Analytic Number Theory

Problem sheet 3

Problem 1. Show that if $\delta > 0$ then the integral in Problem Sheet 2, Question 7, is uniformly convergent for $\operatorname{Re} s \geq \delta + \operatorname{Re} s_0$, and deduce that $F(s)$ is holomorphic in the region $\operatorname{Re} s > \sigma_1$.

Problem 2. Find Euler product representations for $\sum_{n=1}^{\infty} f(n) n^{-s}$ when

(a) $f(n) = (-1)^{n-1}$;

(b) $f(n) = (-1)^{(n-1)/2}$ for n odd and $f(n) = 0$ for n even.

In each case you will have to verify that f is a multiplicative function.

Problem 3. By using Euler products, or otherwise, show that

$$\zeta^2(s) = \sum_{n=1}^{\infty} d(n) n^{-s},$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} d(n^2) n^{-s},$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} d^2(n) n^{-s}.$$

Problem 4. Express in terms of the Riemann Zeta-function the Dirichlet series

(a) $\sum_{n=1}^{\infty} \sigma(n) n^{-s}$;

(b) $\sum_{n=1}^{\infty} \phi(n) n^{-s}$;

(c) $\sum_{n=1}^{\infty} |\mu(n)| n^{-s}$.

Problem 5. Show that for every $s \in \mathbb{C}$ with $\operatorname{Re}(s) = \sigma > 1$ we have

$$\frac{1}{|\zeta(s)|} \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Problem 6. Approximate $\sum_{n \leq X} \log n$, for $X \geq 1$, and use the relation $\Lambda * u(n) = \log n$ (see Lemma 3.6) to deduce that

$$\sum_{m \leq X} \Lambda(m) [\frac{X}{m}] = X \log X + O(X)$$

and hence that

$$\sum_{m \leq X} \frac{\Lambda(m)}{m} \geq \log X + O(1).$$

If $\delta \in (0, 1)$ show that $[\theta] \geq (1 - \delta)\theta$ for $\theta \geq \delta^{-1}$, and conclude that

$$\sum_{m \leq \delta X} \frac{\Lambda(m)}{m} \leq (1 - \delta)^{-1} \log X + O(1).$$

Deduce that

$$\sum_{m \le Y} \frac{\Lambda(m)}{m} \sim \log Y$$

as $Y \to \infty$.

Problem 7. Show that if $\sigma > 1$ then

$$3\frac{\zeta'(\sigma)}{\zeta(\sigma)} + 4\,\mathrm{Re}\,\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \mathrm{Re}\,\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \le 0.$$

Problem 8. The inequality $3+4\cos\theta+\cos(2\theta) \ge 0$ plays a key role in the proof of Theorem 4.5. Suppose one has instead an inequality

$$A_0 + A_1 \cos\theta + \cdots + A_N \cos(N\theta) \ge 0,$$

(with real coefficients A_k) valid for all $\theta \in \mathbb{R}$. If one aims to show that $\zeta(1+it) \ne 0$ for all $t \in \mathbb{R}$ (without necessarily obtaining an estimate for $1/\zeta(1+it)$), what properties will the coefficients A_k need to have in order for the proof to work?

Find a new inequality with $N = 2$ which could be used to prove such a result.