Analytic Number Theory

Problem sheet 4

Problem 1. Let $|t| \geq 2$ and let

$$\alpha \in [\frac{1}{100}, \frac{99}{100}], \quad \beta = \max\{\frac{1}{2}, 1 - \frac{1}{\log|t|}\}.$$

Show that $x^{-\beta} = O(x^{-1})$ for $1 \leq x \leq |t|$.

With $M = [|t|]$, prove that

$$\sum_{n=1}^{M} n^{-\sigma} = O(|t|^{1-\alpha}), \text{ for } \sigma \geq \alpha$$

and that

$$\sum_{n=1}^{M} n^{-\sigma} = O(\log|t|), \text{ for } \sigma \geq \beta.$$

Prove also that

$$\sum_{n=M}^{\infty} n^{-\sigma-1} = O(|t|^{-\alpha}) \text{ for } \sigma \geq \alpha$$

and that

$$\sum_{n=M}^{\infty} n^{-\sigma-1} = O(|t|^{-1}) \text{ for } \sigma \geq \beta.$$

By adapting the proof of Theorem 4.3, deduce that

$$|\zeta(\sigma + it)| = O(|t|^{1-\alpha}) \text{ for } \sigma \geq \alpha$$

and that

$$|\zeta(\sigma + it)| = O(\log|t|) \text{ for } \sigma \geq \beta.$$

Problem 2. Let $|t| \geq 3$ and let

$$\sigma \geq \max\{\frac{3}{4}, 1 - \frac{1}{2\log|t|}\}.$$

Write down Cauchy's integral formula for $\zeta'(s)$ in terms of $\zeta(w)$, using a circular path Γ of radius $(4\log|t|)^{-1}$ about s. Show that $\zeta(w) = O(\log|t|)$ uniformly for w on Γ, and deduce that

$$|\zeta'(\sigma + it)| = O(\log^2|t|)$$

[You may assume that if $w = x + iy$ lies on Γ, then $|y| \geq 2$, and $x \geq 1/2$, and that $1 - (\log|y|)^{-1} \leq x \leq 2$.]

Problem 3. Let

$$\theta(x) = \sum_{p \le x} \log p,$$

where the sum is over all primes $p \le x$. Show that

$$\psi(x) = \theta(x) + O(x^{1/2} \log^2 x).$$

Using partial summation with the arithmetic function

$$f(n) = \begin{cases} \log n & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

and the function $F(x) = (\log x)^{-1}$, show that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t}\, dt.$$

Now suppose that $E(x)$ is an increasing function of x with $E(x) \ge x^{1/2} \log^2 x$, and that $\psi(x) = x + O(E(x))$. Deduce that $\theta(x) = x + O(E(x))$ and hence that $\pi(x) = \mathrm{Li}(x) + O(E(x))$.

Problem 4. Suppose that $\psi_1(x) = \frac{1}{2}x^2 + O(F(x))$, for some non-negative and increasing function $F(x) \le x^2$. By taking $\alpha = 1 - \delta$ and $\beta = 1 + \delta$ in the proof of Theorem 5.3, and choosing δ appropriately, show that

$$\psi(x) = x + O(F(2x)^{1/2}).$$

Problem 5. Recall from Problem Sheet 3 that

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} d^2(n) n^{-s}.$$

Show that if $x > 0$ and $c > 1$ then

$$\sum_{n \le x} d^2(n)(x - n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta^4(s)}{\zeta(2s)} \frac{x^{s+1}}{s(s+1)}\, ds.$$

Move the line of integration to $\sigma = 7/8$ and use the estimate in Question 1 to prove that there is a cubic polynomial $P(X)$ such that

$$\sum_{n \le x} d^2(n)(x - n) = x^2 P(\log x) + O(x^{15/8}).$$

Find the leading coefficient of P.

Problem 6. Apply the technique of the proof of Theorem 5.3 to deduce that

$$\sum_{n\le x} d^2(n) \sim \pi^{-2} x \log^3 x, \text{ as } x \to \infty.$$