Analytic Number Theory

Problem sheet 2

Problem 1. By following the proof of Theorem 2.9 with $\epsilon = 1/4$, show that for every $n \geq 1.9 \times 10^{18}$, $d(n) < \sqrt{n}$. Can you reduce this bound? (For each "small" prime p investigate the maximal value of $(1+e)/p^{\epsilon e}$.)

Problem 2. Use partial summation (Sheet 1, $Q8$), along with Theorem 2.10, to show that as $x \to \infty$,

$$\sum_{n \leq x} \frac{d(n)}{n} \sim \frac{1}{2}(\log x)^2.$$

Problem 3. Taking $f(n) = 1$ for all n in the partial summation formula, show that for any real number $x \geq 1$ we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O(1/x),$$

where,

$$\gamma = 1 - \int_1^\infty \frac{t - [t]}{t^2}\, dt.$$

Problem 4. For any real number $x \geq 1$ show that

$$\sum_{n \leq x} n = \frac{1}{2}x^2 + O(x)$$

and that

$$\sum_{n \leq x} \sigma(n) = \sum_{u \leq x} \sum_{v \leq x/u} v = \frac{1}{2}x^2 \sum_{u \leq x} u^{-2} + O(x(1 + \log x)).$$

Show that

$$\sum_{u > x} \frac{1}{u^2} \leq \frac{2}{x},$$

and conclude that

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12}x^2 + O(x(1 + \log x).$$

Problem 5. Using question 3 above, show that for any real $X \geq 1$,

$$\begin{aligned}\sum_{n \leq X} d(n) &= \sum_{u,v \geq 1, uv \leq X} 1 \\ &= \sum_{u \leq \sqrt{X}} \sum_{v \leq X/u} 1 + \sum_{v \leq \sqrt{X}} \sum_{u \leq X/v} 1 - \sum_{u \leq \sqrt{X}} \sum_{v \leq \sqrt{X}} 1 \\ &= X\big(\log X + 2\gamma - 1\big) + O(\sqrt{X}).\end{aligned}$$

Problem 6. Suppose that $F(s) = \sum_1^\infty f(n)n^{-s}$ is a Dirichlet series for which the abscissa of absolute convergence σ_0 is defined. Write

$$\sigma_1 = \inf\{\mathrm{Re}(s) : s \in \mathbb{C}, \sum_{n=1}^{\infty} f(n)n^{-s} \text{ converges } \}.$$

Show that σ_1 exists, and we have $\sigma_0 - 1 \leq \sigma_1 \leq \sigma_0$.

Problem 7. Under the assumption of Question 6, suppose that $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$ converges at $s = s_0$. Apply the partial summation formula to show that $F(s)$ converges whenever $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, and that

$$F(s) = (s - s_0) \int_1^\infty S(x)x^{s_0 - s - 1}\, dx,$$

where

$$S(x) = \sum_{n \leq x} f(n)n^{-s_0}.$$

Deduce that $F(s)$ converges for any s with $\mathrm{Re}(s) > \sigma_1$.

Problem 8. Show that $\sigma_1 = 0$ and $\sigma_0 = 1$ for the series $\sum_{n=1}^\infty (-1)^{n-1}n^{-s}$. For each $\alpha \in [0, 1]$ construct an example in which $\sigma_1 = \alpha$ and $\sigma_0 = 1$.