Analytic Number Theory

Problem sheet 1

Problem 1. Define the integral

$$I_k(x) = \int_2^x \frac{dt}{(\log t)^{k+1}}.$$

Show that, for every integer $k \geq 1$ there is a constant C_k such that

$$Li(x) = C_k + \sum_{n=0}^{k-1} n! \frac{x}{(\log x)^{n+1}} + k! I_k(x).$$

By splitting the range of integration at $t = \sqrt{x}$, show that

$$I_1(x) \leq \frac{4x}{(\log x)^2} + \frac{\sqrt{x}}{(\log 2)^2}.$$

Then deduce that

$$Li(x) \sim \frac{x}{\log x}.$$

Problem 2. Suppose that $f_1(n)$, $f_2(n)$, $f_3(n)$, and $f_4(n)$ are positive functions for which $f_1(n) \sim f_2(n)$ and $f_3(n) \sim f_4(n)$ as $n \to \infty$. Show that

$$f_1(n)/f_3(n) \sim f_2(n)/f_4(n), \text{ and } f_1(n) + f_3(n) \sim f_2(n) + f_4(n).$$

By taking $f_1(n) = n^2 + n$, $f_2(n) = n^2 + 2n$, $f_3(n) = n^2$, $f_4(n) = n^2$, show that one need not have $f_1(n) - f_3(n) \sim f_2(n) - f_4(n)$.

Problem 3. Let $f_1(n)$ and $f_2(n)$ be positive functions such that $f_1(n) \sim f_2(n)$ as $n \to \infty$. Define

$$F_i(N) = \sum_{n \leq N} f_i(n) \qquad (i = 1, 2)$$

and suppose that $F_2(N) \to \infty$ as $N \to \infty$. Show that $F_1(N) \sim F_2(N)$.

Problem 4. Suppose that f and g are differentiable, with $g(x) > 0$ for $x > 0$.

a) Suppose that $g'(x) > 0$ for $x > 0$. Is it true that $f(x) = O(g(x))$ for $x > 0$ implies $f'(x) = O(g'(x))$ for $x > 0$?

b) Is it true that $f(x) = O(g(x))$ for $x > 0$ implies

$$\int_2^X f(t)dt = O\Big(\int_2^X g(t)dt\Big)$$

for $X > 2$.

Problem 5. Let p_n denote the n-th prime number. Assuming the prime number theorem show that there is no n_0 such that the sequence of differences $p_{n+1} - p_n$ is strictly monotonic for $n \geq n_0$.

Problem 6. Let p_n denote the n-th prime number. Using the prime number theorem show that $p_n/\log p_n \sim n$. Then, by taking logarithms, deduce that $\log p_n \sim \log n$, and hence $p_n \sim n \log n$.

Problem 7. Show that for every $n \in \mathbb{N}$ we have $d(n) \leq d(2^n - 1)$, where d denotes the divisor arithmetic function.

Problem 8. Let $f(n)$ be an arithmetic function, and let $F : [1, X] \to \mathbb{C}$ be a function with a continuous derivative. For every $t \in [1, X]$ define $S(t) = \sum_{1 \leq n \leq t} f(n)$. Show that

$$\sum_{1 \leq n \leq X} f(n)F(n) = S(X)F(X) - \int_1^X S(t)F'(t)dt.$$

[The process in the above problem is known as "summing by parts" or "partial summation". It is useful for estimating $\sum f(n)F(n)$ given information about $\sum f(n)$.]