\alpha = \lim_{N \to \infty} \frac{\log |S(n)|}{\log N} \leq s.$$

Taking infimum over all s with $\operatorname{Re} s \geq \sigma_1$ we conclude from the above inequality that $\alpha \leq \sigma_1$.

Step 2: $\sigma_1 \leq \alpha$. Let δ be an arbitrary positive real number and let $s = \alpha + \sigma$. We aim to prove that $A(s)$ is a convergent series.

Using the partial summation with $f(n) = a_n$ and $F(x) = x^{-s}$ we have

$$\sum_{n=1}^{N} a_n n^{-s} = S(N)N^{-s} + \int_1^N S(x)sx^{-s-1}\,dx$$

By the relation

$$\lim_{n\to\infty} \frac{\log|S(n)|}{\log n} = \alpha$$

there is $n_0 > 1$ such that for all $n \geq n_0$ we have

$$\log|S(n)| \leq (\alpha + \delta/2)\log n.$$

In other words,

$$|S(n)| \leq n^{\alpha+\delta/2}$$

Using this inequality we see that

$$\lim_{N\to\infty} S(N)N^{-s} \leq \lim_{N\to\infty} N^{\alpha+\delta/2}N^{-(\alpha+\delta)} \leq \lim_{N\to\infty} N^{-\delta/2} = 0.$$

Similarly,

$$s\int_{n_0}^{N} S(x)x^{-s-1}\,dx \leq s\int_{n_0}^{N} x^{\alpha+\delta/2}x^{-\alpha-\delta-1}dx \leq s\int_{n_0}^{N} x^{-1-\delta/2}\,dx < \infty.$$

The above bounds prove that $A(s)$ is a convergent series. In particular, $\sigma_1 \leq \alpha + \delta$. As δ was chosen arbitrarily, we may conclude that $\sigma_1 \leq \alpha$.