Analytic Number Theory

Solutions

Solution to Problem 1. We prove the formula for $\mathrm{Li}(x)$ by the induction on k. By the definition,

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\,dt.$$

We use the integration by parts $\int fg' = fg - \int gf'$ with $f(t) = 1/\log t$ and $g(t) = t$ to obtain

$$\int_2^x \frac{1}{\log t}\,dt = \frac{t}{\log t}\Big|_2^x + \int_2^x \frac{1}{t}\frac{1}{(\log t)^2}\cdot t\,dt = \frac{x}{\log x} - \frac{2}{\log 2} + I_1(x).$$

This implies the formula for $k = 1$. Now assume that the formula holds for $k \geq 1$ and we wish to prove it for $k+1$. Again using the integration by parts with $f(t) = 1/(\log t)^{k+1}$ and $g(t) = t$ we have

$$I_k(x) = \frac{t}{(\log t)^{k+1}}\Big|_2^x - \int_2^x \frac{-(k+1)}{t(\log t)^{k+2}}\cdot t\,dt = \frac{x}{(\log x)^{k+1}} - \frac{2}{(\log 2)^{k+1}} + (k+1)I_{k+1}(x).$$

Therefore,

$$\begin{aligned}
\mathrm{Li}(x) &= C_k + \sum_{n=0}^{k-1} n!\frac{x}{(\log x)^{n+1}} + k!I_k(x)\\
&= C_k + \sum_{n=0}^{k-1} n!\frac{x}{(\log x)^{n+1}} + k!\Big(\frac{x}{(\log x)^{k+1}} - \frac{2}{(\log 2)^{k+1}} + (k+1)I_{k+1}(x)\Big)\\
&= \Big(C_k - k!\frac{2}{(\log 2)^{k+1}}\Big) + \sum_{n=0}^{k} n!\frac{x}{(\log x)^{n+1}} + (k+1)!I_{k+1}(x).
\end{aligned}$$

This finishes the proof of the formula by introducing the appropriate constant C_{k+1}.

Since the function $1/(\log t)^2$ is decreasing as t increases, we have

$$\begin{aligned}
I_1(x) &= \int_2^{\sqrt{x}} \frac{1}{(\log t)^2}\,dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^2}\,dt\\
&\leq (\sqrt{x} - 2)\frac{1}{(\log 2)^2} + (x - \sqrt{x})\frac{1}{(\log x^{1/2})^2}\\
&\leq \frac{\sqrt{x}}{(\log 2)^2} + \frac{4x}{(\log x)^2}.
\end{aligned}$$

Using the formula for $\mathrm{Li}(x)$ with $k = 1$ we have

$$\mathrm{Li}(x) = C_1 + \frac{x}{\log x} + I_1(x).$$

Then,

$$\lim_{x\to\infty}\frac{\mathrm{Li}(x)}{x/\log x}=\lim_{x\to\infty}\frac{C_1\log x}{x}+\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{I_1(x)}{x/\log x}=1+\lim_{x\to\infty}\frac{I_1(x)}{x/\log x}$$

However, by the above inequality

$$\lim_{x\to\infty}\frac{I_1(x)}{x/\log x}\leq\lim_{x\to\infty}\frac{\frac{\sqrt{x}}{(\log 2)^2}}{\frac{x}{\log x}}+\lim_{x\to\infty}\frac{\frac{4x}{(\log x)^2}}{\frac{x}{\log x}}=\lim_{x\to\infty}\frac{\log x}{x(\log 2)^2}+\lim_{x\to\infty}\frac{4}{\log x}=0$$

Solution to Problem 2. We have

$$\lim_{n\to\infty}\frac{f_1(n)/f_3(n)}{f_2(n)/f_4(n)}=\lim_{n\to\infty}\frac{f_1(n)f_4(n)}{f_2(n)f_3(n)}=\lim_{n\to\infty}\frac{f_1(n)}{f_2(n)}\cdot\lim_{n\to\infty}\frac{f_4(n)}{f_3(n)}=1\cdot 1=1.$$

and

$$\lim_{n\to\infty}\frac{f_1(n)+f_3(n)}{f_2(n)+f_4(n)}=\lim_{n\to\infty}\frac{f_1(n)+f_3(n)}{f_2(n)+f_4(n)}\cdot\lim_{n\to\infty}\frac{f_4(n)}{f_3(n)}=\lim_{n\to\infty}\frac{f_1(n)/f_3(n)+1}{f_2(n)/f_4(n)+1}=\frac{1+1}{1+1}=1.$$

In the special case given, we have $f_1(n)-f_3(n)=n$ and $f_2(n)-f_4(n)=2n$, thus,

$$\lim_{n\to\infty}\frac{f_1(n)-f_3(n)}{f_2(n)-f_4(n)}=\frac{1}{2}\neq 1.$$

Solution to Problem 3. Let us introduce the notation

$$G_1(N,M)=\sum_{n=N}^{M}f_1(n),\quad G_2(N,M)=\sum_{n=N}^{M}f_2(n).$$

By the hypothesis, for every $\epsilon>0$ there is N_0 such that for all $n\geq N_0$ we have

$$f_2(n)(1-\epsilon)\leq f_1(n)\leq(1+\epsilon)f_2(n).$$

This implies that $F_1(N)\to\infty$ if and only if $F_2(N)\to\infty$, when $N\to\infty$. Also, summing up these inequalities we conclude that for all $M\geq N_0$ we have

$$(1-\epsilon)G_2(N_0,M)\leq G_1(N_0,M)\leq(1+\epsilon)G_2(N_0,M).$$

Hence,

$$\begin{aligned}
\limsup_{M\to\infty}\frac{F_1(M)}{F_2(M)} &= \limsup_{M\to\infty}\frac{G_1(1,N_0-1)+G_1(N_0,M)}{G_2(1,N_0-1)+G_2(N_0,M)}\\
&= \limsup_{M\to\infty}\frac{G_1(1,N_0-1)/G_2(N_0,M)+G_1(N_0,M)/G_2(N_0,M)}{G_2(1,N_0-1)/G_2(N_0,M)+1}\\
&= \limsup_{M\to\infty}\frac{G_1(N_0,M)}{G_2(N_0,M)}\\
&\le 1+\epsilon.
\end{aligned}$$

Similarly, $\liminf_{M\to\infty}\frac{F_1(M)}{F_2(M)} \ge 1-\epsilon$. As $\epsilon>0$ was arbitrary, we conclude that the limit exists and is equal to $+1$.

Solution to Problem 4. a) the answer is no, for instance one can define

$$f(x) = g(x) + sin(e^{g(x)}) + sin(e^x).$$

Then, since $g(x)$ is bounded from below fro $x\ge 1$, then $f(x) = O(g(x)) + O(1) + O(1) = O(g(x))$. On the other hand,

$$f'(x) = g'(x) + g'(x)e^{g(x)}\cos(e^{g(x)}) + e^x\cos(e^x).$$

Depending on whether $g(x) = O(1)$ (and hence $g'(x)\to 0$ as $x\to\infty$) or not, one can see that $f'(x)\ne O(g'(x))$.

b) The answer is yes, since $f(x)\le Cg(x)$ implies that

$$\int_2^X f(t)\,dt \le \int_2^X Cg(t)\,dt \le C\int_2^X g(t)\,dt.$$

Solution to Problem 5. By contradiction, assume that there is n_0 such that for $n\ge n_0$ the differences $p_{n+1}-p_n$ are strictly increasing. Since the differences are integers, then for $n>n_0$, $p_{n+1}-p_n\ge 1+(p_n-p_{n-1})$. In particular, for large values of $n>n_0$ we have

$$p_{n+1}-p_n \ge 1+(p_n-p_{n-1}) \ge 2+(p_{n-1}-p_{n-2}) \ge \cdots \ge (n-n_0)+(p_{n_0+1}-p_{n_0}) \ge (n-n_0)+1.$$

Putting these together, we obtain

$$p_n - p_{n_0} = \sum_{i=n_0}^{n-1}(p_{i+1}-p_i) \ge \sum_{i=n_0}^{n-1}(i-n_0+1) = \sum_{i=1}^{n-n_0} i = \frac{(n-n_0)^2}{2} + \frac{n-n_0}{2} \ge \frac{(n-n_0)^2}{2}.$$

By PNT we have $\pi(x) \sim \frac{x}{\log x}$. For instance, for large values of x we must have $\pi(x) \geq \frac{x}{2\log x}$. Since, for large enough n, p_n is large enough, we must have $n = \pi(p_n) \geq \frac{p_n}{2\log p_n}$.

On the other hand, for large enough x, we know that $\frac{x}{\log x} \geq x^{3/4}$. Then, for large n we must have

$$2n \geq \frac{p_n}{\log p_n} \geq p_n^{3/4} \geq \left(\frac{(n-n_0)^2}{2}\right)^{3/4} \geq \frac{(n-n_0)^{3/2}}{2^{3/4}},$$

which is a contradiction.

Solution to Problem 6. By PNT

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

In particular, we look at the limit along the sequence p_n as $n \to \infty$. By the definition $\pi(p_n) = n$. Then,

$$\lim_{n\to\infty} \frac{n}{p_n/\log p_n} = 1.$$

In other words, $\frac{p_n}{\log p_n} \sim n$. Since log is a continuous function over the positive real numbers, we have

$$\lim_{n\to\infty} (\log n - \log p_n + \log(\log p_n)) = \lim_{n\to\infty} \log\left(\frac{n}{p_n/\log p_n}\right) = 0.$$

This implies that

$$\begin{aligned}\lim_{n\to\infty} \frac{\log n}{\log p_n} &= \lim_{n\to\infty} \left(\frac{\log n - \log p_n + \log(\log p_n)}{\log p_n} + \frac{\log p_n}{\log p_n} + \frac{\log(\log p_n)}{\log p_n}\right)\\ &= 0 + 1 + 0 = 1.\end{aligned}$$

To find the limit of the last ration in the above equation we may for instance use $\log(x) \leq x^{1/2}$, that is, $\log(\log p_n) \leq (\log p_n)^{1/2}$.

Finally,

$$\begin{aligned}\lim_{n\to\infty} \frac{n\log n}{p_n} &= \lim_{n\to\infty} \left(\frac{n\log n}{p_n} \times \frac{\log p_n}{\log p_n}\right)\\ &= \lim_{n\to\infty} \left(\frac{n\log p_n}{p_n} \times \frac{\log n}{\log p_n}\right)\\ &= \lim_{n\to\infty} \frac{n}{p_n/\log p_n} \times \lim_{n\to\infty} \frac{\log n}{\log p_n}\\ &= 1 \times 1 = 1\end{aligned}$$

Solution to Problem 7. Let us define the sets

$$A_n = \{d \leq n; d|n\}, B_n = \{d \leq 2^n - 1; d|2^n - 1\}.$$

We have $d(n) = \#A_n$ and $d(2^n - 1) = \#B_n$. We claim that $\psi(d) = 2^d - 1$ maps A_n into B_n. To see this, let $n = dk$, for some $d \in A_n$. We have

$$2^n - 1 = (2^d)^k - 1 = (2^d - 1)\big((2^d)^{k-1} + (2^d)^{k-2} + \cdots + 1\big)$$

which implies $2^d - 1|2^n - 1$. Since, $\psi : A_n \to B_n$ is one -to-one, we must have $\#A_n \leq \#B_n$.

Solution to Problem 8. Let $N = [X]$.

$$\begin{aligned}
&\int_1^X S(t)F'(t)\,dt \\
&= \sum_{i=1}^{N-1} \int_i^{i+1} S(t)F'(t)\,dt + \int_N^X S(t)F'(t)\,dt \\
&= \sum_{i=1}^{N-1} S(i) \int_i^{i+1} F'(t)\,dt + S(N) \int_N^X F'(t)\,dt \\
&= \sum_{i=1}^{N-1} S(i)\big(F(i+1) - F(i)\big) + S(N)\big(F(X) - F(N)\big) \\
&= \sum_{i=1}^{N-1} S(i)F(i+1) - \sum_{i=1}^{N-1} S(i)F(i) + S(N)\big(F(X) - F(N)\big) \\
&= \sum_{i=1}^{N-1} S(i)F(i+1) - \sum_{i=0}^{N-2} S(i+1)F(i+1) + S(N)\big(F(X) - F(N)\big) \\
&= \sum_{i=1}^{N-2} (S(i) - S(i+1))F(i+1) + \big(S(N-1)F(N) - S(1)F(1)\big) + S(N)\big(F(X) - F(N)\big) \\
&= -\sum_{i=1}^{N-2} f(i+1)F(i+1) + \big(S(N-1)F(N) - S(1)F(1)\big) + \big(S(N-1) + f(N)\big)\big(F(X) - F(N)\big) \\
&= -\sum_{i=1}^{N} f(i)F(i) + S(N-1)F(X) + f(N)F(X) \\
&= -\sum_{i=1}^{N} f(i)F(i) + S(X)F(X).
\end{aligned}$$