Imperial College
London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3P16/M4P16/M5P16

Analytic Number Theory

Date: Wednesday, 25th May 2016 Time: 14:00 – 16:30

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Recall the Euler function $\phi(n) = \#\{k \in \mathbb{N} : k \leq n, (k,n) = 1\}$, the unit function $u(n) \equiv 1$, as well as the Möbius function μ defined by $\mu(1) = 1$ and for distinct primes p_i and positive integers e_i

$$\mu(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}) = \begin{cases} (-1)^k & \text{if } e_i = 1 \text{ for all } i, \\ 0 & \text{if } e_i \geq 2 \text{ for some } i. \end{cases}$$

(a) Prove that

$$(\mu * u)(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $*$ denotes the convolution operation.

(b) Assuming the relation $\sum_{d|n} \phi(d) = n$, for $n \in \mathbb{N}$, prove that ϕ is a *multiplicative* arithmetic function.

[You may use any theorems from the lectures, provided you state them clearly.]

(c) Prove that for $n \in \mathbb{N}$ we have

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$

where the product runs over all primes p.

2. (a) State (without proof) the *Euler product formula* for the *Riemann zeta-function* $\zeta(s)$.

(b) Consider the arithmetic function

$$\nu(n) = \begin{cases} 0 & \text{if } n = 1 \\ k & \text{if } n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} \text{ with } p_i\text{'s distinct primes and all } e_i \geq 1. \end{cases}$$

Prove that for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ we have

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\nu(n)} n^{-s}.$$

3. Assuming the relation

$$\sum_{p \leq x, p \text{ prime}} \frac{\log p}{p} = \log x + O(1), \text{for } x \geq 1,$$

prove that, for some constant A,

$$\sum_{p \leq x, p \text{ prime}} \frac{1}{p} = \log(\log(x)) + A + O(\frac{1}{\log x}), \text{for all } x \geq 2.$$

4. (a) Let $\zeta(s)$ denote the Riemann zeta-function, and $\Gamma(s)$ denote the Euler gamma function. Assuming the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s), \quad \forall s \in \mathbb{C} - \{0\},$$

prove that if $\zeta(s) = 0$ then either $s \in \{-2k : k \in \mathbb{N}\}$, or s lies in the strip $0 \leq \mathrm{Re}(s) \leq 1$.

(b) Prove that if $\zeta(\rho) = 0$ with $0 < \mathrm{Re}(\rho) < 1$, then each of ρ, $1-\rho$, $\overline{\rho}$, and $1-\overline{\rho}$ is a zero of $\zeta(s)$.

[You are required to prove any relation about $\zeta(s)$ you need, except for the functional equation in part (a).]

5. Let $\Psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangold function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e \text{ for some prime } p \text{ and integer } e \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We recall that for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

(a) Prove that for every s with $\mathrm{Re}(s) > 1$ we have

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^{\infty} \frac{\Psi(x)}{x^{s+1}}\, dx.$$

(b) Prove that

$$\lim_{s \to 1}(1-s)\frac{\zeta'(s)}{\zeta(s)} = 1.$$

(c) Let $\delta = \limsup_{x\to\infty} \frac{\Psi(x)}{x}$. Show that for every $\epsilon > 0$ there is a constant $C(\epsilon)$ such that for every real $s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} \leq sC(\epsilon) + \frac{s(\delta+\epsilon)}{s-1}.$$

Using the limit in part (b), or otherwise, deduce that $\delta \geq 1$.

(d) Let $\gamma = \liminf_{x\to\infty} \frac{\Psi(x)}{x}$. By a similar argument, prove that $\gamma \leq 1$.

(e) Deduce Chebycheff's theorem, that is, if $\lim_{x\to\infty} \frac{\Psi(x)}{x}$ exists, it must be equal to 1.