Examination solutions 2015-16
Course: M3P16, M4P16, M5P16

Solution to Question 1.

Part a) The functions μ and u are multiplicative, and by a Theorem in the lectures, the convolution of any two multiplicative functions is multiplicative. **seen, 2**

Let $f(n) = \mu * u(n)$. For an integer $e \geq 1$ and a prime p, we have

$$f(p^e) = \sum_{ab=p^e} \mu(a)u(b) = \sum_{a|p^e} \mu(a)$$

$$= \mu(1) + \mu(p) + \ldots + \mu(p^e) = 1 + (-1) + 0 + \ldots + 0 = 0.$$

seen, 3

Since f is multiplicativ, $f(p_1^{e_1} \cdots p_k^{e_k}) = f(p_1^{e_1}) \cdots f(p_k^{e_k}) = 0$ if at least one of the exponents is greater than or equal to 1. **seen, 1**

For $n = 1$ we have $\mu * u(1) = \mu(1) \cdot u(1) = 1 \cdot 1 = 1$. **seen, 1**

Part b) The equation in part a is equivalent to $\phi * u = u_1$, where $u_1(n) = n$, for $n \in \mathbb{N}$. Since, $\mu * u = u * \mu$ is the identity element of the convolution, we have $\phi = \phi * (u * \mu) = (\phi * u) * \mu = u_1 * \mu$. **seen, 4**

Since the functions u_1 and μ are multiplicative, by the theorem mentioned in part a, ϕ must be multiplicative. **seen, 2**

Part c) For every prime p and integer $e \geq 1$ we have

$$\phi(p^e) = \#\{n : n \leq p^e, (n, p^e) = 1\} = \#\{n : n \leq p^e, p \nmid n\}$$

$$= \#\{n : n \leq p^e\} - \#\{n : n \leq p^e, p \mid n\} = p^e - p^{e-1} = p^e(1 - 1/p).$$

seen, 5

Since ϕ is multiplicative, for distinct primes $p_1, p_2, \ldots, p_k$,

$$\phi(p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}) = \phi(p_1^{e_1})\phi(p_2^{e_2}) \ldots \phi(p_k^{e_k}) = \prod_{i=1}^{k} p_i^{e_i} \prod_{i=1}^{k} (1 - \frac{1}{p})$$

By the prime factorization theorem, $\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$. **seen, 2**

Solution to Question 2.

Part a) By the Euler's product formula, for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (\frac{1}{1 - p^{-s}}),$$

where the product is over all primes p. **seen, 2**

Part b) Let $f(n) = 2^{\nu(n)}$, $n \in \mathbb{N}$. If we show that f is multiplicative, then by a theorem in the lectures, for every s with $\mathrm{Re}(s)$ greater that the AAC of the Dirichlet series $\sum_{n=1}^{\infty} f(n) n^{-s}$ we have

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_{p} \Big(\sum_{e=0}^{\infty} f(p^e) p^{-es} \Big),$$

where the product is over all primes p. **seen, 4**

The function f is multiplicative, if $f(mn) = f(m)f(n)$, for all positive integers m and n with $(m, n) = 1$. First assume that one of m and n, say m, is equal to 1 and $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where p_i's are distinct primes and all $e_i \geq 1$. Then, $f(mn) = f(n) = 2^k$, while $f(m)f(n) = 2^0 \cdot 2^k = 2^k$. **unseen, 2**

Now let $m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ and $n = q_1^{t_1} q_2^{t_2} \dots q_{k'}^{t'_k}$ with all $p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_{k'}$ distinct primes, and $e_1, e_2, \dots, e_k, t_1, t_2, \dots, t_{k'}$ positive integers. Then, $f(mn) = 2^{k+k'} = 2^k \cdot 2^{k'} = f(m)f(n)$. **unseen, 2**

We have $f(1) = 1 \leq 1$ and for $n \geq 1$ with prime factorization $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ we have $f(n) = 2^k \leq p_1 p_2 \dots p_k \leq p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} = n$. Then,

$$\sum_{n=1}^{\infty} |f(n) \cdot n^{-s}| \leq \sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)+1},$$

which is convergent for $\mathrm{Re}(s) > 2$. In particular, AAC of the Dirichlet series for f is ≤ 2. **unseen, 4**

For each prime p and positive integer e we have

$$\sum_{e=0}^{\infty} f(p^e) p^{-es} = f(1) \cdot 1 + f(p) \cdot p^{-s} + f(p^2) \cdot p^{-2s} + \dots =$$

$$2^0 \cdot 1 + 2^1 \cdot p^{-s} + 2^1 \cdot p^{-2s} + 2^1 \cdot p^{-3s} + \dots = -1 + 2(\frac{1}{1 - p^{-s}}) = \frac{1 + p^{-s}}{1 - p^{-s}}.$$

unseen, 3

On the other hand, by the Euler's product formula,

$$\frac{\zeta^2(s)}{\zeta(2s)} = \prod_p \Big((\frac{1}{1-p^{-s}})^2 (1-p^{-2s}) \Big) = \prod_p \Big(\frac{1+p^{-s}}{1-p^{-s}} \Big).$$

unseen, 3

Solution to Question 3

For $n \in \mathbb{N}$ let us define

$$f(n) = \begin{cases} \frac{\log n}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Define, $F(x) = 1/\log x$, for $x \in (0, \infty)$. The function $F(x)$ is C^1 on $(1, \infty)$. By the partial summation formula, for every $X > 1$ we have

$$\sum_{n \leq X} f(n)F(n) = S(X)F(X) - \int_1^X S(t)F'(t)\, dt,$$

where $S(X) = \sum_{n \leq X} f(n)$. **unseen, 5**

We have

$$\sum_{n \leq X} f(n)F(n) = \sum_{p \leq X} \frac{1}{p},$$

and

$$S(X) = \sum_{p \leq X} \frac{\log p}{p} = \log X + O(1),$$

where both sums are over primes p. Moreover, $S(X) = 0$ for $X < 2$. **unseen, 2**

By the formula,

$$\sum_{p \leq X} \frac{1}{p} = \big(\log X + O(1)\big)\frac{1}{\log X} + \int_2^X \frac{S(t)}{t \log^2 t}\, dt = 1 + O(\frac{1}{\log X}) + \int_2^X \frac{S(t)}{t \log^2 t}\, dt.$$

unseen, 3

Let $S(t) = \log t + R(t)$, where $R(t) = O(1)$. Then

$$\int_2^X \frac{S(t)}{t \log^2 t}\, dt = \int_2^X \frac{1}{t \log t}\, dt + \int_2^X \frac{R(t)}{t \log^2 t}\, dt.$$

unseen, 2

We have,

$$\int_2^X \frac{1}{t\log t}\,dt = \log\log X - \log\log 2,$$

using the substitution $u = \log t$, and, **unseen, 3**

$$\int_2^X \frac{R(t)}{t\log^2 t}\,dt = \int_2^\infty \frac{R(t)}{t\log^2 t}\,dt - \int_X^\infty \frac{R(t)}{t\log^2 t}\,dt = C - O(\frac{1}{\log X}),$$

for some constant C, using the same substitution. **unseen, 3**

Combining the above formulas, we obtain

$$\sum_{p\le X}\frac{1}{p} = 1 + O(\frac{1}{\log X}) + \log\log X - \log\log 2 + C - O(\frac{1}{\log X})$$
$$= \log\log X + (1 - \log\log 2 + C) + O(\frac{1}{\log X}).$$

unseen, 2

Solution to Question 4

Part a) If $\operatorname{Re} s > 1$, then by a theorem in the lecture, $1/\zeta(s) = \sum_{n=1}^\infty \mu(n)n^{-s}$, where μ is the Möbius function, and $|\sum_{n=1}^\infty \mu(n)n^{-s}| \le \sum_{n=1}^\infty |n^{-s}| \le \sum_{n=1}^\infty n^{-\operatorname{Re}(s)} < \infty$, for $\operatorname{Re}(s) > 1$. In particular, $\zeta(s) \neq 0$. **unseen, 3**

If $\operatorname{Re}(s) < 0$, then $\operatorname{Re}(1-s) > 1$ and by the above argument, $\zeta(1-s) \neq 0$. Therefore, by the hypothesis, $\zeta(s) = 0$ if and only if $\cos(\frac{\pi s}{2})\Gamma(s)$ has a pole at s. **seen, 2**

The function $\cos(\pi s/2)$ is entire and has no poles, while $\Gamma(s)$ is meromorphic with simple poles at points $k = 0, -1, -2, -3, \dots$. **seen, 2**

When $-k$ is odd, the pole of $\Gamma(s)$ at k is canceled by the zero of $\cos(\pi s/2)$ at k. So, their product has no pole at k. **seen, 2**

When, $-k$ is even, $\cos(\pi k/2) = \pm 1$ and $\Gamma(s)$ has a pole at k. Thus, their product has a pole at k. **seen, 2**

Part b) The function $f(s) = \overline{\zeta(\overline{s})}$ is meromorohic on $\mathbb{C}$. Since for every real $s \ge 1$, $\zeta(s)$ is real, we have $f(s) = \zeta(s)$ on $(1,\infty)$. By the identity theorem, we must have $f(s) = \zeta(s)$ on $\mathbb{C}$. **seen, 3**

The relation $\zeta(\overline{s}) = \overline{\zeta(s)}$ shows that ρ is a zero of $\zeta(s)$ if and only if $\overline{\rho}$ is a zero of $\zeta(s)$. **seen, 3**

Also, for $\rho \in \mathbb{C}$ with $0 < \operatorname{Re}\rho < 1$, $\cos(\pi\rho/2) \neq 0$ and $\Gamma(\rho) \neq 0$. Hence, by the functional equation in part a), for ρ with $0 < \operatorname{Re}\rho < 1$, $\zeta(\rho) = 0$ if and only if $\zeta(1-\rho) = 0$. **seen, 3**

Solution to Question 5
Part a) We have

$$\begin{aligned}
s\int_1^\infty \frac{\Psi(x)}{x^{s+1}}\,dx &= s\sum_{i=1}^\infty \int_i^{i+1} \frac{\Psi(x)}{x^{s+1}}\,dx = s\sum_{i=1}^\infty \int_i^{i+1} \frac{\Psi(i)}{x^{s+1}}\,dx \\
&= -\sum_{i=1}^\infty \Psi(i)\big((i+1)^{-s} - i^{-s}\big) \\
&= -1\Big(\Psi(1)(2^{-s} - 1^{-s}) + \Psi(2)(3^{-s} - 2^{-s}) + \Psi(3)(4^{-s} - 3^{-s}) + \dots\Big) \\
&= \Psi(1)\cdot 1^{-s} + (\Psi(2) - \Psi(1))\cdot 2^{-s} + (\Psi(3) - \Psi(2))\cdot 3^{-s} + \dots \\
&= \sum_{n=1}^\infty \Lambda(n)\cdot n^{-s}.
\end{aligned}$$

unseen, 5

Part b) By a theorem in the lectures, $\zeta(s)$ has a simple pole of order 1 at 1 with residue equal to 1. Hence, we have a convergent Laurent series,

$$\zeta(s) = \frac{1}{s-1} + a_0 + a_1(s-1) + a_2(s-1)^3 + \dots$$

seen, 1

This implies that

$$\zeta'(s) = \frac{-1}{(s-1)^2} + a_1 + 2a_2(s-1) + 3a_3(s-1)^2 + \dots$$

This implies that

$$\lim_{s\to 1} \frac{\zeta'(s)}{\zeta(s)} = -1.$$

unseen, 2

Part c)

By the definition of the lim sup, given $\epsilon > 0$, there is $N(\epsilon) > 0$ such that for $x \geq N(\epsilon)$, $\Psi(x) \leq x(\delta + \epsilon)$. Then, for real s we have

$$\frac{-\zeta'(s)}{\zeta(s)} = s\int_1^{N(\epsilon)} \frac{\Psi(x)}{x^{s+1}}\,dx + s\int_{N(\epsilon)}^{\infty} \frac{\Psi(x)}{x^{s+1}}\,dx \leq sC(\epsilon) - s(\delta+\epsilon)\Big[\frac{x^{s-1}}{-s+1}\Big]$$
$$= sC(\epsilon) + (\delta+\epsilon)\frac{s}{s-1}N(\epsilon)^{-s+1}$$

unseen, 3

Multiplying both sides by $s-1$ and then taking limit we obtain $1 \leq \delta + \epsilon$. As ϵ was arbitrary, we conclude that $1 \leq \delta$. **unseen, 2**

Part d)

By the definition of the lim inf, given $\epsilon > 0$, there is $N'(\epsilon) > 0$ such that for $x \geq N'(\epsilon)$, $\Psi(x) \geq x(\gamma - \epsilon)$. Then, for real s we have

$$\frac{-\zeta'(s)}{\zeta(s)} = s\int_1^{N(\epsilon)} \frac{\Psi(x)}{x^{s+1}}\,dx + s\int_{N(\epsilon)}^{\infty} \frac{\Psi(x)}{x^{s+1}}\,dx$$
$$\geq sC'(\epsilon) + s(\gamma-\epsilon)\Big[\frac{x^{s-1}}{-s+1}\Big] = C'(\epsilon) + (\gamma-\epsilon)\frac{s}{s-1}N(\epsilon)^{-s+1}$$

unseen, 3

Multiplying both sides by $s-1$ and then taking limit we obtain $1 \leq \gamma - \epsilon$. As ϵ was arbitrary, we conclude that $1 \geq \gamma$. **unseen, 2**

Finally, if $\lim_{x\to\infty} \Psi(x)/x$ exists, we must have

$$1 \leq \delta = \limsup_{x\to\infty} \frac{\Psi(x)}{x} = \lim_{x\to\infty} \frac{\Psi(x)}{x} = \liminf_{x\to\infty} \frac{\Psi(x)}{x} = \gamma \leq 1.$$

Hence, $\lim_{x\to\infty} \Psi(x)/x = 1$. **unseen, 2**