Geometric Complex Analysis

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Introduction

The subject of complex variables appears in many areas of mathematics as it has been truly the ancestor of many subjects. It is employed in a wide range of topics, including, algebraic geometry, number theory, dynamical systems, and quantum field theory, to name a few. Basic examples and techniques in complex analysis have been developed over a century into sophistication methods in analysis. On the other hand, as the real and imaginary parts of any analytic function satisfy the Laplace equation, complex analysis is widely employed in the study of two-dimensional problems in physics, for instance in, hydrodynamics, thermodynamics, ferromagnetism, and percolation.

In complex analysis one often starts with a rather weak requirement (regularity) of differentiability. That is, a map $f : U \to \mathbb{C}$ is called *holomorphic* on Ω, if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists at every point in the open set $U \subseteq \mathbb{C}$. Then with little effort one concludes from the above property that f is infinity many times differentiable, and indeed it has a convergent power series. This is in a direct contrast with the notions of C^k regularities we have for real maps of Euclidean spaces. That is, there are C^k real maps that are not C^{k+1}, for any $k \geq 1$. Or, there are C^∞ real maps that have no convergent power series. The difference is rooted in the fact that here h tends to 0 in all directions, and there is a multiplication operation on the plane that interacts nicely with the addition. Due to this difference, complex analysis is not merely "extending the calculus to complex-valued functions"; rather it is a subject of mathematics on its own.

Let Ω be an open set in $\mathbb{C}$ that is bounded by a piece-wise smooth simple closed curve, and let $f : \omega \to \mathbb{C}$ be a holomorphic map. For any C^1 simple closed curve γ in Ω, if we know the values of f on γ, the Cauchy Integral Formula provides a simple formula for the values of f inside γ:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz.$$

Also, there is a similar formula for the higher order derivatives of f at any point inside γ. On the other hand, if we know all derivatives of f at some point $z_0 \in \Omega$, then the infinite series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

is convergent for z close enough to z_0, and the value of the series is equal to $f(z)$.

When the domain Ω enjoys some form of symmetry, for example, when Ω is the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

with rotational symmetry, the objects of interest in complex analysis often find simple algebraic forms. In Chapters 2 we prove some results of this nature, including,

Theorem 0.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a one-to-one and onto holomorphic mapping. Then, there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}.$$

Although the above type of results point to the restrictive nature of holomorphic property, there are also statements that go in the other direction. For example, in Chapter 5, we prove the Riemann mapping theorem, which, as a special case, implies the following.

Theorem 0.2. *Let $\Omega \subset \mathbb{C}$ be an open set that is bounded by a continuous simple closed curve, and let $z_0 \in \Omega$. Then, there is a one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$ with $f(0) = z_0$.*

The domain Ω in the above theorem may have a very complicated shape (geometry), or may have a highly irregular boundary (analysis) obtained from a randomly generated curve. See Figure 1.

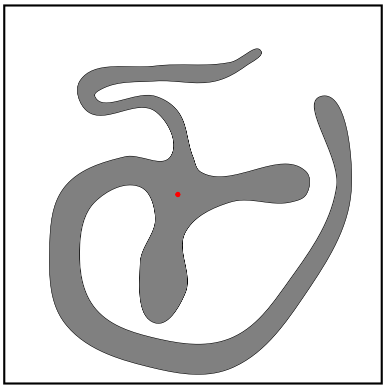

Figure 1: An arbitrary open set Ω bounded by a continuous simple closed curve.

The map f in the above theorem is called the *uniformlization* of the domain Ω. One aim of this course is to study the behavior of the uniformizations in connection with the geometric shape of Ω and its boundary. We also look for such geometric quantities that remain invariant under conformal mappings.

In applications, one often comes across domains Ω that have very complicated shapes, or very irregular boundaries. Although the above theorem provides us with a seemingly nice behaving map, there are little chances that we know the higher order derivatives of f at some $z_0 \in \Omega$ or the behavior of the map f on the boundary of $\mathbb{D}$ in order to use the Taylor series or the Cauchy Integral Formula to study the behavior of f. But, is it still possible to say something about the map f? As we shall see in Chapter 6 there are some *universal laws* that every one-to-one holomorphic map must obey. Let us give an example of this type. For an arbitrary $\theta \in [0, 2\pi)$, one map ask how fast the curve $r \mapsto f(re^{i\theta})$, for $r \in [0, 1)$, move away from 0, or how fast it may spiral about 0? In Chapter 6 we prove some results of the following type.

Theorem 0.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be an arbitrary one-to-one and holomorphic map normalized with $f(0) = 0$ and $f'(0) = 1$. Then, for every $\theta \in [0, 2\pi]$ and $r \in (0, 1)$ we have*

$$|\arg f'(re^{i\theta})| \leq 2 \log \frac{1+r}{1-r}. \tag{1}$$

While Theorem 0.2 is strong and general, its proof is far from constructive. On rare occasions we are able to provide a formula for the map f (a list of such examples appear in Chapter 5). This is a rather general theme in holomorphic mappings that we often know that a holomorphic function with some prescribed conditions exists, but we don't have a constructive approach to it.

In Chapter 7 we introduce the concept of quasi-conformal maps, a generalization of conformal maps. Roughly speaking, these are homeomorphisms whose first partial derivatives exist almost everywhere, and the Cauchy-Riemann condition is nearly satisfied (being small instead of 0). These maps naturally come up in complex analysis in several ways. It turns out that such maps still enjoy many properties of conformal maps, while having a more constructive nature. Many problems related to the behavior of conformal maps through quasi-conformal maps reduce to the study of a certain type of partial differential equation, where there are constructive approaches to the solutions.

Although the above method turns out to be unexpectedly powerful, we must remain humble. It is easy to pose simple looking open (and probably extremely hard) questions in complex analysis, for instance,

Brennans conjecture: For every one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$, and every real p with $-2 < p < 2/3$, we have

$$\int_{\mathbb{D}} |f'(re^{i\theta})|^p \, dxdy < \infty.$$

We will see in this course that geometric complications and irregularities in the boundary of Ω results in large and small values for $|f'|$. The above conjecture suggests some bounds on the average values of $|f'|$. This is part of a set of conjectures knows as *universal integral means spectrum*, and covers some conjectures of Littlewood on the extremal growth rate of the length of the closed curves $f(r \cdot e^{i\theta})$, $\theta \in [0, 2\pi]$, as r tends to 1. These questions are motivated by important problems in statistical physics.

The actual prerequisite for this course is quite minimal. We assume that the students taking this class are familiar with the notions of holomorphic maps and their basic properties. This is a concise math course with ε-δ proofs, and so precise forms of definitions and statements appear in the notes. To rectify the challenge of where we start, we have summarized in Chapter 1 (in three pages) the basic results from complex analysis that we will rely on.

I prepared these notes for the course Geometric Complex Analysis, M3/4/5P60, for the autumn term of 2016 at Imperial College London. I am very pleased with the maths department for agreeing to offer this course for the first time. Complex analysis with its surprises is one of the most beautiful areas of mathematics. You may help me to improve these notes by emailing me any comments or corrections you have.

Contents

Chapter 1

Preliminaries from complex analysis

1.1 Holomorphic functions

In this section we recall the key concepts and results from complex analysis.

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{C}$ denote the set of complex numbers. It is standard to write a point $z \in \mathbb{C}$ as $z = x + iy$, where x and y are real, and $i \cdot i = -1$. Here $x = \operatorname{Re} z$ is called the *real part* of z and $y = \operatorname{Im} z$ is called the *imaginary part* of z. With this correspondence $z \mapsto (x, y)$, $\mathbb{C}$ is heomeomorphic to $\mathbb{R}^2$.

Definition 1.1. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then f is called *differentiable* at a point $z \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is a finite complex number. This limit is denoted by $f'(z)$. The map f is called *holomorphic* (*analytic*) on Ω, if f is differentiable at every point in Ω.

It easily follows that if $f : \Omega \to \mathbb{C}$ is differentiable at $z \in \Omega$, then it is continuous at z.

It is important to note that in Definition 1.1 h tends to 0 in the complex plane. (This is rather an abuse of the terminology "differentiable", as we shall see in a moment!) In particular, h may tend to 0 in any direction. Let us write the map f in the real and imaginary coordinates as $f(x + iy) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real valued functions on Ω. When h tends to 0 in the horizontal direction, then

$$f'(z) = \lim_{x \to 0} \frac{f(z+x) - f(z)}{x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}. \tag{1.1}$$

On the other hand, if h tends to 0 in the vertical direction, that is, in the y direction, then

$$f'(z) = \lim_{y \to 0} \frac{f(z+iy) - f(z)}{iy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial f}{\partial y} \tag{1.2}$$

Then, if $f'(z)$ exists, we must have

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{1.3}$$

In terms of the coordinate functions u and v, we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.4}$$

The above equations are known as the *Cauchy-Riemann* equations. On the other hand, if u and v are real-valued functions on Ω that have continuous first partial derivatives satisfying Equation (1.4), then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on Ω.

Theorem 1.2 (Cauchy-Goursat theorem-first version)**.** *Let Ω be an open set in $\mathbb{C}$ that is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any piece-wise C^1 simple closed curve γ in Ω we have*

$$\int_\gamma f(z)\, dz = 0.$$

There is an important corollary of the above theorem, that we state as a separate statement for future reference.

Theorem 1.3 (Cauchy Integral Formula-first version)**.** *Let Ω be an open set in $\mathbb{C}$ that is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any C^1 simple closed curve γ in Ω and any point z_0 in the region bounded by γ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

The condition Ω bounded by a smooth simple closed curve is not quite necessary in the above two theorem. Indeed, you may have only seen the above theorems when Ω is a disk or a rectangle. We shall see a more general form of these theorems later in this course, where a topological feature of the domain Ω comes into play.

Theorem 1.3 reveals a remarkable feature of holomorphic mappings. That is, if we know the values of a holomorphic function on a simple closed curve, then we know the values of the function in the region bounded by that curve, provided we *a priori* know that the function is holomorphic on the region bounded by the curve.

There is an analogous formula for the higher derivatives of holomorphic maps as well [1]. Under the assumption of Theorem 1.3, and every integer $n \geq 1$, the n-th derivative of f at z_0 is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \tag{1.5}$$

[1] Chauchy had proved Theorem 1.2 when the complex derivative $f'(z)$ exists and is a continuous function of z. Then, Édouard Goursat proved that Theorem 1.2 can be proven assuming only that the complex derivative $f'(z)$ exists everywhere in Ω. Then this implies Theorem 1.3 for these functions, and from that deduce these functions are in fact infinitely differentiable.

In Definition 1.1, we only assumed that the first derivative of f exists. It is remarkable that this seemingly weak condition leads to the existence of higher order derivatives. Indeed, an even stronger statement holds.

Theorem 1.4 (Taylor-series)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on an open set $\Omega \subseteq \mathbb{C}$. For every $z_0 \in \Omega$, the infinite series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

is absolutely convergent for z close to z_0, with the value of the series equal to $f(z)$.

The above theorems are in direct contrast with the regularity properties we know for real maps on $\mathbb{R}$ or on $\mathbb{R}^n$. That is, we have distinct classes of differentiable functions, C^1 functions, C^2 functions, C^∞ functions, real analytic functions (C^ω). For any k, it is possible to have a function that is C^k but not C^{k+1} (Find an example if you already don't know this). There are C^∞ functions that are not real analytic. For example, the function defined as $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$. But these scenarios don't exist for complex differentiable functions.

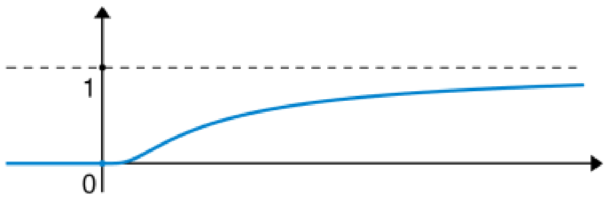

Figure 1.1: The graph of the function f.

Since a holomorphic function $f : \Omega \to \mathbb{C}$ is infinitely differentiable, higher order partial derivatives of u and v exist and are continuous. Differentiating Equations (1.4) with respect to x and y, and using $\partial_x \partial_y v = \partial_y \partial_x v$ and $\partial_x \partial_y u = \partial_y \partial_x u$, we conclude that the real functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are harmonic, that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

hold on Ω. We state this as a separate theorem for future reference.

Theorem 1.5 (Harmonic real and imaginary parts)**.** *Let $f(x + iy) = u(x, y) + iv(x, y)$ be a holomorphic function defined on an open set Ω in $\mathbb{C}$. Then, $u(x, y)$ and $v(x, y)$ are harmonic functions on Ω.*

A pair of harmonic functions u and v defined on the same domain $\Omega \subseteq \mathbb{C}$ are called *harmonic conjugates*, if they satisfy the Cauchy-Riemann equation, in other words, the function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic.

Theorem 1.6 (maximum principle)**.** *If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function defined on an open set Ω, then its absolute value $|f(z)|$ has no maximum in Ω. That is, there is no $z_0 \in \Omega$ such that for all $z \in \Omega$ we have $|f(z)| \leq |f(z_0)|$.*

On the other hand, under the same conditions, either f has a zero on Ω or $|f(z)|$ has no minimum on Ω.

Let K be an open set in Ω such that the closure of K is contained in Ω. If $f : \Omega \to \mathbb{C}$ is an analytic function, $|f(z)|$ is continuous on K and by the extreme value theorem, $|f|$ has a maximum on the closure of K. But by the above theorem, $|f|$ has no maximum on K. This implies that the maximum of $|f|$ must be realized on the boundary of K. Similarly, the minimum of $|f|$ is also realized on the boundary of K.

Chapter 2

Schwarz lemma and automorphisms of the disk

2.1 Schwarz lemma

We denote the disk of radius 1 about 0 by the notation $\mathbb{D}$, that is,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given $\theta \in \mathbb{R}$ the rotation of angle θ about 0, i.e. $z \mapsto e^{i\theta} \cdot z$, preserves $\mathbb{D}$. Due to the rotational symmetry of $\mathbb{D}$ most objects studied in complex analysis find special forms on $\mathbb{D}$ that have basic algebraic forms. We study some examples of these in this section, and will see more on this later on.

A main application of the maximum principle (Theorem 1.6) is the lemma of Schwarz. It has a simple proof, but has far reaching applications.

Lemma 2.1 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then,*

(i) for all $z \in \mathbb{D}$ we have $|f(z)| \leq |z|$;

(ii) $|f'(0)| \leq 1$;

(iii) if either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then f is a rotation about 0.

Proof. Since f is holomorphic on $\mathbb{D}$, we have a series expansion for f centered at 0,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

that is convergent on $\mathbb{D}$. Since $f(0) = 0$, $a_0 = 0$, and we obtain

$$f(z) = a_1 z + a_2 z^2 + \dots = z(a_1 + a_2 z + a_3 z^2 + \dots).$$

In particular, the series in the above parenthesis is convergent on $\mathbb{D}$. In particular, the function $g(z) = f(z)/z = a_1 + a_2 z + a_3 z^2 + \dots$ is defined and holomorphic on $\mathbb{D}$. Note that $g(0) = a_1 = f'(0)$.

On each circle $|z| = r < 1$, as $|f(z)| < 1$, we have

$$|g(z)| < \frac{1}{r}.$$

Then by the maximum principle, we must have the above inequality on $|z| < r$. Taking limit as $r \to 1$ from the left, we conclude that on $\mathbb{D}$, $|g(z)| \leq 1$. This implies part (i) and (ii) of the lemma.

To prove part (iii) of the lemma, note that if any of the two equality holds, then g attains its maximum in the interior of $\mathbb{D}$. Then, by the maximum principle, g must be a constant on $\mathbb{D}$, say $g(z) \equiv a$. Then, either of the relations $|f'(0)| = 1$ and $f(z) = z$ for some $z \neq 0$, implies that $|a| = 1$. This finishes the proof of part (iii). □

As an application of the Schwarz lemma we classify the one-to-one and onto holomorphic mappings of $\mathbb{D}$.

2.2 Automorphisms of the disk

Definition 2.2. Let U and V be open subsets of $\mathbb{C}$. A holomorphic map $f : U \to V$ that is one-to-one and onto is called a *biholomorphism* from U to V. A biholomorphism from U to U is called an *automorphism* of U. The set of all automorphisms of U is denoted by $\mathrm{Aut}(U)$.

Obviously, $\mathrm{Aut}(U)$ contains the identity map and hence is not empty. The composition of any two maps in $\mathrm{Aut}(U)$ is again an element of $\mathrm{Aut}(U)$. Indeed, $\mathrm{Aut}(U)$ forms a group with this operation.

Proposition 2.3. *For every non-empty and open set U in $\mathbb{C}$, $\mathrm{Aut}(U)$ forms a group with the operation being the composition of the maps.*

Proof. The composition of any pair of one-to-one and onto holomorphic maps from U to U is a one-to-one and onto holomorphic map from U to itself. Thus the operation is well defined on $\mathrm{Aut}(U)$. The identity map $z \mapsto z$ is the identity element of the group.

The associativity $(f \circ g) \circ h = f \circ (g \circ h)$ holds because the relation is valid for general maps. The inverse of every $f \in \mathrm{Aut}(U)$ is given by the inverse mapping f^{-1}. Note that the inverse of any one-to-one map is defined and is a holomorphic map. □

We have already seen that every rotation $z \mapsto e^{i\theta} \cdot z$, for a fixed $\theta \in \mathbb{R}$, is an automorphism of the disk. For $a \in \mathbb{D}$ define

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Lemma 2.4. *For every $a \in \mathbb{D}$, φ_a is an automorphism of $\mathbb{D}$.*

Proof. First note that φ_a is defined and holomorphic at every $z \in \mathbb{C}$, except at $z = 1/\overline{a}$ where the denominator becomes 0. However, since $|a| < 1$, we have $|1/\overline{a}| = 1/|a| > 1$, and therefore, $1/\overline{a} \notin \mathbb{D}$. Hence, φ_a is holomorphic on $\mathbb{D}$.

To see that φ_a maps $\mathbb{D}$ into $\mathbb{D}$, fix an arbitrary $z \in \mathbb{C}$ with $|z| = 1$. Observe that

$$|\varphi_a(z)| = \left|\frac{a - z}{1 - \overline{a}z}\right| = \left|\frac{a - z}{1 - \overline{a}z}\right| \cdot \frac{1}{|\overline{z}|} = \left|\frac{a - z}{\overline{z} - \overline{a}}\right| = 1,$$

since $z\overline{z} = |z|^2 = 1$. By the maximum principle (Theorem 1.6), $|\varphi_a(z)| < 1$ on $\mathbb{D}$.

We observe that

$$\varphi_a(a) = 0, \text{ and } \varphi_a(0) = a.$$

Then,

$$\varphi_a \circ \varphi_a(0) = 0, \text{ and } \varphi_a \circ \varphi_a(a) = a.$$

By the Schwarz lemma, this implies that $\varphi_a \circ \varphi_a$ must be the identity map of $\mathbb{D}$. It follows that φ_a is both one-to-one and onto from $\mathbb{D}$ to $\mathbb{D}$. □

It turns out that the composition of rotations and the maps of the form φ_a are all the possible automorphisms of $\mathbb{D}$.

Theorem 2.5. *A map f is an automorphism of $\mathbb{D}$ iff there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \cdot \frac{a - z}{1 - \overline{a}z}.$$

Proof. Let f be an element of $\mathrm{Aut}(\mathbb{D})$. Since f is onto, there is $a \in \mathbb{D}$ with $f(a) = 0$. The map

$$g = f \circ \varphi_a$$

is an automorphism of $\mathbb{D}$ with $g(0) = 0$. By the Schwarz lemma, we must have $|g'(0)| \leq 1$. Applying the Schwarz lemma to g^{-1} we also obtain $|(g^{-1})'(0)| \leq 1$. By the two inequalities, we have $|g'(0)| = 1$. Thus, by the same lemma, $g(z) = e^{i\theta} \cdot z$, for some $\theta \in \mathbb{R}$. That is, $f \circ \varphi_a(z) = e^{i\theta}z$, for $z \in \mathbb{D}$. Since $\varphi_a \circ \varphi_a$ is the identity map, we conclude that $f(z) = e^{i\theta}\varphi_a(z)$.

On the other hand, for any $\theta \in \mathbb{R}$ and any $a \in \mathbb{D}$, f belongs to $\mathrm{Aut}(\mathbb{D})$. That is because, f is the composition of the automorphism φ_a (Lemma 2.4) and a rotation. □

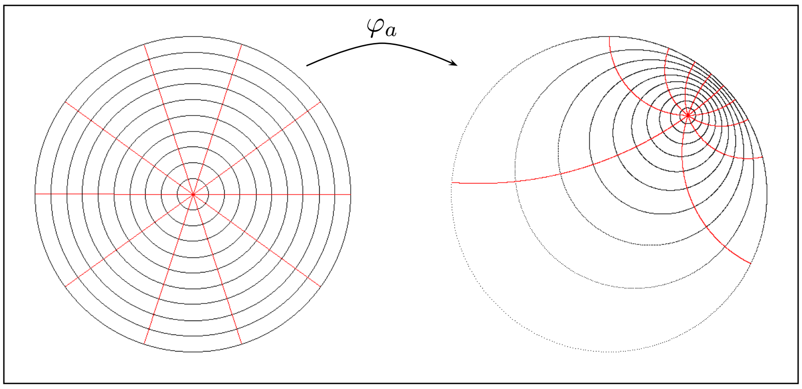

Figure 2.1: The images of the circles and rays under the map φ_a where $a = 0.5 + i0.5$.

2.3 Automorphisms of the half-plane

Define the upper half plane $\mathbb{H}$ as

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

There are biholomorphic maps between $\mathbb{D}$ and $\mathbb{H}$ given by explicit formulae

$$F : \mathbb{H} \to \mathbb{D}, \quad F(z) = \frac{i - z}{i + z}, \quad F(i) = 0 \tag{2.1}$$

and

$$G : \mathbb{D} \to \mathbb{H}, \quad G(w) = i \cdot \frac{1 - w}{1 + w}, \quad G(0) = i.$$

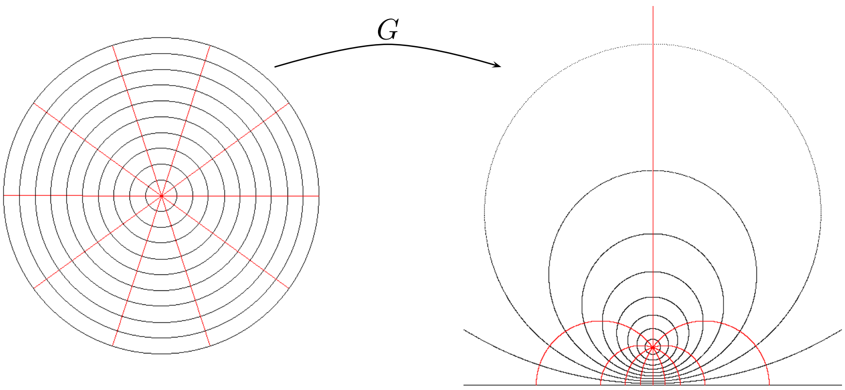

Figure 2.2: Similar line colors are mapped to one another by F.

Lemma 2.6. *The map $F : \mathbb{H} \to \mathbb{D}$ is a biholomorphic map with inverse $G : \mathbb{D} \to \mathbb{H}$.*

The proof of the above lemma is elementary and is left to the reader as an exercise.

The explicit biholomorphic map F allows us to identify $\mathrm{Aut}(\mathbb{H})$ in terms of $\mathrm{Aut}(\mathbb{D})$. That is, define

$$\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H}) \text{ as } \Gamma(\varphi) = F^{-1} \circ \varphi \circ F.$$

It is clear that if $\varphi \in \mathrm{Aut}(\mathbb{D})$ then $\Gamma(\varphi) = F^{-1} \circ \varphi \circ F \in \mathrm{Aut}(\mathbb{H})$. The map Γ is one-to-one and onto with inverse given by $\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$. Indeed, Γ is more than just a bijection, it also preserves the operations on the groups $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$. To see this, assume that φ_1 and φ_2 belong to $\mathrm{Aut}(\mathbb{D})$.

$$\Gamma(\varphi_1 \circ \varphi_2) = F^{-1} \circ (\varphi_1 \circ \varphi_2) \circ F = F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F = \Gamma(\varphi_1) \circ \Gamma(\varphi_2).$$

The isomorphism $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ show that indeed the two groups are the same. However, we still would like to have explicit formulas for members of $\mathrm{Aut}(\mathbb{H})$. Using the explicit formulas for F and G, as well as the explicit formulas for elements of $\mathrm{Aut}(\mathbb{D})$ in Theorem 2.5, a long series of calculations shows that an element of $\mathrm{Aut}(\mathbb{H})$ is of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, and d are real, and $ad - bc = 1$. We shall present an alternative proof of this, but before doing that we introduce some notations that simplify the presentation.

Define

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

The set $\mathrm{SL}_2(\mathbb{R})$ forms a group with the operation of matrix-multiplication. This is called the *special linear group*. To each matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$ we associate the map

$$f_M(z) = \frac{az+b}{cz+d}.$$

It is a straightforward calculation to see that for every M and M' in $\mathrm{SL}_2(\mathbb{R})$ we have

$$f_M \circ f_{M'} = f_{M \cdot M'}. \tag{2.2}$$

That is, the correspondence $M \mapsto f_M$ respects the group operations.

Theorem 2.7. *For every $M \in \mathrm{SL}_2(\mathbb{R})$ the map f_M is an automorphism of $\mathbb{H}$. Conversely, every automorphism of $\mathbb{H}$ is of the form f_M for some M in $\mathrm{SL}_2(\mathbb{R})$.*

Proof. We break the proof into several steps.

Step 1. Let $M \in \mathrm{SL}_2(\mathbb{R})$. The map f_M is holomorphic on $\mathbb{H}$. Moreover, for every $z \in \mathbb{H}$ we have

$$\operatorname{Im} f_M(z) = \operatorname{Im} \frac{az+b}{cz+d} = \operatorname{Im} \frac{(az+b)(\overline{cz+d})}{(cz+d)(\overline{cz+d})} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0.$$

Thus, f_M maps $\mathbb{H}$ into $\mathbb{H}$. As $\mathrm{SL}_2(\mathbb{R})$ forms a group, there is a matrix M^{-1} in $\mathrm{SL}_2(\mathbb{R})$ such that $M \cdot M^{-1} = M^{-1} \cdot M$ is the identity matrix. It follows that $f_M \circ f_{M^{-1}}$ is the identity map of $\mathbb{H}$. In particular, f_M is both one-to-one and onto. This proves the first part of the theorem.

Step 2. Let $h \in \operatorname{Aut}(\mathbb{H})$ with $h(i) = i$. Define $I = F \circ h \circ F^{-1}$, where F is the map in Equation (2.1). Then, $I \in \operatorname{Aut}(\mathbb{D})$ and $I(0) = 0$. Also, I^{-1} is a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ that sends 0 to 0. By the Schwarz lemma, $|I'(0)| \leq 1$, and $|(I^{-1})'(0)| \leq 1$. Thus, by the Schwarz lemma, I must be a rotation about 0, that is, there is $\theta \in \mathbb{R}$ such that

$$F \circ h \circ F^{-1}(z) = e^{i\theta} \cdot z. \tag{2.3}$$

Step 3. Let

$$Q = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

The matrix Q belongs to $\mathrm{SL}_2(\mathbb{R})$, and one can verify that $f_Q(i) = i$ and $f'_Q(i) = e^{i\theta}$. Then the map $F \circ f_Q \circ F^{-1}$ is an automorphism of $\mathbb{D}$ that maps 0 to 0 and has derivative $e^{i\theta}$ at 0. By the Schwarz lemma, $F \circ f_Q \circ F^{-1}$ is the rotation $z \mapsto e^{i\theta} \cdot z$. That is, $F \circ f_Q \circ F^{-1} = F \circ h \circ F^{-1}$. Since F is one-to-one we conclude that $h = f_Q$, where h is the map in Step 1.

Step 4. We claim that for every $z_0 \in \mathbb{H}$ there is $N \in \mathrm{SL}_2(\mathbb{R})$ such that $f_N(i) = z_0$. First we choose a re-scaling about 0 that maps i to a point whose imaginary part is equal to $\operatorname{Im} z_0$. This map is given by the matrix

$$O = \begin{pmatrix} \sqrt{\operatorname{Im} z_0} & 0 \\ 0 & 1/\sqrt{\operatorname{Im} z_0} \end{pmatrix},$$

that is, $\operatorname{Im} f_O(i) = \operatorname{Im} z_0$. Then we use the translation $z + (z_0 - f_O(i))$ to map $f_O(i)$ to z_0. The latter map is obtained from the matrix

$$P = \begin{pmatrix} 1 & z_0 - f_M(i) \\ 0 & 1 \end{pmatrix}.$$

The map $f_P \circ f_O = f_{P \cdot O}$ maps i to z_0. Set $N = P \cdot O$.

Step 5. Let g be an automorphism of $\mathbb{H}$. There is z_0 in $\mathbb{H}$ with $g(z_0) = i$. By Step 4, there is $N \in \mathrm{SL}_2(\mathbb{R})$ such that $f_N(i) = z_0$. The composition $h = g \circ f_N$ belongs to $\mathrm{Aut}(\mathbb{H})$ and sends i to i. Thus, by Steps 2 and 3, $h = f_Q$. Now, using Equation (2.2), we have

$$g = h \circ (f_N)^{-1} = f_Q \circ f_{N^{-1}} = f_{Q \cdot N^{-1}}.$$

This shows that g has the desired form. $\square$

2.4 Exercises

Exercise 2.1. For $a \in \mathbb{C}$ and $r > 0$ let $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, that is, the open disk of radius r about a. Let a and b be arbitrary points in $\mathbb{C}$, and let r and s be positive real numbers. Prove that for every holomorphic map $f : B(a, r) \to B(b, s)$ with $f(a) = b$ we have $|f'(a)| \leq s/r$.

Exercise 2.2. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map.

(i) Prove that for every $a \in \mathbb{D}$ we have

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2},$$

(ii) Prove that for every a and b in $\mathbb{D}$ we have

$$\left| \frac{f(a) - f(b)}{1 - f(a)\overline{f(b)}} \right| \leq \left| \frac{a - b}{1 - a\bar{b}} \right|.$$

The above inequalities are known as the Schwarz-Pick lemma.

Exercise 2.3. Let $h : \mathbb{H} \to \mathbb{H}$ be a holomorphic map. Prove that for every $a \in \mathbb{H}$ we have

$$|h'(a)| \leq \frac{\operatorname{Im} h(a)}{\operatorname{Im} a}.$$

Exercise 2.4. Prove that for every z and w in $\mathbb{D}$ there is $f \in \mathrm{Aut}(\mathbb{D})$ with $f(z) = w$.

[For an open set $U \subseteq \mathbb{C}$, we say that $\mathrm{Aut}(U)$ acts *transitively* on U, if for every z and w in U there is $f \in \mathrm{Aut}(U)$ such that $f(z) = w$. By the above statements, $\mathrm{Aut}(\mathbb{D})$ act transitively on $\mathbb{D}$.

Exercise 2.5. Prove Lemma 2.6

Chapter 3

Riemann sphere and rational maps

3.1 Riemann sphere

It is sometimes convenient, and fruitful, to work with holomorphic (or in general continuous) functions on a compact space. However, we wish to still "keep" all of $\mathbb{C}$ in the space we work on, but see it as a subset of a compact space. There are sequences in $\mathbb{C}$ that have no sub-sequence converging to a point in $\mathbb{C}$. The least one needs to do is to add the limiting values of convergent sub-sequences to $\mathbb{C}$. It turns out that one may achieve this by adding a single point to $\mathbb{C}$ in a suitable fashion. We denote this point with the notation ∞. Below we discuss the construction in more details.

Let us introduce the notation $\hat{\mathbb{C}}$ for the set $\mathbb{C} \cup \{\infty\}$, where ∞ is an element not in $\mathbb{C}$. The arithmetic on $\mathbb{C}$ may be extended, to some extent, by assuming that

- for all finite $a \in \mathbb{C}$, $\infty + a = a + \infty = \infty$.
- for all non-zero $b \in \mathbb{C} \cup \{\infty\}$, $b \cdot \infty = \infty \cdot b = \infty$.

Remark 3.1. It is not possible to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. But, by convention, for $a \in \mathbb{C} \setminus \{0\}$ we write $a/0 = \infty$, and for $b \in \mathbb{C}$ we write $b/\infty = 0$.

We "attach" the point ∞ to $\mathbb{C}$ by requiring that every sequence $z_i \in \mathbb{C}$, for $i \geq 1$, with $|z_i|$ diverging to infinity converges to ∞. This is rather like adding the point $+1$ to the set $(0, 1)$. With this definition, it is easy to see that every sequence in $\hat{\mathbb{C}}$ has a convergent sub-sequence. We have also kept a copy of $\mathbb{C}$ in $\hat{\mathbb{C}}$.

There is a familiar model for the set $\mathbb{C} \cup \{\infty\}$ obtain from a process known as "stereographic projection". To see that, let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let $N = (0, 0, 1) \in S$. We define a homeomorphism $\pi : S \to \hat{\mathbb{C}}$ as follows. Let $\pi(N) = \infty$, and for every point $(x_1, x_2, x_3) \neq N$ in S define

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}. \tag{3.1}$$

By the above formula,

$$|\pi(X)|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3},$$

which implies that

$$x_3 = \frac{|\pi(X)|^2 - 1}{|\pi(X)|^2 + 1}, \quad x_1 = \frac{\pi(X) + \overline{\pi(X)}}{1 + |\pi(X)|^2}, \quad x_2 = \frac{\pi(X) - \overline{\pi(X)}}{i(1 + |\pi(X)|^2)}.$$

The above relations imply that π is one-to-one and onto.

The continuity of π at every point on $S \setminus \{N\}$ is evident from the formula. To see that π is continuous at N, we observe that if X tends to N on S, then x_3 tends to $+1$ from below. This implies that $|\pi(X)|$ tends to $+\infty$, that is, $\pi(X)$ tends to ∞ in $\hat{\mathbb{C}}$.

If we regard the plane $(x_1, x_2, 0)$ in $\mathbb{R}^3$ as the complex plane $x_1 + ix_2$, there is a nice geometric description of the map π, called stereographic projection. That is the points N, X, and $\pi(X)$ lie on a straight line in $\mathbb{R}^3$. See Figure 3.1.

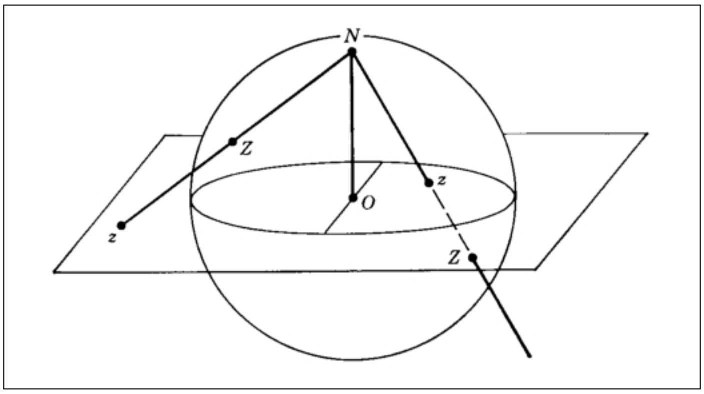

Figure 3.1: Presentation of the map π.

The set $\hat{\mathbb{C}}$, with the convergence of sequences described above, is known as the *Riemann sphere*. In view of the above construction, as we know S as a symmetric space, $\hat{\mathbb{C}}$ should be also viewed as a symmetric space. To discuss this further, we need to give some basic definitions.

Let Ω be an open set in $\mathbb{C}$. Recall that $f : \Omega \to \mathbb{C}$ is called continuous at a point $z \in \Omega$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z' \in \Omega$ with $|z - z'| < \delta$ we have $|f(z) - f(z')| < \varepsilon$. This is equivalent to saying that f is continuous at z if and only if for every sequence z_n, $n \geq 1$, in Ω that converges to z, the sequence $f(z_n)$ converges to $f(z)$.

We use the above idea to define the notion of continuity for maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. That is, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called continuous at $z \in \hat{\mathbb{C}}$, if every sequence that converges to z is mapped by f to a sequence that converges to $f(z)$.

When f maps ∞ to ∞, the continuity of f at ∞ is equivalent to the continuity of the map $z \mapsto 1/f(1/z)$ at 0. Similarly, when $f(\infty) = a \neq \infty$, the continuity of f at ∞ is equivalent to the continuity of the map $z \mapsto f(1/z)$ at 0. When $f(a) = \infty$ for some $a \in \mathbb{C}$, the continuity of f at a is equivalent to the continuity of the map $z \mapsto 1/f(z)$ at a.

As usual, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called continuous, if it is continuous at every point in $\hat{\mathbb{C}}$.

Definition 3.2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a continuous map and $a \in \hat{\mathbb{C}}$. Then,

(i) When $a = \infty$ and $f(a) = \infty$, we say that f is holomorphic at a if the map $z \mapsto 1/f(1/z)$ is holomorphic at 0.

(ii) If $a = \infty$ and $f(a) \neq \infty$, then f is called holomorphic at a if the map $z \mapsto f(1/z)$ is holomorphic at 0.

(iii) If $a \neq \infty$ but $f(a) = \infty$, then f is called holomorphic at a if the map $z \mapsto 1/f(z)$ is holomorphic at a.

Continuous and Holomorphic maps from $\mathbb{C}$ to $\hat{\mathbb{C}}$, from $\mathbb{D}$ to $\hat{\mathbb{C}}$, and vice versa, are defined accordingly.

Example 3.3. You have already seen that every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is holomorphic from $\mathbb{C}$ to $\mathbb{C}$. As z tends to ∞ in $\mathbb{C}$, $P(z)$ tends to ∞ in $\mathbb{C}$. Hence, we may extend P to a continuous map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ by defining $P(\infty) = \infty$. To see whether P is holomorphic at ∞ we look at

$$\frac{1}{P(1/z)} = \frac{z^n}{a_n + a_{n-1}z + \cdots + a_0 z^n},$$

which is well-defined and holomorphic near 0. When $n > 1$, the complex derivative of the above map at 0 is equal to 0. When $n = 1$, its derivative becomes $1/a_1$. Thus, P is a holomorphic map of $\hat{\mathbb{C}}$.

Proposition 3.4. *If $f : \hat{\mathbb{C}} \to \mathbb{C}$ is a holomorphic map, then f is a constant map.*

Proof. We break the proof into several steps.

Step1. There is $z_0 \in \hat{\mathbb{C}}$ such that for all $z \in \hat{\mathbb{C}}$ we have $|f(z)| \leq |f(z_0)|$. That is, $|f|$ attains its maximum value at some point.

To prove the above statement, first we note that there is $M > 0$ such that for all $z \in \hat{\mathbb{C}}$, we have $|f(z)| \leq M$. If this is not the case, there are $z_n \in \hat{\mathbb{C}}$, for $n \geq 1$, with $|f(z_n)| \geq n$. As $\hat{\mathbb{C}}$ is a compact set, the sequence z_n has a sub-sequence, say z_{n_k} that converges to some point $w \in \hat{\mathbb{C}}$. By the continuity of f we must have $f(w) = \lim_{k\to+\infty} f(z_{n_k}) = \infty$. This contradicts with $f : \hat{\mathbb{C}} \to \mathbb{C}$.

The set $V = \{|f(z)| : z \in \hat{\mathbb{C}}\}$ is a subset of $\mathbb{R}$, and by the above paragraph, it is bounded from above. In particular, V has a supremum, say s. For any $n \geq 1$, since s is the least upper bound, there is $z_n \in \hat{\mathbb{C}}$ such that $|f(z_n)| \geq s - 1/n$. The sequence z_n is contained in the compact set $\hat{\mathbb{C}}$. Thus, there is a sub-sequence z_{n_l}, for $l \geq 1$, that converges to some point z_0 in $\hat{\mathbb{C}}$. It follows from the continuity of $|f(z)|$ that $|f(z_0)| = s$. Therefore, for all $z \in \hat{\mathbb{C}}$, $|f(z)| \leq |f(z_0)|$.

Step 2. If $z_0 \in \mathbb{C}$, then the map $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f|$ attains its maximum value at a point inside $\mathbb{C}$. By the maximum principle, f must be constant on $\mathbb{C}$. Then, by the continuity of $f : \hat{\mathbb{C}} \to \mathbb{C}$, we conclude that f is constant on $\hat{\mathbb{C}}$.

Step 3. If $z_0 = \infty$, then we look at the map $h(z) = f(1/z)$. Buy definition, $h : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|h|$ attains its maximum value at 0. Again, by the maximum principle, h must be constant on $\mathbb{C}$. Equivalently, f is constant on $\hat{\mathbb{C}} \setminus \{0\}$. As in the above paragraph, the continuity of $f : \hat{\mathbb{C}} \to \mathbb{C}$, implies that indeed f is constant on $\hat{\mathbb{C}}$. □

Example 3.5. The exponential map $z \mapsto e^z$ is holomorphic from $\mathbb{C}$ to $\mathbb{C}$. As z tends to infinity along the positive real axis, e^z tends to ∞ along the positive real axis. But as z tends to ∞ along the negative real axis, e^z tends to 0. Hence there is no continuous extension of the exponential map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Definition 3.6. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map. We say that f has a *zero of order* $k \in \mathbb{N}$ at $z_0 \in \Omega$, if $f^{(i)}(z_0) = 0$ for $0 \leq i \leq k-1$, and $f^{(k)}(z_0) \neq 0$. Similarly, we can say that f attains value w_0 at z_0 of order k, if z_0 is a zero of order k for the function $z \mapsto f(z) - w_0$. Here, the series expansion of f at z_0 has the form $f(z) = w_0 + a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} \dots$, with $a_k \neq 0$.

Definition 3.7. Definition 3.6 may be extended to holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. That is, we say that f attains ∞ at $z_0 \in \mathbb{C}$ of order k, if z_0 is a zero of order k for the map $z \mapsto 1/f(z)$. Then, near z_0 we have

$$1/f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + a_{k+2}(z-z_0)^{k+2} + \dots.$$

This implies that

$$\begin{aligned}
f(z) &= \frac{1}{a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + a_{k+2}(z-z_0)^{k+2} + \dots} \\
&= \frac{1}{z^k(a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \dots)} \\
&= \frac{1}{(z-z_0)^k}(b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots) \\
&= \frac{b_0}{(z-z_0)^k} + \frac{b_1}{(z-z_0)^{k-1}} + \frac{b_2}{(z-z_0)^{k-2}} + \dots.
\end{aligned}$$

Recall that z_0 is also called *a pole of order* k for f.

Similarly, if $f(\infty) = \infty$, we say that f attains ∞ at ∞ of order k, if the map $z \mapsto 1/f(1/z)$ has a zero of order k at 0

Proposition 3.8. *Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map such that for every $z \in \mathbb{C}$, $g(z) \in \mathbb{C}$. Then, g is a polynomial.*

Proof. The map g has a convergent power series on all of $\mathbb{C}$ as

$$g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots.$$

We consider two possibilities.

If $g(\infty) \neq \infty$, then $g : \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic and by Proposition 3.4, g must be constant on $\hat{\mathbb{C}}$. Therefore, $g(z) \equiv a_0$ is a polynomial.

The other possibility is that $g(\infty) = \infty$. To understand the behavior of g near ∞, we consider the map $h(w) = 1/g(1/w)$ near 0. We have $h(0) = 0$. Let $n \geq 1$ be the order of 0 at 0 for the map h, that is, $h(w) = a_n w^n + a_{n+1} w^{n+1} + \dots$ near 0. This implies that there is $\delta > 0$ such that for $|w| \leq \delta$ we have

$$|h(w)| \geq \frac{|a_n w^n|}{2}.$$

In terms of g, this means that for $|z| \geq 1/\delta$ we have $|g(z)| \leq 2|z^n|/|a_n|$. Then, by the Cauchy integral formula for the derivatives, for every $j \geq n+1$ and $R > 0$ we have

$$g^{(j)}(0) = \frac{j!}{2\pi i} \int_{\partial B(0,R)} \frac{g(z)}{z^{j+1}}\, dz.$$

Then, for $R > 1/\delta$,

$$|g^{(j)}(0)| \leq \frac{2 \cdot j!}{2\pi |a_n|} \int_{\partial B(0,R)} \frac{|z|^n}{|z^{j+1}|}\, dz \leq \frac{2 \cdot j!}{2\pi |a_n|} \cdot 2\pi R \cdot \frac{1}{R^{j+1-n}}.$$

Now we let $R \to +\infty$, and conclude that for all $j \geq n+1$, $g^{(j)}(0) = 0$. Therefore, for all $j \geq n+1$, $a_j = g^{(j)}(0)/j! = 0$, and thus, g is a polynomial of degree n. □

3.2 Rational functions

Example 3.9. If $Q(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a polynomial, then Q attains ∞ of order n at ∞. The map $1/Q(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is well-defined and holomorphic. At every point z_0 where $Q(z_0) \neq 0$, $1/Q(z)$ is well defined near z_0. If z_0 is a zero of order k for $Q(z)$, then $1/Q(z_0) = \infty$ and z_0 is a pole of order k.

Definition 3.10. If P and Q are polynomials, the map $z \mapsto P(z)/Q(z)$ is a well-defined holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Any such map is called a *rational function*.

Theorem 3.11. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Then, there are polynomials $P(z)$ and $Q(z)$ such that*

$$f(z) = \frac{P(z)}{Q(z)}.$$

Before we present a proof of the above theorem, we recall a basic result from complex analysis.

Proposition 3.12. *Let Ω be a connected and open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map. Assume that there is a sequence of distinct points z_j in Ω converging to some $z \in \Omega$ such that f takes the same value on the sequence z_j. Then, f is constant on Ω.*

In the above proposition, the connectivity of Ω is necessary and is imposed to avoid trivial counter examples. For example, one may set $\Omega = \mathbb{D} \cup (\mathbb{D} + 5)$ and defined f as $+1$ on $\mathbb{D}$ and as -1 on $\mathbb{D}+5$. It is also necessary to assume that the limiting point z belongs to Ω. For instance, the map $\sin(1/z)$ is defined and holomorphic on $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and has a sequence of zeros at points $1/(2\pi n)$, but it is not identically equal to 0.

Proof. Without loss of generality we may assume that the value of f on the sequence z_j is 0 (otherwise consider $f - c$). Since f is holomorphic at z, it has a convergent power series for ζ in a neighborhood of z as

$$f(\zeta) = a_1(\zeta - z) + a_2(\zeta - z)^2 + a_3(\zeta - z)^3 + \dots.$$

If f is not identically equal to 0, there the smallest integer $n \geq 1$ with $a_n \neq 0$. Then, $f(\zeta) = (\zeta - z)^{n-1} \cdot h(\zeta)$, for some holomorphic function h defined on a neighborhood U of z with $h(z) \neq 0$. But for large enough j, z_j belongs to U and we have $f(z_j) = 0$. This is a contradiction that shows for all $n \geq 1$, $a_n = 0$. In particular, f is identically 0 on U.

Let us define the set $E \subseteq \Omega$ as the set of points w in Ω such that for all $n \geq 1$ we have $f^{(n)}(w) = 0$. By the above paragraph, E contains z and hence it is not empty. Also, the argument shows that E is an open subset of Ω (see Theorem 1.4).

If $E = \Omega$ then we are done and f is identically equal to 0. Otherwise, there must be an integer $n \geq 1$ and $w \in \Omega$ such that $f^{(n)}(w) \neq 0$. Let us define the sets

$$F_n = \{w \in \Omega : f^{(n)}(w) \neq 0\}, \text{for } n \geq 1.$$

By the continuity of the map $z \mapsto f^{(n)}(z)$ on Ω, each F_n is an open set. In particular, the union $F = \cup_{n \geq 1} F_n$ is an open set. Now, $\Omega = E \cup F$, where E and F are non-empty and open sets. This contradicts the connectivity of Ω. □

By the above proposition, if holomorphic functions f and g defined on Ω are equal on a sequence converging to some point in Ω, they must be equal. This follows from considering the function $f - g$ in the above proposition. In other words, a holomorphic function is determined by its values on a sequence whose limit is in the domain of the function. However, this does not mean that we know how to identify the values of the function all over the domain.

Proof of Theorem 3.11. If the map f is identically equal to a constant $c \neq \infty$ we choose $P \equiv c$ and $Q \equiv 1$. If the map f is identically equal to ∞ we choose, $P \equiv 1$ and $Q \equiv 0$. Below we assume that f is not constant on $\hat{\mathbb{C}}$.

If f does not attain ∞ at any point on $\hat{\mathbb{C}}$, then $f : \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic, and by Proposition 3.4 it must be constant on $\hat{\mathbb{C}}$. So, if f is not constant, it must attain ∞ at some points in $\hat{\mathbb{C}}$.

There are at most a finite number of points in $\mathbb{C}$, denoted by a_1, a_2, $\dots$, a_n, where $f(a_i) = \infty$. That is because, if f attains ∞ at an infinite number of distinct points in $\mathbb{C}$, since $\hat{\mathbb{C}}$ is a compact set, there will be a sub-sequence of those points converging to some z_0 in $\hat{\mathbb{C}}$. Then, we apply proposition 3.12 to the map $1/f(z)$ or $1/f(1/z)$ (depending on the value of z_0), and conclude that f is identically equal to ∞.

Each pole a_i of Q has some finite order $k_i \geq 1$. Define

$$Q(z) = (z - a_1)^{k_1}(z - a_2)^{k_2} \dots (z - a_n)^{k_n}.$$

Consider the map $g(z) = f(z)Q(z)$. Since f and Q are holomorphic functions from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, g is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. The map g is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Moreover, since the order of zero of Q at a_i is equal to the order of the pole of f at a_i, g is finite at any point in $\mathbb{C}$. Thus, by Proposition 3.8, g is a polynomial in variable z. This finishes the proof of the theorem. □

The degree of a rational map $f = P/Q$, where P and Q have no common factors, is defined as the maximum of the degrees of P and Q. There is an intuitive meaning of the degree of a rational map as in the case of polynomials. Recall that by the fundamental theorem of algebra, for $c \in \mathbb{C}$, the equation $P(z) = c$ has $\deg(P)$ solutions, counted with the multiplicities given by the orders of the solutions. As the equation $f(z) = c$ reduces to $cQ(z) - P(z) = 0$, the number of solutions of $f(z) = c$ counted with multiplicities is given by $\deg(f)$.

Theorem 3.13. *A holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an automorphism of $\hat{\mathbb{C}}$, iff there are constants a, b, c, and d in $\mathbb{C}$ with $ad - bc = 1$ and*

$$f(z) = \frac{az + b}{cz + d}. \tag{3.2}$$

Proof. By Example 3.9, every map f of this form is holomorphic from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Moreover, one can verify that the map $g(z) = (dz - b)/(-cz + a)$ satisfies $f \circ g(z) = g \circ f(z) = z$, for all $z \in \hat{\mathbb{C}}$. Hence, f is both on-to-one and onto from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. This proves one side of the theorem.

On the other hand, if f is an automorphism of $\hat{\mathbb{C}}$, by Theorem 3.11, there are polynomials P and Q such that $f = P/Q$. Let us assume that P and Q have no common factors. Since f is one-to-one, every point has a single pre-image. Thus, by the paragraph preceding the theorem, we must have $\max\{\deg(P), \deg(Q)\} = 1$. Then, there are complex constants a, b, c, d such that $P(z) = az + b$ and $Q(z) = cz + d$, where at least one of a and c is non-zero.

Since P and Q have no common factors, we must have $ad - bc \neq 0$. We may multiply both P and Q by some constant to make $ad - bc = 1$. □

Definition 3.14. Every map of the form in Equation (3.2), where a, b, c, and d are constants in $\mathbb{C}$ with $ad - bc = 1$ is called a *Möbius* transformation. By Theorems 2.5 and 2.7, every automorphism of $\mathbb{D}$ and $\mathbb{C}$ is a Möbius transformation.

Theorem 3.15. *Every automorphism of $\mathbb{C}$ is of the form $az + b$ for some constants a and b in $\mathbb{C}$ with $a \neq 0$.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be an automorphism. We claim that when $|z| \to +\infty$, $|f(z)| \to +\infty$. If this is not the case, there is an infinite sequence of distinct points z_i with $|z_i| \to +\infty$ but $|f(z_i)|$ are uniformly bounded. As f is one-to-one, the values $f(z_i)$ are distinct for distinct values of i. There is a sub-sequence of $f(z_i)$ that converges to some point in $\mathbb{C}$, say w'. Since $f : \mathbb{C} \to \mathbb{C}$ is onto, there is $z' \in \mathbb{C}$ with $f(z') = w'$.

There is a holomorphic map $g : \mathbb{C} \to \mathbb{C}$ with $f \circ g(z) = g \circ f(z) = z$ on $\mathbb{C}$. We have $g(w') = z'$, and by the continuity of g, the points $z_i = g(f(z_i))$ must be close to z'. This contradicts $|z_i| \to +\infty$.

We extend f to the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by defining $f(\infty) = \infty$. By the above paragraph, f is continuous at ∞, and indeed holomorphic (see Exercise 3.2) from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. It follows that $f \in \mathrm{Aut}(\hat{\mathbb{C}})$, and by Theorem 3.13, it must be of the form $(az+b)/(cz+d)$. However, since f maps $\mathbb{C}$ to $\mathbb{C}$, we must have $c = 0$. This finishes the proof of the theorem. $\square$

Definition 3.16. The automorphisms $z \mapsto z + c$, for c constant, are called *translations*, and the automorphisms $z \mapsto c \cdot z$, for c constant, are called *dilations*. When c is real, these are also automorphisms of $\mathbb{H}$. When $|c| = 1$, the map $z \mapsto c \cdot z$ is called a *rotation* of $\mathbb{C}$.

3.3 Exercises

Exercise 3.1. Prove that

(i) for every z_1, z_2, w_1, and w_2 in $\mathbb{C}$ with $z_1 \neq z_2$ and $w_1 \neq w_2$, there is $f \in \mathrm{Aut}(\mathbb{C})$ with $f(z_1) = w_1$ and $f(z_2) = w_2$;

(ii) for all distinct points z_1, z_2, and z_3 in $\hat{\mathbb{C}}$ and all distinct points w_1, w_2, and w_3 in $\hat{\mathbb{C}}$, there is $f \in \mathrm{Aut}(\hat{\mathbb{C}})$ with $f(z_i) = w_i$, $i = 1, 2, 3$.

Exercise 3.2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a map whose restriction to $\mathbb{C}$ is holomorphic, and has a continuous extension to ∞. Show that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic.

Exercise 3.3. Let $f : \Omega \to \mathbb{C}$ be a holomorphic map that has a zero of order $k \geq 1$ at some $z_0 \in \Omega$.

(i) Prove that there is $\delta > 0$ and a holomorphic function $\psi : B(z_0, \delta) \to \mathbb{C}$ such that $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, and $f(z) = (\psi(z))^k$ on $B(z_0, \delta)$.

(ii) Conclude from part (i) that near 0 the map f is k-to-one, that is, every point near 0 has exactly k pre-images near z_0.

Exercise 3.4. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a holomorphic map.

(i) Using Exercise 3.3, prove that if f is not constant, it is an *open map*, that is, f maps every open set in Ω to an open set in $\mathbb{C}$.

(ii) Using part (i), prove the maximum principle, Theorem 1.6.

Exercise 3.5. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Prove that the image of every straight line in $\mathbb{C}$ is either a straight line or a circle in $\mathbb{C}$. Also, the image of every circle in $\mathbb{C}$ is either a straight line or a circle in $\mathbb{C}$.

Exercise 3.6. Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map which maps $\mathbb{D}$ into $\mathbb{D}$ and maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Prove that there are points a_1, a_2, $\dots$, a_d (not necessarily distinct) in $\mathbb{D}$ and $\theta \in [0, 2\pi]$ such that

$$g(z) = e^{2\pi i\theta} \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}.$$

The maps of the above form are called *Blaschke products of degree d.*

Chapter 4

Conformal geometry on the disk

4.1 Poincare metric

Let X be a set. Recall that a *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that for all x, y, and z in X we have

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric on a space X allows us to talk about distances on X. The most familiar example is probably the *Euclidean distance* on $\mathbb{R}$ given by the absolute value. That is, $d(x, y) = |x-y|$, for x and y in $\mathbb{R}$. This notion of distance respects the underlying operation of addition that is described by the relation $d(x, y) = d(x + c, y + c)$, for all $c \in \mathbb{R}$. That is, d is *invariant* under translations.

Another example of a metric on $\mathbb{R}$ is given by the function

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

The above notion of metric is rather general for our purpose. There is a more restrictive notion of metric that is suitable in the world of complex analysis. Before we define that, recall that a set $A \subset \mathbb{C}$ is called a discrete set, if for every $z \in A$ there is an open set U in $\mathbb{C}$ with $A \cap U = \{z\}$.

Definition 4.1. Let Ω be a domain in $\mathbb{C}$. A *conformal metric* on Ω is a continuously differentiable (C^1) function

$$\rho : \Omega \to [0, \infty),$$

where $\rho(z) \neq 0$ except on a discrete subset of Ω. If $z \in \Omega$ and $\xi \in \mathbb{C}$ is a vector, we define the *length of ξ at z with respect to the metric ρ* as

$$\| \xi \|_{\rho,z} = \rho(z) \cdot |\xi|.$$

Here, $|\xi|$ denotes the Euclidean norm of ξ, i.e. $\sqrt{\xi\overline{\xi}}$.

Remark 4.2. In contrast to what we learn in a calculus course that a vector has only direction and magnitude, in the above notion of the metric, a vector has direction, magnitude, and position. That is the length of a vector also depends on its position.

Definition 4.3. Assume that $\gamma : [a, b] \to \Omega$ is a C^1 curve. The *length of* γ *with respect to the metric* ρ is defined as

$$\ell_\rho(\gamma) = \int_a^b \left\| \frac{\partial \gamma(t)}{\partial t} \right\|_{\rho,\gamma(t)} dt = \int_a^b \rho(\gamma(t)) \cdot \left| \frac{\partial \gamma(t)}{\partial t} \right| dt.$$

The length of a piece-wise C^1 curve is defined as the sum of the lengths of its C^1 parts.

As in the definition of the integration along a curve in complex analysis, the above notion of length is independent of the parameterization of the curve.

It is convenient to think of the tangent vector to $\gamma'(t)$ at $\gamma(t)$ as a vector based at $\gamma(t)$. See Figure 4.1

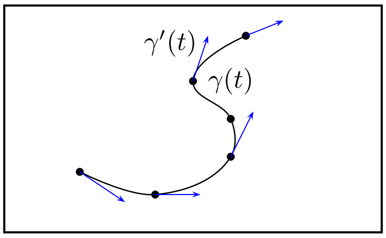

Figure 4.1: The tangent vectors to a C^1 curve γ in the calculation of the length of γ with respect to a conformal metric.

For every $c \in \mathbb{C}$ with $|c| = 1$ we have

$$\| \, c \cdot \xi \, \|_{\rho,z} = \| \, \xi \, \|_{\rho,z} \, .$$

By the above relation, the length of a vector ξ at some $z \in \Omega$ is independent of its direction. This feature makes conformal metrics natural in complex analysis, as we shall see in this section.

In the classical literature in analysis, sometimes you find the notation

$$\ell_\rho(\gamma) = \int_\gamma \rho(z) \, |dz|,$$

for the length of γ with respect to the metric ρ. This is consistent with the definition of integration along curves you learn in complex analysis.

Definition 4.4. A set A in $\mathbb{C}$ is called *path connected* if for every two points z and w in A there is a continuous map $\gamma : [0,1] \to A$ with $\gamma(0) = z$ and $\gamma(1) = w$.

It follows that any path connected subset of $\mathbb{C}$ is connected, but there are connected subsets of $\mathbb{C}$ that are not path connected.

Definition 4.5. Let ρ be a conformal metric defined on an open and path connected set $\Omega \subseteq \mathbb{C}$. Given z and w in Ω let $\Gamma_{z,w}$ denote the set of all piece-wise C^1 curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = z$ and $\gamma(1) = w$. Define

$$d_\rho(z,w) = \inf\{\ell_\rho(\gamma) : \gamma \in \Gamma_{z,w}\}.$$

It follows (Exercise 4.1) that d_ρ defines a metric on Ω, that is, $d_\rho : \Omega \times \Omega \to [0,\infty)$ satisfies the required properties listed at the beginning of this chapter.

Remark 4.6. One should not confuse the notion of $d_\rho(z,w)$ with the notion of the length of the vector $w - z$ at z with respect to ρ. In general, these are different values and not related.

Example 4.7. When $\rho(z) \equiv 1$ on Ω, the length of a piece-wise C^1 curve γ with respect to ρ, $\ell_\rho(\gamma)$, becomes the Euclidean length of γ (which we learn in calculus). When $\Omega = \mathbb{C}$, d_ρ becomes the Euclidean distance. In general, when Ω is a convex set, that is, the line segment connecting any two points in Ω lies in Ω, then d_ρ is the restriction of the Euclidean metric to Ω. But in general, there may not be a curve of shortest length between two points in Ω. See Figure 4.2.

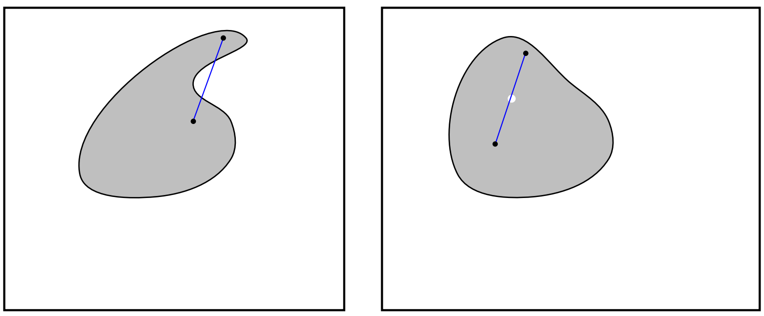

Figure 4.2: Examples of non-convex domains; one with a point omitted, and the other with a special shape.

Definition 4.8. The Poincaré metric on $\mathbb{D}$ is defined as

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

The Poincaré metric has been used to gain deep understanding into the complex analysis on the disk and beyond. We shall study this metric in details.

For any vector $\xi \in \mathbb{C}$ we have

$$\begin{aligned} \| \xi \|_{\rho,0} &= \rho(0) \cdot |\xi| = |\xi|, \\ \| \xi \|_{\rho,(1/2+0i)} &= \rho(1/2+0i) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\ \| \xi \|_{\rho,(0+i/2)} &= \rho(0+i/2) \cdot |\xi| = \frac{4}{3} \cdot |\xi|, \\ \| \xi \|_{\rho,(0.99+0i)} &= \rho(0.99+0i) \cdot |\xi| = (50.251256\dots) \cdot |\xi|. \end{aligned}$$

The metric ρ has a rotational symmetry about 0, i.e. $\rho(c \cdot z) = \rho(z)$ for all $c \in \mathbb{C}$ with $|c| = 1$. Also, $\rho(z) \to \infty$ as $|z|$ tends to $+1$ from below. There are many conformal metrics on $\mathbb{D}$ with rotational symmetry and diverging to $+\infty$ near the boundary, but the speed of divergence in the Poincaré metric is chosen to guarantee some significant properties.

Example 4.9. Let us calculate the length of the curve $[0, 1-\varepsilon]$ with respect to the Pioncaré metric ρ on $\mathbb{D}$. Define $\gamma(t) = t + 0i$, for $t \in [0, 1-\varepsilon]$. Then,

$$\ell_\rho(\gamma) = \int_0^{1-\varepsilon} \rho(\gamma(t)) \cdot |\gamma'(t)|\, dt = \int_0^{1-\varepsilon} \frac{1}{1-t^2}\, dt = \frac{1}{2} \log\Big(\frac{1+t}{1-t}\Big)\Big|_{t=0}^{t=1-\varepsilon} = \frac{1}{2} \log\Big(\frac{2-\varepsilon}{\varepsilon}\Big).$$

We note that the above quantity tends to $+\infty$ as ε tends to 0. This means that the point $+1$ is at distance ∞ from the point 0 along the curve γ, with respect to the Poincaré metric on $\mathbb{D}$.

Proposition 4.10. *Let ρ be the Poincaré metric on $\mathbb{D}$. We have*

$$d_\rho(0, 1-\varepsilon) = \frac{1}{2} \log\Big(\frac{2-\varepsilon}{\varepsilon}\Big).$$

Proof. Let $\eta : [a, b] \to \mathbb{D}$ be a C^1 curve with $\eta(a) = 0$ and $\eta(b) = 1-\varepsilon+0i$. In coordinates, let $\eta(t) = \eta_1(t) + i\eta_2(t)$, for $t \in [a, b]$. Both η_1 and η_2 are C^1 and moreover, for all $t \in [a, b]$,

$$|\eta'(t)| = |\eta_1'(t) + i\eta_2'(t)| \geq |\eta_1'(t)|.$$

Also, since $|\eta(t)| \geq |\eta_1(t)|$, for all $t \in [a, b]$ we have

$$\rho(\eta(t)) \geq \rho(\eta_1(t)).$$

Note that $\eta_1 : [a, b] \to (-1, 1)$ is a C^1 curve with $\eta_1(a) = 0$ and $\eta_1(b) = 1-\varepsilon$. However, η_1 may not a monotone function of $t \in [a, b]$. Its image may cover some parts of $[0, 1-\varepsilon]$ several times. If necessary, we may throw away parts of this curve and keep a piece-wise

C^1 and monotone part of η_1 that maps a subset of $[a,b]$ to $[0,1-\varepsilon]$. Let $A \subseteq [a,b]$ be that set. Using the above inequalities,

$$\begin{aligned}
\ell_\rho(\eta) &= \int_a^b \rho(\eta(t)) \cdot |\eta'(t)|\,dt \\
&\geq \int_a^b \rho(\eta_1(t)) \cdot |\eta_1'(t)|\,dt \\
&\geq \int_A \rho(\eta(t)) \cdot |\eta'(t)|\,dt \\
&= \frac{1}{2}\log(\frac{2-\varepsilon}{\varepsilon}).
\end{aligned}$$

□

Figure 4.3 shows the graph of the function $r \mapsto d_\rho(0,r)$, on $(0,1)$. Note how on a large interval $(0,r)$ (with r close to 1) the distance of the points from 0 is less than 5.

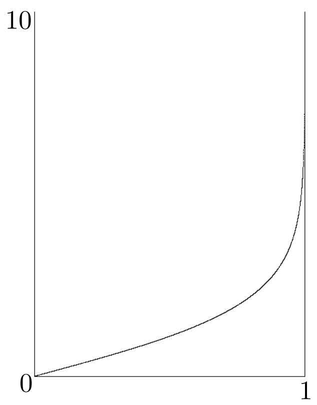

Figure 4.3: The graph of the function $r \mapsto d_\rho(0,r)$, for $r \in (0,1)$.

4.2 Isometries

Definition 4.11. Assume that Ω_1 and Ω_2 are open sets in $\mathbb{C}$ and

$$f : \Omega_1 \to \Omega_2$$

is a holomorphic map. Let ρ_2 be a conformal metric on Ω_2. The *pull-back* of ρ_2 by f is defined as

$$(f^*\rho_2)(z) = \rho_2(f(z)) \cdot |f'(z)|.$$

It is clear that if ρ_2 is C^1 then $f^*\rho_2$ is also C^1. On the other hand, $f^*\rho_2(z) = 0$ if and only if either $\rho(f(z)) = 0$ or $f'(z) = 0$. Since, f' is a holomorphic function on Ω_1, the set of points where it becomes 0 is a discrete set. These imply that the pull-back of a conformal metric under a holomorphic map is a conformal metric. Indeed, this is the reason for the name conformal metric. These are metrics that behave well under holomorphic transformations.

By the above definition, if ξ is a vector in $\mathbb{C}$ and $z \in \Omega_1$, then

$$\| \xi \|_{f^*\rho_2,z} = \rho_2(f(z)) \cdot |f'(z)| \cdot |\xi| = \rho_2(f(z)) \cdot |f'(z) \cdot \xi|.$$

Let us denote the metric $f^*\rho_2$ on Ω_1 by ρ_1. If γ_1 is a C^1 curve in Ω_1, and $\gamma_2 = f \circ \gamma_1$, then it follows that $\ell_{\rho_1}(\gamma_1) = \ell_{\rho_2}(\gamma_2)$.

$$\ell_{\rho_2}(\gamma_2) = \int_a^b \rho_2(\gamma_2(t)) \cdot |\gamma_2'(t)|\, dt = \int_a^b \rho_2(f(\gamma_1(t))) \cdot |(f \circ \gamma_1)'(t)|\, dt = \int_a^b \rho_2(f(\gamma_1(t))) \cdot |f'(\gamma_1(t))| \cdot |\gamma_1'(t)|\, dt = \int_a^b \rho_1(\gamma_1(t))|\gamma_1'(t)|\, dt = \ell_{\rho_1}(\gamma_1)$$

Definition 4.12. Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ and $f : \Omega_1 \to \Omega_2$ be an onto holomorphic map. Let ρ_i be a conformal metric on Ω_i, for $i = 1, 2$. Then, f is called an *isometry* from (Ω_1, ρ_1) to (Ω_2, ρ_2) if for all $z \in \Omega_1$ we have

$$f^*\rho_2(z) = \rho_1(z).$$

Proposition 4.13. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ with conformal metrics ρ_1 and ρ_2, respectively. Assume that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2). Then for every C^1 curve $\gamma : [0, 1] \to \Omega_1$ we have*

$$\ell_{\rho_1}(\gamma) = \ell_{\rho_2}(f \circ \gamma).$$

The curve $f \circ \gamma$ is often called the *push-forward* of the curve γ by f, and is denoted by $f_*\gamma$. That is,

$$f_*\gamma(t) = f \circ \gamma(t), \text{for } t \in [0, 1].$$

Proof. By Definition 4.3 we have

$$\ell_{\rho_1}(\gamma) = \int_0^1 \| \gamma'(t) \|_{\rho_1,\gamma(t)}\ dt, \quad \ell_{\rho_2}(f \circ \gamma) = \int_0^1 \| (f \circ \gamma)'(t) \|_{\rho_2, f\circ\gamma(t)}\ dt.$$

To prove that the above integrals give the same value, it is enough to show that the integrands are equal on $[0,1]$. That is,

$$\begin{aligned}
\| (f\circ\gamma)'(t) \|_{\rho_2, f\circ\gamma(t)} &= \rho_2(f\circ\gamma(t)) \cdot |(f\circ\gamma)'(t)| \\
&= \rho_2(f\circ\gamma(t)) \cdot |f'(\gamma(t)) \cdot \gamma'(t)| \\
&= \rho_2(f\circ\gamma(t)) \cdot |f'(\gamma(t))| \cdot |\gamma'(t)| \\
&= (f^*\rho_2)(\gamma(t)) \cdot |\gamma'(t)| \\
&= \| \gamma'(t) \|_{\rho_1, \gamma(t)} \, .
\end{aligned}$$

□

Note that if f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2) we may not conclude that for every z and w in Ω_1 we have

$$d_{\rho_1}(z,w) = d_{\rho_2}(f(z), f(w)). \tag{4.1}$$

That is because not every curve in Ω_2 from $f(z)$ to $f(w)$ is obtained from push-forward of a curve in Ω_1 from z to w. We illustrate this by an example.

Example 4.14. Let

$$\Omega_1 = \{z \in \mathbb{C} : e^{-1} < |z| < e\}, \qquad \Omega_2 = \{z \in \mathbb{C} : e^{-2} < |z| < e^2\}$$

and define

$$f : \Omega_1 \to \Omega_2, \quad f(z) = z^2.$$

Consider the conformal metrics

$$\rho_1(z) = \frac{\pi}{2} \cdot \frac{1}{|z| \cdot \cos(\frac{\pi \log |z|}{2})},$$

$$\rho_2(z) = \frac{\pi}{4} \cdot \frac{1}{|z| \cdot \cos(\frac{\pi \log |z|}{4})}.$$

We have

$$\begin{aligned}
f^*\rho_2(z) &= \rho_2(f(z)) \cdot |f'(z)| \\
&= \frac{\pi}{4} \cdot \frac{1}{|z|^2 \cdot \cos(\frac{\pi \log |z|^2}{4})} \cdot 2|z| \\
&= \frac{\pi}{2} \cdot \frac{1}{|z| \cdot \cos(\frac{\pi \log |z|}{2})} = \rho_1(z).
\end{aligned}$$

That means that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2).

Note that f is not one-to-one, for example, $f(+1) = f(-1) = +1$. If γ is a curve in Ω_1 connecting $+1$ to -1, $f \circ \gamma$ is a curve in Ω_2 that connects $+1$ to itself and wraps around in Ω_2 at least once. But, there is a constant curve with zero length from $+1$ to $+1$ in Ω_2. The constant curve is not the image of any continuous curve from $+1$ to -1 in Ω_1. This is the only issue that prevents us from having Equation (4.1). As you will show in Exercise 4.5, if $f : \Omega_1 \to \Omega_2$ is one-to-one, then Equation (4.1) holds for all z and w in Ω_1.

Let Ω_1, Ω_2, and Ω_3 be open sets in $\mathbb{C}$ with conformal metrics ρ_1, ρ_2, and ρ_3, respectively. Assume that f is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2), and g is an isometry from (Ω_2, ρ_2) to (Ω_3, ρ_3). You can show that $g \circ f$ is an isometry from (Ω_1, ρ_1) to (Ω_3, ρ_3).

Theorem 4.15. *Every automorphism of $\mathbb{D}$ is an isometry from $(\mathbb{D}, \rho)$ to $(\mathbb{D}, \rho)$, where ρ is the Poincaré metric on $\mathbb{D}$.*

Proof. By Theorem 2.5 every automorphism of $\mathbb{D}$ is of the form

$$z \mapsto e^{i\theta} \cdot \frac{z - a}{1 - \overline{a}z},$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. We shall prove the theorem in two steps.

First assume that $h(z) = e^{i\theta} \cdot z$. We have

$$(h^*\rho)(z) = \rho(h(z)) \cdot |h'(z)| = \frac{1}{1 - |h(z)|^2} \cdot |e^{i\theta}| = \frac{1}{1 - |z|^2} = \rho(z).$$

Thus, h is an isometry of $(\mathbb{D}, \rho)$.

Now assume that $h(z) = (z - a)/(1 - \overline{a}z)$. we have

$$\begin{aligned}
(h^*\rho)(z) &= \rho(h(z)) \cdot |h'(z)| \\
&= \frac{1}{1 - \left|\frac{z-a}{1-\overline{a}z}\right|^2} \cdot \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \\
&= \frac{1 - |a|^2}{|1 - \overline{a}z|^2 - |z - a|^2} \\
&= \frac{1 - |a|^2}{1 - |z|^2 - |a|^2 + |a|^2|z|^2} \\
&= \rho(z).
\end{aligned}$$

That is, h is an isometry of $(\mathbb{D}, \rho)$. Since the composition of isometries is an isometry, see Exercise 4.4, we conclude that any member of $\mathrm{Aut}(\mathbb{D})$ is an isometry of $(\mathbb{D}, \rho)$. $\square$

As a corollary of the above theorem, and Proposition 4.10, we are able to calculate the Poincaré distant between any two points on $\mathbb{D}$.

Proposition 4.16. *Let p and q be two points in $\mathbb{D}$ equipped with Poincaré metric ρ. We have*

$$d_\rho(p,q) = \frac{1}{2}\log\left(\frac{1+\left|\frac{p-q}{1-\overline{p}q}\right|}{1-\left|\frac{p-q}{1-\overline{p}q}\right|}\right).$$

Proof. When $p = 0$ and q is a positive real number the formula in the proposition reduces to the one in Proposition 4.10. Now, define

$$\varphi_1(z) = \frac{z-p}{1-\overline{p}z},$$

and

$$\varphi_2(z) = \frac{|\varphi_1(q)|}{\varphi_1(q)} \cdot z.$$

Note that both of φ_1 and φ_2 belong to $\mathrm{Aut}(\mathbb{D})$. Then, by Theorem 4.15 and Exercise 4.5, we must have

$$d_\rho(p,q) = d_\rho(\varphi_1(p),\varphi_1(q)) = d_\rho(0,\varphi_1(q)) = d_\rho(\varphi_2(0),\varphi_2(\varphi_1(q))) = d_\rho(0,|\varphi_1(q)|).$$

Using Proposition 4.10 with $1-\varepsilon = |\varphi_1(q)|$ we obtain

$$d_\rho(0,|\varphi_1(q)|) = \frac{1}{2}\log\left(\frac{1+|\varphi_1(q)|}{1-|\varphi_1(q)|}\right).$$

This finishes the proof of the proposition. □

The proof of the above proposition also provides us with the shortest curve connecting the two points p and q. We state this in a separate proposition.

Proposition 4.17. *Let p and q be two distinct points in $\mathbb{D}$. The shortest curve with respect to ρ connecting p to q is given by the formula*

$$\gamma_{p,q}(t) = \frac{t\frac{q-p}{1-q\overline{p}}+p}{1+t\overline{p}\frac{q-p}{1-q\overline{p}}}, \quad 0 \le t \le 1.$$

Proof. In Proposition 4.10 and its preceding example, the shortest curve connecting 0 to a point z on $(0,1) \subset \mathbb{D}$ is given by the interval $[0,z]$. As the rotation $z \mapsto e^{i\theta}\cdot z$, for each fixed $\theta \in \mathbb{R}$, is an isometry of $(\mathbb{D},\rho)$, we conclude that the shortest curve connecting 0 to a given point $z \in \mathbb{D}$ is the curve $t \mapsto t\cdot z$, $0 \le t \le 1$.

Consider the automorphism

$$\varphi_1(z) = \frac{z-p}{1-\overline{p}z}.$$

We have $\varphi_1(p) = 0$ and $\varphi_1(q) \in \mathbb{D}$. The inverse of this map is given by the formula

$$\varphi_1^{-1}(z) = \frac{z+p}{1+\overline{p}z}.$$

By the above paragraph, the shortest curve connecting 0 to $\varphi_1(q)$ is given by the formula $\theta(t) = t \cdot \varphi_1(q)$. Since φ_1^{-1} is an isometry of the pair $(\mathbb{D}, \rho)$, the image of this curve under φ^{-1} is the shortest curve connecting p to q. The formula for this curve is

$$t \mapsto \varphi_1^{-1}(t \cdot \varphi_1(q)) = \frac{t \cdot \varphi_1^{-1}(q) + p}{1 + \overline{p}t \cdot \varphi^{-1}(q)}.$$

This finishes the proof of the proposition. □

Definition 4.18. Let Ω be an open set in $\mathbb{C}$ and ρ be a conformal metric on Ω. A continuous curve $\gamma : [a, b] \to \Omega$ is called *geodesic* if for every $t \in [a, b]$ there is $\varepsilon_t > 0$ such that for all x and y in $[a, b] \cap [t - \varepsilon_t, t + \varepsilon_t]$ we have

$$d_\rho(\gamma(x), \gamma(y)) = \ell_\rho(\gamma : [x, y] \to \Omega).$$

In other words, the curve γ is locally the shortest curve connecting points together. For example, straight lines on $\mathbb{C}$ are geodesics with respect to the conformal metric $\rho \equiv 1$. The curves $\gamma_{p,q}$ in the above proposition provide examples of geodesics with respect to the Poincaré metric on $\mathbb{D}$.

There is an intuitive way to visualize the curve $\gamma_{p,q}$. To present this, we need to recall a basic property of holomorphic maps.

Definition 4.19. A holomorphic map $f : \Omega \to \mathbb{C}$ is called *conformal* at $z \in \Omega$ if $f'(z) \neq 0$. A holomorphic map $f : \Omega \to \mathbb{C}$ is called *conformal*, if it is conformal at every point in Ω.

If $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is one-to-one, it follows that f is conformal at every point in U; see Exercise 3.3. In particular, biholomorphic maps are conformal at every point in their domain of definition. However, note that a map that is conformal at every point in its domain of definition is not necessarily one-to-one from its domain to its range. For example, the map $z \mapsto z^2$ is conformal on the set

$$\{w \in \mathbb{C} \mid \arg(w) \in (-3\pi/4, 3\pi/4), |w| \in (1, 2)\}$$

but is not one-to-one on this set.

Recall from complex analysis that conformal maps preserve angles. We state this below for future reference.

Proposition 4.20. *Let U and V be two open subsets of $\mathbb{C}$ and $f : U \to V$ be a holomorphic map that is conformal at some $z \in U$. Then f preserves angles at z.*

The curve $t \mapsto t \cdot \varphi_1(q)$ is part of a straight line segment in $\mathbb{D}$. By Exercise 3.5, the image of any line segment in $\mathbb{D}$ under φ_1^{-1} is either a line segment or part of a circle. The image may be a line segment only if the three points p, q, and 0 lie on a straight line segment, and other wise the curve is part of a circle. Moreover, since the line segment passing through 0 is orthogonal to the boundary of $\mathbb{D}$, and conformal maps preserve angles, the circle passing through p and q is orthogonal to the circle $|z| = 1$. See Figure 4.4.

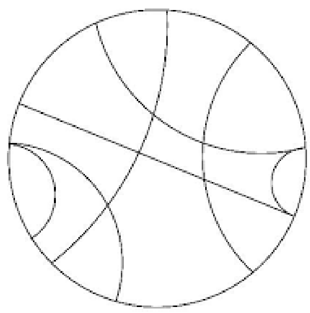

Figure 4.4: Some examples of geodesics with respect to the Poincaré metric ρ on $\mathbb{D}$.

4.3 Hyperbolic contractions

Theorem 4.21 (Schwarz-Pick Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and ρ denote the Poincaré metric on $\mathbb{D}$. Then, f is distance decreasing with respect to ρ, that is, for every $z \in \mathbb{D}$ we have*

$$f^* \rho(z) \leq \rho(z).$$

In particular, if $\gamma : [0, 1] \to \mathbb{D}$ is a C^1 curve then

$$\ell_\rho(f_* \gamma) \leq \ell_\rho(\gamma).$$

Therefore, if z and w belong to $\mathbb{D}$, then

$$d_\rho(f(z), f(w)) \leq d_\rho(z, w).$$

Proof. Recall that

$$f^* \rho(z) = \rho(f(z)) \cdot |f'(z)| = \frac{1}{1 - |f(z)|^2} \cdot |f'(z)|,$$

and

$$\rho(z) = \frac{1}{1 - |z|^2}.$$

Hence, the inequality in the theorem reduces to the Schwarz-Pick lemma we saw earlier in Exercise 2.2.

The latter parts of the theorem follow directly from the definitions. $\square$

Theorem 4.22 (Farkas-Ritt)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and assume that $f(\mathbb{D})$ has a compact closure in $\mathbb{D}$, that is, every sequence in $f(\mathbb{D})$ has a sub-sequence converging to some point in $\mathbb{D}$. Then,*

i) there is a unique point $p \in \mathbb{D}$ such that $f(p) = p$;

ii) for every w_0 in $\mathbb{D}$ the sequence of points w_i defined as $w_{i+1} = f(w_i)$, for $i \geq 0$, converges to p in the Euclidean metric.

Proof. Define

$$A = \{f(z) : z \in \mathbb{D}\}.$$

By the hypothesis, the closure of A is contained in $\mathbb{D}$. This implies that there is $\delta > 0$ such that for every $z \in \mathbb{C}$ with $|z| \geq 1$ and every $w \in A$ we have $|w - z| > \delta$.

Fix an arbitrary $z_0 \in \mathbb{D}$. Define the map

$$g(z) = f(z) + \frac{\delta}{2}(f(z) - f(z_0)), \quad \forall z \in \mathbb{D}.$$

The map g is holomorphic on $\mathbb{D}$, and maps $\mathbb{D}$ into $\mathbb{D}$ since

$$|g(z)| \leq |f(z)| + \frac{\delta}{2}|f(z) - f(z_0)| < (1 - \delta) + \frac{\delta}{2} \cdot 2 = 1.$$

We have $g(z_0) = f(z_0)$ and $g'(z_0) = (1 + \delta/2)f'(z_0)$. By Theorem 4.21, g is non-expanding the Poincaré metric at z_0, that is,

$$g^*\rho(z_0) \leq \rho(z_0).$$

Writing the definition of g^*, this yields

$$(1 + \delta/2) \cdot |f'(z_0)| \cdot \rho(f(z_0)) \leq \rho(z_0).$$

As $z_0 \in \mathbb{D}$ was arbitrary, we conclude that the above inequality holds for all $z_0 \in \mathbb{D}$. In particular, if γ is any C^1 curve in $\mathbb{D}$, then

$$(1 + \delta/2) \cdot \ell_\rho(f \circ \gamma) \leq \ell_\rho(\gamma).$$

This implies that for arbitrary points z and w in $\mathbb{D}$ we have

$$d_\rho(f(z), f(w)) \leq (1 + \delta/2)^{-1} \cdot d_\rho(z, w).$$

Fix an arbitrary w_0 in $\mathbb{D}$ and define the sequence of points $w_{i+1} = f(w_i)$, for $i \geq 0$. Inductively using the above inequality we conclude that for every $i \geq 2$ we have

$$d_\rho(w_{i+1}, w_i) \leq (1 + \delta/2)^{-1} \cdot d_\rho(w_i, w_{i-1}) \leq \cdots \leq (1 + \delta/2)^{-i} \cdot d_\rho(w_1, w_0).$$

Since $\sum_{i=0}^{\infty}(1 + \delta/2)^{-i}$ is finite, the sequence w_i is Cauchy with respect to d_ρ. The space $\mathbb{D}$ with respect to d_ρ is a complete metric space, see Exercise 4.3. This means that any Cauchy sequence (w.r.t d_ρ) in $\mathbb{D}$ converges (w.r.t d_ρ) to some point in $\mathbb{D}$. By Exercise 4.2, the sequence w_i converges with respect to the Euclidean metric on $\mathbb{D}$. Let p denote the limit of this sequence. Taking limit from the relation $w_{i+1} = f(w_i)$ as i tends to $+\infty$, we conclude that $f(p) = p$.

If there is q in $\mathbb{D}$ with $f(q) = q$, by the above inequalities,

$$d_\rho(p, q) = d_\rho(f(p), f(q)) \leq (1 + \delta/2)^{-1} d_\rho(p, q).$$

As $\delta > 0$, this is only possible if $p = q$. This shows the uniqueness of p. So far we have completed the proof of Part i).

By the above arguments, $d_\rho(w_i, p) \leq (1 + \delta/2)^{-i} d_\rho(w_0, p)$. Hence, w_i converges to p with respect to d_ρ. It follows that w_i converges to p with respect to the Euclidean metric, see Exercise 4.2. $\square$

4.4 Exercises

Exercise 4.1. Show that $d_\rho : \Omega \times \Omega \to \mathbb{R}$ defined in Definition 4.5 is a metric on Ω.

Exercise 4.2. Let z_i, $i \geq 1$, be an infinite sequence in $\mathbb{D}$, and ρ be the Poincaré metric on $\mathbb{D}$. Show that z_i converges to some point z in $\mathbb{D}$ with respect to d_ρ iff it converges to $z \in \mathbb{D}$ with respect to the Euclidean metric.

Exercise 4.3. Show that the disk $\mathbb{D}$ equipped with the Poincaré metric ρ is a complete metric space. That is, every Cauchy sequence in $\mathbb{D}$ with respect to d_ρ converges to some point in $\mathbb{D}$ with respect to the distance d_ρ.

Exercise 4.4. Let ρ be the Poincaré metric on $\mathbb{D}$. For $z \in \mathbb{D}$ and $r > 0$, the circle of radius r about z with respect to the metric d_ρ is defined as

$$\{w \in \mathbb{D} : d_\rho(z, w) = r\}.$$

Show that for every $z \in \mathbb{D}$ and $r > 0$, the circle of radius r about z is an Euclidean circle. Find the center of this circle.

Exercise 4.5. Let Ω_1 and Ω_2 be open sets in $\mathbb{C}$ with conformal metrics ρ_1 and ρ_2, respectively. Assume that $f : \Omega_1 \to \Omega_2$ is a one-to-one holomorphic map that is an isometry from (Ω_1, ρ_1) to (Ω_2, ρ_2). Prove that for all z and w in Ω_1 we have

$$d_{\rho_1}(z, w) = d_{\rho_2}(f(z), f(w)).$$

Exercise 4.6. Recall the biholomorphic map $F : \mathbb{H} \to \mathbb{D}$ given in Equation (2.1), and let ρ be the Poincaré metric on ρ. Show that for all $w \in \mathbb{H}$ we have

$$(F^*\rho)(w) = \frac{1}{2|\operatorname{Im} w|}.$$

Chapter 5

Conformal Mappings

In the previous chapters we studied automorphisms of $\mathbb{D}$, and the geometric behavior of holomorphic maps from $\mathbb{D}$ to $\mathbb{D}$ using the Poincaré metric. A natural question is whether similar methods can be used for other domains in $\mathbb{C}$. A possible approach is the idea we used to describe $\mathrm{Aut}(\mathbb{H})$. To employ that idea for an open set $\Omega \subset \mathbb{C}$ we need a biholomorphic map $f : \mathbb{D} \to \Omega$. Then, elements of $\mathrm{Aut}(\Omega)$ obtain the form $f \circ \varphi \circ f^{-1}$, for $\varphi \in \mathrm{Aut}(\mathbb{D})$, and the Poincaré metric can be pulled back by f^{-1} to a conformal metric on Ω, etc. To carry out this idea we face the following key questions:

(i) for which domains $\Omega \subseteq \mathbb{C}$ there is a biholomorphic map from $\mathbb{D}$ to Ω;

(ii) if the answer to question (i) is positive for some Ω, when is there an explicit biholomorphic map from $\mathbb{D}$ to Ω. For instance, a biholomorphic map given by an algebraic formula, trigonometric functions, or a combination of such maps;

(iii) what if there are no elementary biholomorphic maps from $\mathbb{D}$ to Ω, but a biholomorphic map exists.

In this chapter we study the questions in parts (i) and (ii). We shall study the question in part (iii) in the next chapters.

5.1 Conformal mappings of special domains

Example 5.1. The exponential map $z \mapsto e^z = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y)$, where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$. The exp map is biholomorphic from the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \pi\}$ to the upper half plane. It maps horizontal lines $\mathrm{Im}\, z = y_0$ to straight rays $\{z \in \mathbb{C} : \arg z = y_0\}$, and the vertical lines $\mathrm{Re}\, z = x_0$ to the arcs $\{z \in \mathbb{C} : |z| = e^{x_0}, \mathrm{Im}\, z > 0\}$.

The inverse of exp is log which is only determined up to translations by $2\pi i$. We often fix a branch of the inverse to determine which inverse of exp we are considering. For example, to map $\mathbb{H}$ to the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \pi\}$ we may work with the branch that

$0 < \operatorname{Im} \log w < 2\pi$. If we write $z = re^{i\theta}$, then

$$\log z = \log r + i\theta = \log |z| + i \arg z.$$

Also, the restriction of log provides a biholomorphic map

$$\log : \{z \in \mathbb{C} \mid |z| < 1, 0 < \operatorname{Im} z\} \to \{z \in \mathbb{C} \mid \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}.$$

Example 5.2. Let n be a positive integer. The power map $p_n(z) = z^n$ is biholomorphic from the sector $S_n = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/n\}$ to the upper half plane. The inverse of this map is given by $w \mapsto w^{1/n}$. This is defined using a branch of log as $w \mapsto e^{\frac{1}{n} \cdot \log w}$, where $0 < \operatorname{Im} \log(w) < 2\pi$.

The map p_n has a rather simple behavior on S_n. To see this, we note that S_n is the union of the straight rays $R_\theta = \{re^{i\theta} : r > 0\}$, for $0 < \theta < \pi/n$. Then, p_n maps R_θ to $R_{n\theta}$. Also, we may consider S_n as the union of the arcs $C_r = \{re^{i\theta} : 0 < \theta < \pi/n\}$, for $r > 0$. Then p_n maps C_r to the arc $\{r^n e^{i\theta} : 0 < \theta < \pi\}$.

In general, for $\alpha > 0$ the power map $p_\alpha(z) = z^\alpha = e^{\alpha \log z}$ is defined and biholomorphic from the sector $\{z \in \mathbb{C} : 0 < \arg(z) < \pi/\alpha\}$ to the upper half plane. The inverse of this map is given by the formula $w \mapsto w^{1/\alpha} = e^{\frac{1}{\alpha} \log w}$, where $0 < \operatorname{Im} \log(w) < 2\pi$.

Example 5.3. Recall the biholomorphic map $G(w) = i\frac{1-w}{1+w}$ from $\mathbb{D}$ to $\mathbb{H}$ we introduced in Equation 2.1. The restriction of this map provides a biholomorphic map

$$G : \{z \in \mathbb{D} : \operatorname{Im} z > 0\} \to \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}.$$

The composition of G with the map $z \mapsto z^2$ provides a biholomorphic map from the upper half-disk to $\mathbb{H}$.

Example 5.4. Let $f(z) = z + 1/z$. For non-zero $z \in \mathbb{C}$, z and $1/z$ are mapped to the same point . On the other hand, for each non-zero $w_0 \in \mathbb{C}$, the equation $f(z) = w_0$ reduces to $z^2 - w_0 z + 1 = 0$ that has two solutions, counted with multiplicity. The two solutions are the same if and only if $z = 1/z$, which is only possible if $z = \pm 1$. This implies that f is one-to-one on the set $\{z \in \mathbb{C} : |z| > 1\}$.

When $|z| = 1$,

$$f(z) = z + 1/z = z + \overline{z} = 2 \operatorname{Re} z.$$

That is, f maps the circle $|z| = 1$ in a two-to-one fashion to the interval $[-2, +2]$. Then, the restriction of f to the set $|z| > 1$ covers $\mathbb{C} \setminus [-2, +2]$. It follows that

$$f : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C} \setminus [-2, +2].$$

is biholomorphic. By the same arguments,

$$f : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus [-2, +2]$$

is also biholomorphic. Since the map $w \mapsto 1/w$ is biholomorphic from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$, the map

$$z \mapsto \frac{1}{f(z)} = \frac{z}{1+z^2} : \mathbb{D} \to \mathbb{C} \setminus \big((-\infty, -1/2] \cup [1/2, +\infty)\big)$$

is biholomorphic.

Example 5.5. The Koebe map

$$k(z) = \frac{z}{(1-z)^2} : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4]$$

is biholomorphic. To see this, we write

$$k(z) = \frac{1}{4}\Big(\frac{1+z}{1-z}\Big)^2 - \frac{1}{4}.$$

and observe that $(1+z)/(1-z)$ is biholomorphic from $\mathbb{D}$ to the right half-plane $\operatorname{Re} z > 0$. As we shall see in the next chapter, the Koebe function has some extreme behavior among biholomorphic maps defined on $\mathbb{D}$.

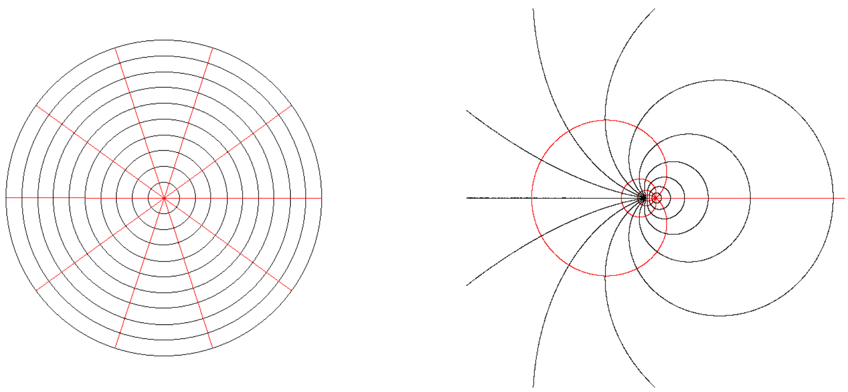

Figure 5.1: The images of the rays and circles by the Koebe function discussed in Example 5.5.

Example 5.6. Using the relations $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ we obtain a formula for the sine function $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, for real values of θ. This can be used to extend sin onto the whole complex plane, i.e.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Let $g(\zeta) = -i \log \zeta$. We have

$$\sin \circ g(\zeta) = \frac{1}{2i}\Big(\zeta - \frac{1}{\zeta}\Big) = \frac{-1}{2}\Big(i\zeta + \frac{1}{i\zeta}\Big) = \frac{-1}{2} f(i\zeta),$$

where f is the function in Example 5.4. Using Example 5.4, we obtain a biholomorphic map

$$\sin : \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\} \to \mathbb{H}.$$

See Figure 5.6

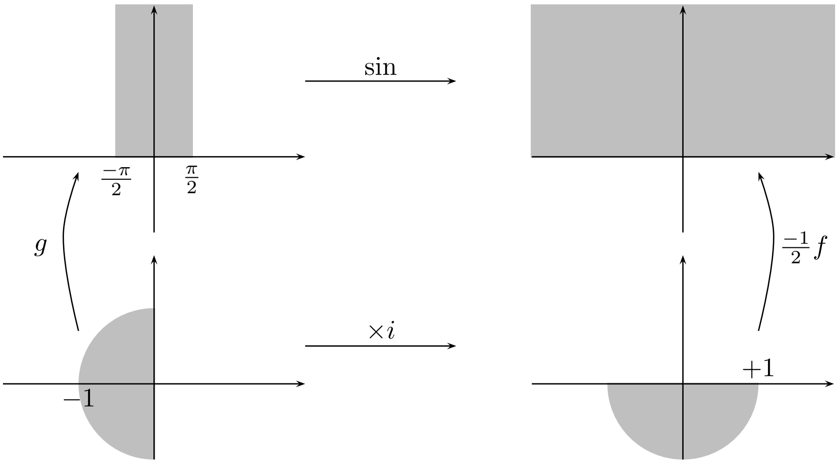

Figure 5.2: The sine function.

5.2 Normal families, Montel's theorem

Let Ω be an open set in $\mathbb{C}$. We would like to introduce a limiting process to build new holomorphic maps on Ω using limits of known holomorphic maps (just like how one build real numbers as limits of rational numbers). There are many notions of convergence of functions in analysis, but it turns out that the natural notion of limit in this setting is the "uniform convergence on compact set".

Definition 5.7. Let $f_n : \Omega \to \mathbb{C}$, for $n \geq 1$, be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. Assume that E is a subset of Ω. We say that the sequence f_n *converges uniformly on* E to some function $f : E \to \mathbb{C}$, if for every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$ and all $z \in E$ we have $|f_n(z) - f(z)| \leq \varepsilon$.

Note that the above convergence is stronger than the *point-wise convergence* where we say that the sequence f_n converges to f on E if for every $z \in E$ and every $\varepsilon > 0$ there is

n_0 such that for all $n \geq n_0$ we have $|f_n(z) - f(z)| \leq \varepsilon$. Here ε may depend on z, but in the uniform convergence ε works for all $z \in E$. For example, the functions $f_n(z) = (1 + 1/n)z$ converge to the function $f(z) = z$ at every point $z \in \mathbb{C}$ bu the convergence is not uniform on unbounded sets $E \subset \mathbb{C}$.

Definition 5.8. Let $f_n : \Omega \to \mathbb{C}$, for $n \geq 1$, be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. We say that f_n *converges uniformly on compact sets* to $f : E \to \mathbb{C}$, if for every compact set $E \subset \Omega$ the sequence f_n converges uniformly to f on E.

It is a simple exercise to show that if $f_n : \Omega \to \mathbb{C}$, $n \geq 1$, are continuous and converge uniformly on compact sets to some $f : \Omega \to \mathbb{C}$, then $f : \Omega \to \mathbb{C}$ is also continuous. We shall prove in Theorem 5.10 that the holomorphic property also survives under convergence on compact sets.

The Cauchy's criterion also has a counter part here. The sequence f_n converges uniformly on E if and only if for every $\varepsilon > 0$ there is an $n_0 > 0$ such that for all $n, m \geq n_0$ and all $z \in E$ we have $|f_n(z) - f_m(z)| \leq \varepsilon$.

Definition 5.9. Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ be a family (set) of maps that are defined on Ω. We say that the family $\mathcal{F}$ is *normal* if every sequence of maps f_n, $n \geq 1$, in $\mathcal{F}$ has a sub-sequence that converges uniformly on every compact subset of Ω.

Note that in the above definition we do not require the limiting map to be in $\mathcal{F}$. This is often a consequence of the uniform convergence. For instance, when the maps involved in the definition are holomorphic we may use the following theorem of Weierstrass.

For example, you can show that the sequence of functions $f_n(z) = z^n$ is normal on $\mathbb{D}$. But this sequence is not normal on the ball $|z| < 2$.

Theorem 5.10. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. Assume that the sequence f_n converges uniformly on compact sets to some function $f : \Omega \to \mathbb{C}$. Then, f is holomorphic on Ω.*

Moreover, $f_n' : \Omega \to \mathbb{C}$ converges uniformly on compact sets to $f' : \Omega \to \mathbb{C}$.

Proof. Let z_0 be an arbitrary point in Ω and choose $r > 0$ such that the disk $|z - z_0| \leq r$ is contained in Ω. Let us denote the circle $|z - z_0| = r$ by γ. Since each f_n is holomorphic on Ω, by Cauchy integral formula, for every z_1 in the disk $|z - z_0| < r$ we have

$$f_n(z_1) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(z)}{z - z_1} \, dz.$$

We wish to take limits as n tends to infinity. To that end we observe that

$$\begin{aligned}
\Big|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-z_1}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_1}\,dz\Big| &= \Big|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)-f(z)}{z-z_1}\,dz\Big| \\
&\leq \frac{1}{2\pi}\int_\gamma \Big|\frac{f_n(z)-f(z)}{z-z_1}\Big|\,|dz| \qquad (5.1)\\
&\leq \frac{r}{r-|z_1-z_0|}\max_{z\in\gamma}|f_n(z)-f(z)|.
\end{aligned}$$

The expression on the last line of the above equation tends to 0 as n tends to ∞. That is because γ is compact in Ω. Taking limits in the Cauchy integral formula for f_n, we obtain

$$f(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_1}\,dz.$$

Note that the above equation holds for every point z_1 enclosed by γ. It is easy to conclude from the above formula that

$$f'(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_1)^2}\,dz. \qquad (5.2)$$

In particular, $f'(z_1)$ exists and is a continuous function of z_1, as long as z_1 moves within the disk $|z-z_0|<r$. That is, $f(z)$ is holomorphic on the disk $|z-z_0|<r$. Since $z_0\in\Omega$ was arbitrary we conclude that f is holomorphic on Ω.

To prove the uniform convergence in the last part of the theorem, let E be a compact set in Ω. For every $z\in E$ there is $r_z>0$ such that the closure of the disk $B(z,2r_z)$ is contained in Ω. Thus the union of $B(z,r_z)$, for $z\in E$, provides a cover of E by open sets. Since E is compact, a finite number of such open sets covers E. Let $z_i\in E$ and $r_i>0$ be a finite collection such that the closure of each $B(z_i,2r_i)$ is contained in Ω and $E\subset\cup_i B(z_i,r_i)$.

Fix an arbitrary i and let γ_i denote the circle $|z-z_i|=2r_i$. We may repeat the inequalities in Equation 5.1 for the integral formula in Equation 5.2 to conclude that for every $z\in B(z_i,r_i)$ we have

$$\begin{aligned}
|f_n'(z)-f'(z)| &= \Big|\frac{1}{2\pi i}\int_{\gamma_i}\frac{f_n(\zeta)}{(\zeta-z)^2}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_i}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\Big| \\
&\leq \frac{1}{r_i}\max_{\zeta\in\gamma_i}|f_n(\zeta)-f(\zeta)|.
\end{aligned}$$

Since f_n converges to f uniformly on the compact set γ_i, the above inequality implies that $f_n'(z)$ converges to $f'(z)$ uniformly on the ball $B(z_i,r_i)$. As there are a finite number of such balls that covers E, and the convergence on each ball is uniform, we conclude that f_n' converges to f' uniformly on E. $\square$

When $\mathcal{F}$ is a class of one-to-one maps on Ω, we need the following theorem of A. Hurwitz to conclude that the limiting maps are also one-to-one.

Theorem 5.11. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$ such that $f_n(z) \neq 0$ for all n and all $z \in \Omega$. If f_n converges uniformly on compact subsets of Ω to some $f : \Omega \to \mathbb{C}$, then either f is identically equal to 0 or has no 0 on Ω.*

Proof. By Theorem 5.10, f is holomorphic on Ω. Thus, by Proposition 3.12, either $f(z) \equiv 0$, or the set of solutions of $f(z) = 0$ forms a discrete subset of Ω. If the latter happens, we show that the set of solutions is empty.

Let z_0 be a solution of $f(z) = 0$. By the above paragraph, there is $r > 0$ such that the ball $B(z_0, r) \subset \Omega$ and the equation $f(z) = 0$ has no solution with $0 < |z - z_0| \leq r$. Let γ denote the circle $|z - z_0| = r$. As γ is compact in Ω, f_n converges uniformly on γ to f. Also, by Theorem 5.10, f_n' converges uniformly on γ to f'. Then, it follows that

$$\lim_{n \to +\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

For each $n \geq 1$, $f_n(z) = 0$ has no solutions in Ω. It follows that $f_n'(z)/f_n(z)$ is holomorphic on Ω. In particular, by Cauchy-Goursat Theorem (Thm 1.2), the integrals on the left hand side of the above equation are equal to 0. Then, the right hand side integral is equal to 0. However, the integral on the right hand side counts the number of points z within γ such that $f(z) = 0$. That is, $f(z) = 0$ has no solution within γ. This contradiction proves that $f(z) = 0$ has no solutions in Ω. $\square$

Corollary 5.12. *Let Ω be an open set in $\mathbb{C}$ and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions that converge uniformly on compact sets to some function $f : \Omega \to \mathbb{C}$. If every f_n is one-to-one on Ω, then either f is a constant function, or f is one-to-one on Ω.*

Proof. Assume that there are distinct points a and b in Ω with $f(a) = f(b)$. Since each f_n is one to one on Ω, $f_n(z) - f(b)$ does not vanish on the ball $|z - a| < r$. Therefore, by Theorem 5.11, either $f(z) - f(b)$ has no zero in $|z - a| < r$ or is identically constant. As a is a zero of this function, f is a constant function. $\square$

It is often possible that a sequence of holomorphic maps has no convergent sub-sequence, or the point-wise limit exists but is not holomorphic. We are looking for criteria on a family that imply a convergent (uniformly on compact sets) sub-sequence exists.

There are some natural conditions that a family of maps must fulfill in order to be uniformly convergent on compact sets. We state these as definitions.

Definition 5.13. Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$. We say that the family $\mathcal{F}$ is *uniformly bounded on compact subsets* of Ω, if for every compact set $E \subset \Omega$, there is a constant M such that for all $z \in E$ and all $f \in \mathcal{F}$ we have $|f(z)| \leq M$.

Definition 5.14. Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$, and let $K \subseteq \Omega$. We say that the family $\mathcal{F}$ is *equicontinuous* on K, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all z and z' in Ω with $|z - z'| \leq \delta$ and all $f \in \mathcal{F}$ we have $|f(z) - f(z')| \leq \varepsilon$.

By the above definition, each map in an equicontinuous family is uniformly continuous.

One can see that any normal family must satisfy the properties in Definitions 5.13 and 5.14. It turns out that the condition in Definition 5.13 is the key to the normality of a family. The following theorem is due to P. Montel.

Theorem 5.15. *Let $\mathcal{F}$ be a family of holomorphic maps defined on an open set $\Omega \subseteq \mathbb{C}$. If $\mathcal{F}$ is uniformly bounded on every compact subset of Ω, then*

(i) $\mathcal{F}$ is equicontinuous on every compact subset of Ω;

(ii) $\mathcal{F}$ is a normal family.

Before we prove the above theorem we give a basic property of compact and closed sets in the plane.

Lemma 5.16. *Let A be a compact set in $\mathbb{C}$ and B be a closed set in $\mathbb{C}$ such that $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap B = \emptyset$. There is $r > 0$ such that for every $z \in A$ and $w \in B$ we have $|z - w| > r$.*

Proof. If there is no such $r > 0$, for each $n \geq 1$ there are $z_n \in A$ and $w_n \in B$ such that $|z_n - w_n| \leq 1/n$. Since A is compact, there is a sub-sequence of z_n, say z_{n_k}, that converges to some $z \in A$. The sequence w_{n_k} is bounded since $|w_{n_k}| \leq |w_{n_k} - z_{n_k}| + |z_{n_k}| \leq 1 + |z_{n_k}|$, and z_{n_k} belong to a compact set. Thus, there is a sub-sequence of w_{n_k} that converges to some w. As B is closed, $w \in B$, and since $|z_{n_k} - w_{n_k}| \leq 1/n$ we have $w = z$. This contradicts $A \cap B = \emptyset$. $\square$

The above lemma allows us to define the distance between a non-empty compact set and a non-empty closed set in $\mathbb{C}$. That is,

$$d(A, B) = \inf\{|z - w| : z \in A, w \in B\}.$$

The above set is bounded from below, and by Lemma 5.16, the infimum is strictly positive when $A \cap B = \emptyset$.

Proof of Theorem 5.15. Part (i): Let K be an arbitrary compact set in Ω. By Lemma 5.16 there is $r > 0$ such that for every $z \in K$ the ball $B(z, 3r) \subset \Omega$.

Let z and w be in K with $|z-w| \leq r$. By the choice of r, the closure of the ball $B(z, 2r)$ is contained in Ω. Let γ_z denote the boundary of the ball $B(z, 2r)$. By the Cauchy integral formula, for every $f \in \mathcal{F}$ we have

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\gamma_z} f(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\Big)\, d\zeta.$$

We note that for $\zeta \in \gamma_z$ we have

$$\Big|\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\Big| \leq \frac{|z - w|}{|\zeta - z||\zeta - w|} \leq \frac{|z - w|}{r^2}.$$

Therefore,

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} |z - w| \sup_{\zeta \in \gamma_z} \cdot |f(\zeta)|.$$

Define

$$\Gamma_K = \{a \in \Omega : d(a, K) \leq 2r\}.$$

This is a compact subset of Ω. By the hypothesis of the theorem, there is $C > 0$, depending on Γ_K, such that for all $f \in \mathcal{F}$ and every $a \in \Gamma_K$ we have $|f(a)| \leq C$. In particular, $\sup_{\zeta \in \gamma_z} |f(\zeta)| \leq C$.

By the above paragraphs, we have shown that for all $f \in \mathcal{F}$and all z and w in K with $|z-w| \leq r$ we have $|f(z)-f(w)| \leq C/r$. This implies that the family $\mathcal{F}$ is equicontinuous on K (given $\varepsilon > 0$ let $\delta = \min\{r, r\varepsilon/C\}$). As K was an arbitrary compact set in Ω, we have proved the first part of the theorem.

Part (ii): Let f_n be an arbitrary sequence in $\mathcal{F}$. There is a sequence of points $\{w_i\}$, for $i \geq 1$, that is dense in Ω. We first extract a sub-sequence of f_n that converges at each of these points w_j. The process we are going to use is known as the Cantor's diagonal process.

By the hypothesis of the theorem, $|f_n(w_1)|$, for $n \geq 1$, is uniformly bounded from above. Hence there is a sub-sequence $f_{1,1}(w_1), f_{2,1}(w_1), f_{3,1}(w_1), \dots$ of this sequence that converges to some point in $\mathbb{C}$. For the same reason, there is a sub-sequence $f_{1,2}, f_{2,2}, f_{3,2}, \dots$ of the sequence $f_{1,1}, f_{2,1}, f_{3,1}, \dots$ such that $\lim_{k\to\infty} f_{k,2}(w_2)$ exists in $\mathbb{C}$. Inductively, for $l \geq 1$, we build a sub-sequence $f_{1,l}, f_{2,l}, f_{3,l}, \dots$ of the sequence $f_{1,l-1}, f_{2,l-1}, f_{3,l-1}, \dots$ such that $\lim_{k\to\infty} f_{k,l}(w_l)$ exists in $\mathbb{C}$.

Let us define the sequence of maps $g_n = f_{n,n}$, for $n \geq 1$. This is a sub-sequence of $\{f_n\}$, and for each $j \geq 1$ the limit $\lim_{n\to\infty} g_n(w_j)$ exists and is finite. We are going to show that this sequence is uniformly convergent on compact sets of Ω. To this end we shall show that this sequence is Cauchy on compact sets. Let K be an arbitrary compact set in Ω and fix $\varepsilon > 0$.

Since $\mathcal{F}$ is equicontinuous on K, for $\varepsilon/3$ there is $\delta > 0$ such that for all $f \in \mathcal{F}$ and every z and w in K with $|z - w| < \delta$ we have $|f(z) - f(w)| \leq \varepsilon/3$. Since K is compact, there are a finite number of points $w_1, w_2, \dots, w_l$ such that $K \subset \cup_{i=1}^{l} B(w_i, \delta)$.

The sequence $\lim_{n\to\infty} g_n(w_i)$ exists for each $i = 1, 2, 3, \dots, l$. In particular each of these sequences is Cauchy. As there are a finite number of points w_i, given $\varepsilon/3$ there is $N > 0$ such that for all $m, n \geq N$ and all $i = 1, 2, 3, \dots, l$ we have

$$|g_n(w_i) - g_m(w_i)| \leq \varepsilon/3.$$

Now, for an arbitrary $w \in K$ there is $i \in \{1, 2, 3, \dots, l\}$ with $w \in B(w_i, \delta/2)$. Then for all $n, m \geq N$, we have

$$\begin{aligned} |g_n(w) - g_m(w)| &\leq |g_n(w) - g_n(w_i)| + |g_n(w_i) - g_m(w_i)| + |g_m(w_i) - g_m(w)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

The above property implies that for each $w \in \Omega$ the sequence of points $g_n(w)$ is convergent. So, the sequence g_n converges at every point in Ω. The convergence is uniform on compact sets. $\square$

Corollary 5.17. *Let $\mathcal{F}$ be a family of holomorphic maps $f : \Omega \to \mathbb{D}$. Then, $\mathcal{F}$ is a normal family.*

The notion of normality corresponds to the notion of pre-compactness in a metric space. To explain the correspondence we define a notion of metric on holomorphic maps such that uniform convergence on compact sets becomes equivalent to the convergence with respect to that metric.

Lemma 5.18. *Let Ω be an open set in $\mathbb{C}$. There are compact sets $E_i \subset \Omega$, for $i \in \mathbb{N}$, such that*

(i) for all $i \geq 1$, E_i is contained in the interior of E_{i+1};

(ii) for every compact set E in Ω there is $i \geq 1$ with $E \subset E_i$. In particular,

$$\Omega = \bigcup_{i=1}^{\infty} E_i.$$

A sequence of sets E_i satisfying the properties in the above lemma is called *an exhaustion of* Ω by compact sets.

Proof of Lemma 5.18. Let us first assume that Ω is bounded in the plane, that is, there is $M > 0$ such that $\Omega \subseteq B(0, M)$. For $l \geq 1$ we define the set

$$E_i = \{z \in \Omega \mid \forall w \in \partial\Omega, |w - z| \geq 1/i\}.$$

Each E_i is a bounded and closed subset of $\mathbb{C}$. Therefore, each E_i is compact. Clearly, every compact subset of Ω is contained in some E_i.

If Ω is not a bounded subset of $\mathbb{C}$, we defined the sets

$$E_i = \{z \in \Omega \mid \forall w \in \partial\Omega, |w - z| \geq 1/i\} \bigcap \{z \in \Omega \mid |z| \leq i\}.$$

One can verify that these sets satisfy the properties in the lemma. □

Define the new metric d' on $\mathbb{C}$ as

$$d'(z, w) = \frac{|z - w|}{1 + |z - w|}. \tag{5.3}$$

One can see that the above function on $\mathbb{C} \times \mathbb{C}$ is a metric, see Exercise 5.5.

Let Ω be an open set in $\mathbb{C}$ and E_i, for $i \in \mathbb{N}$, be an exhaustion of Ω with compact sets. Let $C^0(\Omega)$ be the set of continuous functions on Ω with values in $\mathbb{C}$. Define the function $d'' : C^0(\Omega) \times C^0(\Omega) \to [0, \infty)$ as

$$d''(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\sup_{z \in E_i} |f(z) - g(z)|}{1 + \sup_{z \in E_i} |f(z) - g(z)|}. \tag{5.4}$$

In an exercise you will show that d'' is a metric on the class $\mathcal{F}$.

Theorem 5.19. *A class of holomorphic maps $\mathcal{F}$ defined on an open set Ω is compact with respect to d'' if and only if the family $\mathcal{F}$ is normal and the limiting functions are contained in $\mathcal{F}$.*

See Exercise 5.7 for the proof.

5.3 General form of Cauchy integral formula

Definition 5.20. Let Ω be an open set in $\mathbb{C}$, and $\gamma_1, \gamma_2 : [0,1] \to \Omega$ be continuous maps with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. We say that γ_1 is *homotopic to* γ_2 in Ω if there is a continuous map

$$F : [0,1] \times [0,1] \to \Omega,$$

such that

(i) for all $t \in [0,1]$, $F(t,0) = \gamma_1(t)$;

(ii) for all $t \in [0,1]$, $F(t,1) = \gamma_2(t)$;

(iii) for all $s \in [0,1]$, $F(0,s) = \gamma_1(0)$ and $F(1,s) = \gamma_1(1)$.

In other words, the curves γ_1 and γ_2 with the same end points are homotopic in Ω if one can continuously move one of them to the other one without moving the end points.

Definition 5.21. Let Ω be a path connected subset of $\mathbb{C}$. We say that Ω is *simply connected* if every continuous closed curve $\gamma_1 : [0,1] \to \Omega$ is homotopic to the constant curve $\gamma_2(t) \equiv \gamma(0)$, $t \in [0,1]$ in Ω.

Note that any connected open set in $\mathbb{C}$ is path connected.

Example 5.22. The open unit disk $\mathbb{D}$ is simply connected. To see this, let $\gamma : [0,1] \to \mathbb{D}$ be a closed curve. We define $F : [0,1] \times [0,1] \to \mathbb{D}$ as $F(t,s) = (1-s)\gamma(t) + s\gamma(0)$. This is clearly a continuous map satisfying the three conditions for being a homotopy from γ to the constant curve $\gamma(0)$.

The above example shows that any convex set in $\mathbb{C}$ is simply connected. But this condition is far from necessary as we see in the next example.

Example 5.23. The set $\Omega = \mathbb{C} \setminus [0,\infty)$ is simply connected. Let $\gamma[0,1] \to \Omega$ be an arbitrary continuous map. First we move the curve γ continuously to the constant curve $\gamma_1 \equiv -1$, by the homotopy $F : [0,1] \times [0,1] \to \Omega$ defined as $F(t,s) = (1-s)\gamma(t) - s$. This we move the constant curve γ_1 to the constant curve $\gamma(0)$.

Recall that a curve $\gamma : [a,b] \to \mathbb{C}$ is called a *simple closed curve* if γ is continuous, one-to-one, and $\gamma(a) = \gamma(b)$. It is a non-trivial theorem due to Jordan that every simple closed curve in $\mathbb{C}$ divides the complex plane into two connected components. That is, $\mathbb{C} \setminus \gamma$ has two connected components.

Intuitively, Ω is simply connected if for every simple closed curve γ in Ω, the bounded connected component of $\mathbb{C} \setminus \gamma$ is contained in Ω.

One can show that the set $\Omega = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ is not simply connected. In general, any open set with "holes" is not simply connected.

The above definition allows us to generalize a number of theorems you have already seen in complex analysis. We state these below without proof. The proof of these statements can be found in any standard book on complex analysis.

Theorem 5.24 (Cauchy-Goursat-theorem-general-form)**.** *Assume that Ω is a simply connected open set in $\mathbb{C}$ and f is an analytic map defined on Ω. Let γ be a simple closed curve in Ω which is piece-wise C^1. Then,*

$$\int_\gamma f(z)\,dz = 0.$$

The inverse of the above theorem is also true and is known as the Morera Theorem. That is, if $f(z)$ is defined and continuous in a region Ω, and for all closed curves γ in Ω we have $\int_\gamma f(z)\,dz = 0$, then f is holomorphic on Ω. We shall not use this theorem in this course.

Theorem 5.25 (Cauchy Integral Formula-general version)**.** *Assume that Ω is a simply connected open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is holomorphic. Let γ be a piece-wise C^1 simple closed curve in Ω. Then, for every z_0 in the bounded connected component of $\mathbb{C} \setminus \gamma$,*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Proposition 5.26. *Assume that Ω is a simply connected domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a holomorphic function such that for all $z \in \Omega$, $f(z) \neq 0$. Then, there is a well-defined continuous branch of $\log f(z)$ defined on Ω.*

In particular, there are well-defined continuous branches of $\sqrt[n]{f(z)}$ defined on Ω, for each $n \geq 1$.

Proof. Fix a point z_0 in Ω. By the assumptions, $f(z_0) \neq 0$. So, $\log(f(z_0))$ is defined up to an additive constant in $2\pi i\mathbb{Z}$. Let us fix a value for $\log(f(z_0))$.

By the assumption, the function $f'(z)/f(z)$ is defined and holomorphic on Ω. We define a new function $g(z)$ on Ω as follows. For z in Ω choose a continuous curve $\gamma_z : [0,1] \to \Omega$ with $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. Then define the integral

$$g(z) = \int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta + \log(f(z_0)). \tag{5.5}$$

The above integral is independent of the choice of γ_z. To see this let $\gamma : [0,1] \to \Omega$ be another continuous curve with $\gamma(0) = z_0$ and $\gamma(1) = z$. The curve γ_z followed by the curve $\gamma(1-t)$, for $t \in [0,1]$, is a closed curve in Ω. Then, since Ω is simply connected, by Theorem 5.24,

$$\int_{\gamma_z(t)\cup\gamma(1-t)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = 0.$$

This implies that

$$\int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta.$$

Therefore, $g(z)$ is a well-defined function on Ω.

Since $f'(\zeta)/f(\zeta)$ is continuous on Ω, $g(z)$ is differentiable on Ω with derivative

$$g'(z) = \frac{f'(z)}{f(z)}.$$

This yields

$$\frac{d}{dz}(f(z)\cdot e^{-g(z)}) = 0.$$

As Ω is connected, $f(z)e^{-g(z)}$ must be a constant. Evaluating this function at z_0 we obtain the value of the constant $f(z_0)e^{-\log(f(z_0))} = 1$. Hence, for all $z \in \Omega$ we have $f(z) = e^{g(z)}$. This finishes the proof of the first part.

In the last part of the corollary we define $\sqrt[n]{f(z)} = e^{\frac{1}{n}\log f(z)}$, where $\log f(z)$ is a continuous branch defined on Ω. $\square$

5.4 Riemann mapping theorem

There are some obvious restrictions on a subsets Ω of $\mathbb{C}$ that is biholomorphic to $\mathbb{D}$. If there is a biholomorphic map $\phi : \mathbb{D} \to \Omega$, then Ω must be connected, since the image of any connected set under a continuous map is connected. Also, Ω cannot be equal to $\mathbb{C}$, since the inverse map $\phi^{-1} : \mathbb{C} \to \mathbb{D}$ is bounded and must be constant by Liouville's theorem. Also, Ω must be simply connected, since $\mathbb{D}$ is simply connected, and homeomorphisms map homotopic curves to homotopic curves. It turns out that these three conditions on Ω are enough to guarantee the existence of a biholomorphic map from $\mathbb{D}$ to Ω.

Bernhard Riemann was the first person to state the following important theorem for domains Ω with piece-wise smooth boundaries (1851). However, the proof he presented contained a gap. The first proof of the theorem for arbitrary domains is due to William Osgood (1900), but it did not attract the attention it deserved. The proof we present here is mostly due to Carathéordory with some simplifications due to Paul Koebe.

Theorem 5.27. *Let Ω be a non-empty, connected, and simply connected open subset of $\mathbb{C}$ which is different from $\mathbb{C}$. Then, there is a biholomorphic map $f : \mathbb{D} \to \Omega$.*

Proof. Fix a point $z_0 \in \Omega$. We shall prove that there is a biholomorphic map $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) \in (0, \infty)$. To this end we define the class of maps

$$\mathcal{F} = \Big\{ h : \Omega \to \mathbb{D} \mid h \text{ is holomorphic, one to one}, h(z_0) = 0, \text{ and } h'(z_0) \in (0, \infty) \Big\}.$$

We break the proof into several steps.

Step 1. The class $\mathcal{F}$ is not empty.

If the set Ω is bounded in $\mathbb{C}$, we may translate and re-scale the set Ω to find a map in $\mathcal{F}$. That is, there is $a \in (0, \infty)$ such that the map $f(z) = a(z - z_0)$ maps Ω into $\mathbb{D}$. Clearly f is holomorphic, one-to-one, and $f'(z_0) = a \in (0, \infty)$.

If there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $B(w_0, r) \cap \Omega = \emptyset$, then $g(z) = 1/(z - w_0)$ is holomorphic and one-to-one on Ω and maps Ω to a bounded region in $\mathbb{C}$. Then, there is a linear map f as in the above paragraph such that $f \circ g$ belongs to $\mathcal{F}$.

In general, first we note that since Ω is not equal to $\mathbb{C}$ there is $w_0 \in \mathbb{C} \setminus \Omega$. Hence the function $z \mapsto z - w_0$ never vanishes on the simply connected set Ω. By Proposition 5.26, there is a continuous (and holomorphic) branch

$$f(z) = \log(z - w_0)$$

defined on Ω. As a consequence, we have $e^{f(z)} = z - w_0$. This implies that f is one-to-one on Ω. Fix a point $a \in \Omega$, and observe that for all $z \in \Omega$ we have $f(z) \neq f(a) + 2\pi i$. That is because, otherwise $z - w_0 = e^{f(z)} = e^{f(a)+2\pi i} = e^{f(a)} = a - w_0$. This implies that $z = a$, and hence $f(z) = f(a)$, which is a contradiction.

We claim that there is $r > 0$ such that $f(\Omega) \cap B(f(a) + 2\pi i, r) = \emptyset$. If this is not the case, there is a sequence of points $z_i \in \Omega$ such that $f(z_i)$ converges to $f(a) + 2\pi i$. Since the exponential map is continuous, we conclude that $z_i - w_0 = e^{f(z_i)} \to e^{f(a)+2\pi i} = e^{f(a)} = a - w_0$. This implies that $z_i \to a$, and hence $f(z_i) \to f(a)$. This is a contradiction.

The map

$$I(z) = \frac{1}{f(z) - (f(a) + 2\pi i)}$$

is holomorphic and one-to-one on Ω and $I(\Omega) \subseteq B(0, 1/r)$. As in the first paragraph, we may compose I with a linear transformation to obtain a map of the from $k(z) = a(I(z) - I(z_0))$ that is holomorphic, one-to-one, and $k(\Omega) \subset \mathbb{D}$. Finally, the map $h(z) = k(z)/k'(z_0)$ is holomorphic, one-to-one, $k(z_0) = 0$, and $k'(z_0) \in (0, \infty)$.

Step 2. There is $f \in \mathcal{F}$ such that

$$f'(z_0) = \sup\{h'(z_0) : h \in \mathcal{F}\}.$$

Let $A = \{h'(z_0) : h \in \mathcal{F}\}$. This is a subset of $(0, \infty)$, and by step 1, A is a non-empty set.

As Ω is open, there is $r > 0$ such that $B(z_0, r) \subset \Omega$. Let γ denote the circle $|z - z_0| = r$. By the Cauchy integral formula for the derivative, for every $h \in \mathcal{F}$, we have

$$|h'(z_0)| = \Big|\frac{1}{2\pi i} \int_\gamma \frac{h(\zeta)}{(\zeta - z_0)^2}\, d\zeta\Big| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} \cdot \sup_{\zeta\in\gamma} |h(\zeta)| \leq \frac{1}{r}.$$

This proves that A is bounded from above. In particular, $\sup A$ exists and is finite.

By the definition of supremum, there is a sequence of maps $f_n \in \mathcal{F}$, for $n \geq 1$, such $f'_n(0) \to \sup A$.

Every map in $\mathcal{F}$ maps Ω into the bounded set $\mathbb{D}$. In particular, the family $\mathcal{F}$ is uniformly bounded on compact sets. By Montel's theorem, $\mathcal{F}$ is a normal family. Therefore, $\{f_n\}$ has a sub-sequence converging uniformly on compact sets to some map $f : \Omega \to \mathbb{C}$.

By Theorem 5.10, f is holomorphic. In particular, $f'(z_0)$ exists, and $f'(z_0) = \sup A$ is non-zero. This implies that f is not a constant function.

Each map f_n is one-to-one. Since f is not a constant function, it follows from Corollary 5.12 that f is one-to-one. For each $z \in \Omega$, $|f_n(z)| < 1$. Hence, taking limit along the convergent sub-sequence, we conclude that $|f(z)| \leq 1$. As f is not constant, by the maximum principle, for all $z \in \Omega$, $|f(z)| < 1$, that is, f maps Ω into $\mathbb{D}$.

By the above paragraph $f \in \mathcal{F}$. This finishes the proof of Step 2.

Step 3. The map f obtained in Step 2 is onto.

We shall prove that if f is not onto, there is $h \in \mathcal{F}$ with $h'(z_0) > f'(z_0)$, contradicting the extremality of f in Step 2.

Assume that there is $a \in \mathbb{D} \setminus f(\Omega)$. Consider the automorphism of $\mathbb{D}$ that maps a to 0 and 0 to a, that is,

$$\psi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Since Ω is simply connected, the set $U = \psi_a \circ f(\Omega)$ is simply connected, and since $f(\Omega)$ does not contain a, U does not contain 0. Then, by Proposition 5.26, there is a continuous branch of the square root function defined on U, that is,

$$g(w) = e^{\frac{1}{2}\log w}, w \in U.$$

Consider the function

$$h = \psi_{g(a)} \circ g \circ \psi_a \circ f.$$

The map h is holomorphic on Ω and maps Ω into $\mathbb{D}$. The latter is because, f maps Ω into $\mathbb{D}$, all of the other functions in the composition map $\mathbb{D}$ into $\mathbb{D}$.

As each of the maps f, ψ_a, $\psi_{g(a)}$ and g are one-to-one, h is also one-to-one. We also have $h(z_0) = 0$.

Let $p_2(z) = z^2$ denoted the square function. Define the map $I : \mathbb{D} \to \mathbb{D}$ as

$$I = \psi_a^{-1} \circ p_2 \circ \psi_{g(a)}^{-1}.$$

The function I maps 0 to 0, since $g(a)$ is the square root of a.

We have

$$I \circ h = \big(\psi_a^{-1} \circ p_2 \circ \psi_{g(a)}^{-1}\big) \circ \big(\psi_{g(a)} \circ g \circ \psi_a \circ f\big) = f.$$

In particular, we have

$$I'(0) \cdot h'(z_0) = f'(z_0).$$

On the other hand, by the Schwarz lemma $|I'(0)| < 1$, unless I is a rotation of the circle. However, since, $p_2 : \mathbb{D} \to \mathbb{D}$ is not one-to-one, $I : \mathbb{D} \to \mathbb{D}$ is not one-to-one. In particular, I may not be a rotation of the circle, and we have $|I'(0)| < 1$. This implies that $h'(z_0) > f'(z_0)$. This finishes the proof of Step 3.

All together we have proved that $f : \Omega \to \mathbb{D}$ is holomorphic, one-to-one, and onto. Thus, f is biholomorphic. $\square$

5.5 Exercises

Exercise 5.1. Prove that the function $z \mapsto z + 1/z$ maps the circle $|z| = r > 1$ to the ellipse

$$\frac{x^2}{(r+1/r)^2} + \frac{y^2}{(r-1/r)^2} = 1.$$

Exercise 5.2. Show that

$$\tan z = \frac{1}{i}\Big(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\Big) = i\Big(\frac{1 - e^{2iz}}{1 + e^{2iz}}\Big)$$

Write tan as a composition of the map G in Equation (2.1) and the map $z \mapsto e^{2iz}$. Conclude that

$$\tan : \{z \in \mathbb{C} : -\pi/4 < \operatorname{Re} z < \pi/4, \operatorname{Im} z > 0\} \to \mathbb{H}$$

is biholomorphic.

Exercise 5.3. Let $\Omega = \mathbb{D} \setminus (1/2, 1)$. Find a biholomorphic map from Ω to $\mathbb{D}$ as a composition of some elementary maps.

Exercise 5.4. Prove that if a family of holomorphic maps defined on the same domain Ω is normal then the family is uniformly bounded on compact sets.

Exercise 5.5. Prove that the function d' defined in Equation (5.3) defines a metric on $\mathbb{C}$.

Exercise 5.6. Prove that the function d'' in Equation (5.4) defines a metric on the space $C^0(\Omega)$.

Exercise 5.7. Let Ω, $\mathcal{F}$, and d'' be as in Equation 5.4. Prove that a sequence of functions $f_n \in \mathcal{F}$, $n \geq 1$, converges to some function $f : \Omega \to \mathbb{C}$ uniformly on compact sets if and only if $d''(f_n, f) \to 0$. In particular, the statement of Theorem 5.19 is independent of the choice of the exhaustion E_i in the definition of the metric d''.

Exercise 5.8. Let $\mathrm{Hol}(\Omega)$ denote the space of all holomorphic maps from Ω to $\mathbb{C}$. Define the map $D : \mathrm{Hol}(\Omega) \to \mathrm{Hol}(\Omega)$ as $D(f) = f'$, that is, $D(f)(z) = f'(z)$. Prove that D is continuous from $\mathrm{Hol}(\Omega)$ to $\mathrm{Hol}(\Omega)$ with respect to the metric d''.

Exercise 5.9. Let $\mathcal{F}$ denote the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the from

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$

such that for all $n \geq 2$ we have $|a_n| \leq n$. Prove that the family $\mathcal{F}$ is normal.

Exercise 5.10. Let $f_n : \Omega \to \mathbb{C}$, $n \geq 1$, be a sequence of holomorphic functions that is uniformly bounded on compact sets. Assume that for every $z \in \Omega$ the sequence $f_n(z)$ converges in $\mathbb{C}$. Prove that the sequence f_n converges uniformly on compact sets.

Chapter 6

Growth and Distortion estimates

6.1 The classes of maps $\mathcal{S}$ and Σ

Definition 6.1. Let U be an open subset of $\mathbb{C}$. A holomorphic map $f : U \to \mathbb{C}$ that is one to one is called a *univalent* map. These are also called *schlicht* maps.

In this section we are concerned with the class of maps

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is univalent on } \mathbb{D}, f(0) = 0, f'(0) = 1\}. \tag{6.1}$$

That is, holomorphic and univalent maps defined on $\mathbb{D}$ that are normalized by the condition $f(0) = 0$ and $f'(0) = 1$. Each member of $\mathcal{S}$ has a Taylor series expansion about 0

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \tag{6.2}$$

which is convergent for $|z| < 1$.

By virtue of the Riemann mapping theorem, elements of $\mathcal{S}$ correspond to simply connected regions in $\mathbb{C}$, distinct from $\mathbb{C}$ itself, modulo some translations and re-scaling. The translations and re-scalings allows us to imposed the two conditions $f(0) = 0$ and $f'(0) = 1$. Thus, theorems about elements of $\mathcal{S}$ often translate to geometric features of the simply connected domains obtain as the images of such elements. Before we discuss such results we give some simple, but key, examples of maps in $\mathcal{S}$.

(i) The identity map $f(z) = z$ is univalent on $\mathbb{D}$. Hence, $\mathcal{S}$ is not empty.

(ii) The Koebe function we discussed in Example 5.5

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \dots.$$

The map f is univalent from $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$. In many ways, as we shall see in this section, f is a leading example in the class $\mathcal{S}$.

(iii) The map
$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \dots$$
which maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/2] \cup [1/2, \infty)$. This is obtained from the map in Example 5.4 using the transformation $z \mapsto -if(iz)$.

(iv) The map
$$f(z) = \frac{1}{2} \log \frac{1+z}{1-z},$$
which maps $\mathbb{D}$ onto the strip $-\pi/4 < \operatorname{Im} w < \pi/4$.

(v) The map
$$f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}(1 - (1-z)^2),$$
which maps $\mathbb{D}$ onto a cardioid.

Note that the class of maps $\mathcal{S}$ is not closed under addition and multiplication. For example, the maps $z \mapsto \frac{z}{1-z}$ and $z \mapsto \frac{z}{1+iz}$ are in class $\mathcal{S}$, but their sum is not univalent as it has a critical point at $(1+i)/2$.

However, the class of maps $\mathcal{S}$ is preserved under a number of transformations. We list these below.

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be an arbitrary element of $\mathcal{S}$.

(i) Conjugation: The map
$$g(z) = \overline{f(\overline{z})} = z + \overline{a_2} z^2 + \overline{a_3} z^3 + \dots$$
belongs to $\mathcal{S}$. This property implies that for every integer $k \geq 1$ the set
$$\{f^{(k)}(0) : f \in \mathcal{S}\}$$
is invariant under the complex conjugation. That is, symmetric with respect to the real axis.

(ii) Rotation: For every $\theta \in \mathbb{R}$, the map
$$g(z) = e^{-i\theta} f(e^{i\theta} z) = z + e^{i\theta} a_2 z^2 + e^{i2\theta} a_3 z^3 + \dots$$
belongs to $\mathcal{S}$. This property implies that for every integer $k \geq 1$ the set
$$\{f^{(k)}(0) : f \in \mathcal{S}\}$$
is invariant under the rotations about 0.

(iii) Dilation: For every $r \in (0,1)$, the map

$$g(z) = r^{-1} f(rz) = z + r a_2 z^2 + r^2 a_3 z^3 + \dots$$

belongs to $\mathcal{S}$.

(iv) Disk automorphism: For every fixed $\alpha \in \mathbb{D}$, the map

$$g(z) = \frac{f\Big(\dfrac{z+\alpha}{1+\overline{\alpha} z}\Big) - f(\alpha)}{(1-|\alpha|^2) f'(\alpha)}$$

belongs to $\mathcal{S}$.

(v) Range transformation: If ψ is a function that is analytic and univalent on the range of f with $\psi(0) = 0$ and $\psi'(0) = 1$ then the map $g = \psi \circ f$ belongs to $\mathcal{S}$.

(vi) Omitted value transformation: If w does not belong to the range of f then the map

$$g(z) = \frac{w f(z)}{w - f(z)}$$

belongs to $\mathcal{S}$. This is a special case of the transformation in (v), where we have post composed the map f with the transformation $z \mapsto (wz)/(w-z)$.

(vi) Square-root transformation: There is a well-defined and continuous branch of the map

$$g(z) = \sqrt{f(z^2)}$$

that belongs to $\mathcal{S}$. To see this first note that $f(z)$ has a unique zero at 0 which implies that $f(z^2)$ has a unique zero at 0 and this zero is of order 2. Thus, if we expand the map

$$f(z^2) = z^2 + a_2 z^4 + a_3 z^6 + a_4 z^8 + \dots = z^2(1 + a_2 z^2 + a_3 z^4 + a_4 z^6 + \dots).$$

In particular, the expression in the above parenthesis never becomes zero on $\mathbb{D}$. By Proposition 5.26, there is a continuous branch of the square root of $(1 + a_2 z^2 + a_3 z^4 + a_4 z^6 + \dots)$ defined on $\mathbb{D}$. There are two such branches, with values equal to $+1$ and -1 at 0. We choose the branch with value $+1$ at 0, and denote it by $h(z)$. Then

$$g(z) = \sqrt{f(z^2)} = z \cdot h(z).$$

We have $g(0) = 0$ and $g'(0) = 1 \cdot h(z) + z \cdot h'(z)|_{z=0} = 1$. It remains to show that g is univalent on $\mathbb{D}$.

The map h is an even functions, as $h(z) = h(-z)$. Hence, g is an odd function, that is $g(-z) = -g(z)$, for all $z \in \mathbb{D}$. Let z_1 and z_2 be two points in $\mathbb{D}$ with $g(z_1) = g(z_2)$. Thus, $f(z_1^2) = f(z_2^2)$. As f is one-to-one, we must have $z_1^2 = z_2^2$. This implies that $z_1 = \pm z_2$. However, if $z_1 = -z_2$, then $g(z_2) = g(-z_1) = -g(z_1)$. Hence, combining with $g(z_1) = g(z_2)$, we must have $g(z_1) = 0$, which is only possible if $z_1 = 0$.

Using $(1+x)^{1/2} = 1 + x/2 - x^2/4 + \dots$, we can see that

$$h(z) = 1 + \frac{a_2}{2} z^2 + \dots.$$

This implies that

$$g(z) = \sqrt{f(z^2)} = z + \frac{a_2}{2} z^3 + \dots.$$

The symmetrization of f into g leads to eliminating the second derivative at 0.

Define

$$\Delta = \{w \in \mathbb{C} : |w| > 1\}.$$

A closely related class of maps to $\mathcal{S}$ is the class of maps

$$\Sigma = \big\{g : \Delta \to \mathbb{C} : g \text{ is univalent on } \Delta, \lim_{z\to\infty} g(z) = \infty, g'(\infty) = 1\big\}.$$

Recall that the condition $\lim_{z\to\infty} g(z) = \infty$ implies that g is holomorphic from a neighborhood of ∞ to a neighborhood of infinity. The derivative of g at ∞ is calculated by looking at the derivative of the map $f(z) = 1/g(1/z)$ at 0. That is,

$$g'(\infty) = f'(0).$$

An element of Σ has a series expansion

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \tag{6.3}$$

that is convergent for $|z| > 1$. Each $g \in \Sigma$ maps Δ onto the complement of some compact and connected set in $\mathbb{C}$. It is useful to consider the subclass of maps

$$\Sigma' = \{f : \Delta \to \mathbb{C} : f \in \Sigma, 0 \notin f(\Delta)\}.$$

Note that every element of Σ can be adjusted by adding a constant term to make it an element of Σ'. Such a transformation only translates the image of the element by a constant, and does not change the shape of the image.

There is a one-to-one correspondence between $\mathcal{S}$ and Σ' obtained by inversion. That is, for each $f \in \mathcal{S}$ the map

$$g(z) = \frac{1}{f(1/z)}, |z| > 1,$$

belongs to Σ'. One can see that if f has the series expansion given in Equation (6.2), then

$$g(z) = z - a_2 + \frac{a_2^2 - a_3}{z} + \dots.$$

In particular the class of maps Σ' is invariant under the square-root transformation,

$$G(z) = \sqrt{g(z^2)} = z(1 + b_0 z^{-2} + b_1 z^{-4} + \dots)^{1/2}.$$

Note that the square-root transformation may not be applied to elements of Σ. That is because if $g(z^2)$ has a zero at some point in Δ, then G will necessary have a singularity at that point.

Recall that a set $E \subset \mathbb{C}$ is said to have Lebesgue measure zero, or of zero area, if for every $\varepsilon > 0$ there are $z_i \in \mathbb{C}$ and $r_i > 0$ such that $E \subseteq \cup B(z_i, r_i)$ and $\sum_i \pi r_i^2 \leq \varepsilon$.

A relevant subclass of Σ is

$$\tilde{\Sigma} = \{f : \Delta \to \mathbb{C} : f \in \Sigma, \mathbb{C} \setminus f(\Delta) \text{ has zero Lebegue measure.}\}$$

The functions in the above class are sometimes referred to as *full mappings*.

6.2 Area theorem

Gronwall in 1914 discovered the following restriction on the coefficients of the functions in class Σ.

Theorem 6.2 (Area theorem)**.** *If*

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

belongs to Σ, then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1, \tag{6.4}$$

with the equality if and only if $g \in \tilde{\Sigma}$.

The above theorem is the basis of a theory of univalent functions, parts of which we shall present in this section. The reason for the name area theorem comes from the proof.

Proof. For $r > 1$, let C_r denote the image of the circle $|z| = r$ under g. Each C_r is a simple, closed, and smooth curve. Let E_r denote the bounded connected component of

$\mathbb{C} \setminus C_r$. Let $w = x + iy$ be the coordinate in the image of g. Then, by Green's theorem, for every $r > 1$,

$$
\begin{aligned}
\operatorname{area}(E_r) = \int_{C_r} x\, dy &= \frac{1}{2i} \int_{C_r} \overline{w}\, dw \\
&= \frac{1}{2i} \int_{|z|=r} \overline{g(z)} g'(z)\, dz \\
&= \frac{1}{2} \int_0^{2\pi} \Big(re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n} r^{-n} e^{in\theta} \Big) \Big(1 - \sum_{\nu=1}^{\infty} \nu b_\nu r^{-\nu-1} e^{-i(\nu+1)\theta} \Big) re^{i\theta}\, d\theta \\
&= \pi \Big(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \Big).
\end{aligned}
$$

Taking limits as r tends to 1 from above in the above equation, we conclude that

$$
\operatorname{area}(\mathbb{C} \setminus g(\Delta)) = \pi \Big(1 - \sum_{n=1}^{\infty} n |b_n|^2 \Big).
$$

(Note that we are allows to take limit of the infinite sum, since the infinite sum is a decreasing function of r and is uniformly bounded from above. See the dominated convergence theorem.) Since the left-hand side of the above equation is ≥ 0, we obtain the inequality in the theorem. $\square$

As each term in the sum in Equation (6.4) is positive, we conclude that for every $n \geq 1$ we must have

$$
|b_n| \leq \frac{1}{\sqrt{n}}.
$$

However, these bounds are not sharp for values of $n \geq 2$. For example, for $n \geq 2$, the function

$$
g_n(z) = z + n^{-1/2} z^{-n}
$$

is not univalent on Δ. That is because,

$$
g_n'(z) = 1 - n^{1/2} z^{-n-1}
$$

vanishes at some points in Δ. The inequality for $n = 1$ is sharp, as stated below.

Corollary 6.3. *If $g \in \Sigma$, then $|b_1| \leq 1$, with equality if and only if g has the form*

$$
g(z) = z + b_0 + b_1/z, |b_1| = 1.
$$

The above map g is a conformal mapping of Δ onto the complement of a line segment of length 4.

Proof. By Theorem 6.2, we must have $|b_1| \leq 1$.

If the equality $|b_1| = 1$ occurs, we must have $b_n = 0$ for all $n \geq 2$. Thus, g has the desired form in the corollary.

Indeed, we can show that for any b_0 and b_1 with $|b_1| = 1$, the map g belongs to Σ. Given b_0 and b_1, let $a_1 = \sqrt{b_1}$, for some choice of the square root, and then let $a_2 = 1/a_1$. Define the maps $h_1(z) = a_1 z$ and $h_2(z) = a_2 z - a_2 b_0$. The maps h_1 and h_2 are automorphisms of $\mathbb{C}$. The map $f = h_2 \circ g \circ h_1$ is defined and univalent on Δ. A simple calculation shows that $f(z) = z + 1/z$, for $z \in \Delta$. In Example 5.4 we have seen that f is univalent on Δ with image equal to $\mathbb{C} \setminus [-2, 2]$. This implies that g is univalent on Δ and its image is equal to some line segment of length 4.

It is also clear that $g(\infty) = \infty$, and $g'(\infty) = 1$. □

As a consequence of Corollary 6.3, we obtain a short proof of the Bieberbach estimate on the second coefficient.

Theorem 6.4 (Bieberbach's Theorem)**.** *If $f \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

Proof. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. We apply the square root transformation to obtain

$$h(z) = \sqrt{f(z^2)} = z + \frac{a_2}{2} z^3 + \dots.$$

We saw in Section 6.1 that this is an element of $\mathcal{S}$. Applying an inversion to the map h we obtain

$$g(z) = \frac{1}{h(1/z)} = \frac{1}{f(1/z^2)^{1/2}} = \frac{1}{1/z + \frac{a_2}{2z^3} + \dots} = z\Big(\frac{1}{1 + \frac{a_2}{2z^2} + \dots}\Big) = z - \frac{a_2}{2}\frac{1}{z} + \dots.$$

The map g belongs to Σ. Thus by Corollary 6.3, $|a_2| \leq 2$.

If $|a_2| = 2$, then g reduces to the form

$$g(z) = z - e^{i\theta}/z,$$

which is equivalent to

$$f(1/z^2) = \frac{z^2}{z^4 - 2e^{i\theta} z^2 + e^{2i\theta}}.$$

Using the coordinate $w = 1/z^2$ on $\mathbb{D}$ we conclude that

$$f(w) = \frac{w}{(1 - e^{i\theta} w)^2} = e^{-i\theta} \frac{e^{i\theta} w}{(1 - e^{i\theta} w)^2} = e^{-i\theta} k(e^{i\theta} w),$$

where k is the Koebe function. □

Recall that any holomorphic map is an open mapping. That is, the image of every open set under a holomorphic map is open. In particular, this implies that for every $f \in \mathcal{S}$, $f(\mathbb{D})$ contains some disk of positive radius centered at 0. Around 1907, Koebe discovered that there is a uniform constant ρ such that the image of every map in $\mathcal{S}$ contains the open disk $B(0, \rho)$. The Koebe map suggests that ρ must be less than or equal to 1/4. Koebe conjectured that $\rho = 1/4$. Bieberbach later established this conjecture.

Theorem 6.5 (Koebe 1/4-Theorem)**.** *For every $f \in \mathcal{S}$, $f(\mathbb{D})$ contains the ball $|w| < 1/4$.*

Proof. Let $f(z) = z + a_2 z^2 + \dots$ be a function in $\mathcal{S}$ that omits a value $w \in \mathbb{C}$. Using the omitted value transformation, we build the map

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \big(a_2 + \frac{1}{w}\big)z^2 + \dots$$

in class $\mathcal{S}$. By Theorem 6.4, we must have

$$\Big|a_2 + \frac{1}{w}\Big| \leq 2.$$

Combining with the estimate $|a_2| \leq 2$, we conclude that $|1/w| \leq 4$. That is, $|w| \geq 1/4$. This finishes the proof of the theorem. □

The above proof shows that the Koebe function, and its rotations, are the only functions omitting a w with $|w| = 1/4$. Thus, any other function in $\mathcal{S}$ covers a larger disk.

6.3 Growth and Distortion theorems

Shapes in $\mathbb{D}$ are distorted under a map $f \in \mathcal{S}$ according to the changes in $f'(z)$. For instance, fast changes in the size of $|f'(z)|$ cause nearby curves of the same length to be mapped to curves of very different length, or fast changes in $\arg f'(z)$ make straight line segments to be mapped to curves with sharp bends. The upper bound on the size of the second derivative at 0, that is $|a_2| \leq 2$, leads to a collection of uniform bounds on the changes of $f'(z)$ as z varies in $\mathbb{D}$. Here uniform means that estimates that are independent of the map in $\mathcal{S}$. The bounds we discuss in this section are known as the Koebe distortion theorems.

We first formulate a basic theorem that leads to the distortion estimates and related results.

Theorem 6.6. *For each $f \in \mathcal{S}$, we have*

$$\Big|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1 - r^2}\Big| \leq \frac{4r}{1 - r^2}, \quad r = |z| < 1. \tag{6.5}$$

Proof. Given $f \in \mathcal{S}$ and $z \in \mathbb{D}$, we use the disk automorphism transformation to build the map

$$F(w) = \frac{f\big(\frac{w+z}{1+\overline{z}w}\big) - f(z)}{(1-|z|^2)f'(z)} = w + \frac{1}{2}\Big((1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\Big)w^2 + \dots.$$

Since the map $F \in \mathcal{S}$, by Theorem 6.4, the absolute value of the coefficient of w^2 in the above expansion is bounded from above by 2. Thus,

$$\Big|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\Big| \leq 4,$$

which implies the desired inequality in the theorem. $\square$

Theorem 6.7 (Distortion Theorem)**.** *For each $f \in \mathcal{S}$, we have*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad r = |z| < 1. \tag{6.6}$$

Moreover, one of the equalities hold at some $z \neq 0$, if and only if f is a suitable rotation of the Koebe function.

In order to prove the above theorem we need a lemma on calculating derivatives with respect to the polar coordinates.

Lemma 6.8. *There is a continuous branch of* $\log f'(z)$ *defined on* $\mathbb{D}$ *that maps* 0 *to* 0. *Moreover, for all $z = re^{i\theta}$ in $\mathbb{D}$ we have*

$$\frac{zf''(z)}{f'(z)} = r\frac{\partial}{\partial r}(\log|f'(z)|) + ir\frac{\partial}{\partial r}(\arg f'(z)).$$

Proof. Recall that $f'(0) = 1$, and since f is univalent on $\mathbb{D}$, for all $z \in \mathbb{D}$, $f'(z) \neq 0$. Thus, by Proposition 5.26, there is a continuous branch of $\log f'(z)$ defined on $\mathbb{D}$ which maps 0 to 0.

Let $u(z) = u(re^{i\theta})$ be an arbitrary holomorphic function defined on some open set $U \subset \mathbb{C}$. Using the relation $z = r\cos\theta + ir\sin\theta$ we have

$$r\frac{\partial u}{\partial r} = r\frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r} = r\frac{\partial u}{\partial z} \cdot (\cos\theta + i\sin\theta) = z \cdot \frac{\partial u}{\partial z}.$$

Applying the above formula to the function $\log f'(z)$, and using $\log z = \log|z| + i\arg z$, we obtain the desired relation

$$\frac{zf''(z)}{f'(z)} = z \cdot \frac{\partial}{\partial z}(\log f'(z)) = r\frac{\partial}{\partial r}(\log f'(z)) = r\frac{\partial}{\partial r}(\log|f'(z)|) + ir\frac{\partial}{\partial r}(\arg f'(z)).$$

$\square$

Proof of Theorem 6.7. Note that inequality $|w-c|<R$ implies $c-R\leq \operatorname{Re} w\leq c+R$. In particular, by Equation (6.5), for $|z|=r$, we have

$$\frac{2r^2}{1-r^2}-\frac{4r}{1-r^2}\leq \operatorname{Re}\Big(\frac{zf''(z)}{f'(z)}\Big)\leq \frac{2r^2}{1-r^2}+\frac{4r}{1-r^2},$$

which simplifies to

$$\frac{2r^2-4r}{1-r^2}\leq \operatorname{Re}\Big(\frac{zf''(z)}{f'(z)}\Big)\leq \frac{2r^2+4r}{1-r^2}. \tag{6.7}$$

By Lemma 6.8, there is a continuous branch of $\log f'(z)$ defined on $\mathbb{D}$ that maps 0 to 0. Moreover, the relation in the lemma, and the above inequality implies that

$$\frac{2r-4}{1-r^2}\leq \frac{\partial}{\partial r}\log|f'(re^{i\theta})|\leq \frac{2r+4}{1-r^2}. \tag{6.8}$$

Now we fix θ and integrate the above equation from 0 to R to obtain

$$\log\frac{1-R}{(1+R)^3}\leq \log|f'(Re^{i\theta})|\leq \log\frac{1+R}{(1-R)^3}. \tag{6.9}$$

Above we have used the explicit calculation

$$\int_0^R \frac{2r+4}{1-r^2}\,dr=\int_0^R \frac{3}{1-r}+\frac{2}{1+r}\,dr=-3\log(1-r)+\log(1+r)\Big|_{r=0}^{r=R}=\log\frac{1+R}{(1-R)^3}.$$

As the map $x\mapsto e^x$ is monotone, Equation (6.9) implies the desired inequality in the theorem.

Assume that for some $z=Re^{i\theta}\in\mathbb{D}$, $z\neq 0$, we have an equality in Equation 6.6. Then, we must have the corresponding equality in Equation (6.9) for R. The latter condition implies the corresponding equality in Equation (6.8) and then in Equation (6.7), for all $r\in(0,R)$. Now let r tend to 0 from above, to obtain one of the equalities

$$\operatorname{Re}\Big(e^{i\theta}f''(0)\Big)=+4,\qquad \text{or}\qquad \operatorname{Re}\Big(e^{i\theta}f''(0)\Big)=-4.$$

Recall that since $f\in\mathcal{S}$, by Theorem 6.4, $|f''(0)|\leq 4$. Therefore, by the above equation we must have $|f''(0)|=4$. By the same theorem, we conclude that f must be a rotation of the Koebe function.

For the Koebe function $k(z)=z/(1-z)^2$, we have

$$k'(z)=\frac{1+z}{(1-z)^3},$$

so we have the right-hand equality at every $z=r\in(0,1)$.

On the other hand, for the function $h(z)=e^{i\pi}k(e^{-i\pi}z)$, where k is the Koebe function we have

$$h'(z)=k'(e^{-i\pi}z)=\frac{1-z}{(1+z)^3},$$

so we have the left-hand equality at any $z \in (0,1)$. This finishes the proof of the if and only if statement. $\square$

Theorem 6.9 (Growth Theorem)**.** *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r. \tag{6.10}$$

Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof. An upper bound on $|f'(z)|$ as in Theorem 6.7 gives an upper bound on $|f(z)|$. That is, fix $z = re^{i\theta} \in \mathbb{D}$. Observe that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) d\rho.$$

Then,

$$|f(z)| =\leq \int_0^r |f'(\rho e^{i\theta})|\, d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3}\, d\rho = \frac{r}{(1-r^2)}.$$

However, since we are working in dimension 2, a lower bound on $|f'|$ does not give a lower bound $|f|$. Let z be an arbitrary point in $\mathbb{D}$. We consider two possibilities:

(i) $|f(z)| \geq 1/4$,

(ii) $|f(z)| < 1/4$.

Assume that (i) occurs. Since for all $r \in (0,1)$, $r/(1+r)^2 \leq 1/4$, we trivially have $r/(1+r^2) \leq |f(z)|$.

Now assume that (ii) occurs. By the Koebe 1/4-Theorem, the radial line rz, for $r \in [0,1]$ is contained in the image of of f. As f is one-to-one, the pre-image of this radial line, is a simple smooth curve in $\mathbb{D}$ connecting 0 to z. Let C denoted this curve. We have

$$f(z) = \int_C f'(w)\, dw.$$

By the definition of C, for any point w on C, $f'(w)dw$ has the same argument as the argument of z. Thus,

$$|f(z)| = \Big| \int_C f'(w)\, dw \Big| = \int_C |f'(w)|\, |dw| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3}\, d\rho = \frac{r}{(1+r)^2}.$$

It follows from the above arguments that an inequality in either side of Equation (6.10) implies the equality in the corresponding side of Equation (6.6), which by Theorem 6.7 implies that f is a suitable rotation of the Koebe function.

Also, as in the proof of the previous theorem, suitable rotations of the Koebe function lead to the equality on either side of Equation (6.10). Thus, the bounds in the theorem are sharp. $\square$

It is possible to prove a distortion estimate involving both of $|f(z)|$ and $|f'(z)|$.

Theorem 6.10 (combined growth-distortion Theorem). *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{1+r} \leq \Big|\frac{zf'(z)}{f(z)}\Big| \leq \frac{1+r}{1-r}, |z| = r. \tag{6.11}$$

Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

It is not possible to conclude the above theorem as a combination of the bounds in Theorems 6.7 and 6.9. But the proof is obtained from applying the Beiberbach Theorem 6.4 to a suitable disk automorphism applied to f. As we have already seen this technique we skip the proof of the above theorem.

Theorem 6.11 (Radial distortion Theorem). *For each $f \in \mathcal{S}$,*

$$|\arg f'(z)| \leq 2\log\frac{1+r}{1-r}, |z| = r. \tag{6.12}$$

Proof. By considering the imaginary part of the inequality in Theorem 6.7, we obtain

$$-\frac{4r}{1-r^2} \leq \operatorname{Im}\Big(\frac{zf''(z)}{f'(z)}\Big) \leq \frac{4r}{1-r^2}.$$

By Lemma 6.8, this implies that

$$-\frac{4}{1-r^2} \leq \frac{\partial}{\partial r}\arg f'(re^{i\theta}) \leq \frac{4}{1-r^2}.$$

Integrating the above equation from $r = 0$ to $r = |z|$ we obtain

$$|\arg f'(z)| \leq \int_{r=0}^{r=|z|} \frac{4}{1-r^2}dr = 2\log\frac{1+r}{1-r}.$$

This finishes the proof of the theorem. □

The quantity $\arg f'(z)$ has a geometric interpretation as the "local rotation" factor of f at z. Unfortunately, in contrast to the other bounds we proved in this section, the upper bound in Theorem 6.11 is not optimal. The optimal bound is

$$|\arg f'(z)| \leq \begin{cases} 4\sin^{-1} r & r \leq 1/\sqrt{2}, \\ \pi + \log\frac{r^2}{1-r^2} & r \geq 1/\sqrt{2}. \end{cases}$$

This lies much deeper than the arguments we have seen so far. The proof relies on a more powerful method known as Loewner evolution.

We have seen so far that for every $f \in \mathcal{S}$ we have $|a_2| \leq 2$. This naturally raises the question of finding the quantities

$$A_n = \sup_{f \in \mathcal{S}} |a_n|.$$

In Exercise 6.3 you will show that these are finite numbers. The Koebe function has coefficients

$$K(z) = \sum_{n=1}^{\infty} n z^n,$$

as the Koebe function is the extreme example in the distortion theorems, Bieberbach in 1916 conjectured that $A_n = n$, for all n. This conjecture motivated the development of many techniques in complex analysis and eventually settled by Louis de Branges in 1985.

6.4 Exercises

Exercise 6.1. Show that the class of maps $\mathcal{S}$ forms a normal family.

Exercise 6.2. Let $f : \mathbb{D} \to \mathbb{C} \setminus \{c\}$ be a one-to-one and holomorphic map. Prove that for every $z \in \mathbb{D}$ we have

$$|f(z)| \leq \frac{4|cz|}{(1-|z|)^2}.$$

Exercise 6.3. Let $k \geq 2$ be an integer and define

$$\Lambda_k = \{f^{(k)}(0) : f \in \mathcal{S}\}.$$

Prove that

(i) for every $k \geq 2$, there is $r_k > 0$ such that $\Lambda_k = \{w \in \mathbb{C} : |w| \leq r_k\}$;

(ii) there is a constant $C > 0$ such that for all $n \geq 1$ we have $r_n \leq C n^2 n!$.

Exercise 6.4. Show that for every integer $n \geq 1$, the function

$$h_n(z) = \frac{1}{n}(e^{nz} - 1),$$

satisfies $f_n(0) = 0$, and $f_n'(0) = 1$, but f_n omits value $-1/n$.

Exercise 6.5. Let Ω be a non-empty simply connected subset of $\mathbb{C}$ that is not equal to $\mathbb{C}$. For $z \in \Omega$, the *conformal radius* of Ω at z is defined as

$$\mathrm{rad}_{\mathrm{conf}}(\Omega, z) = |\varphi'(0)|,$$

where $\varphi : \mathbb{D} \to \Omega$ is the Riemann mapping with $\varphi(0) = z$.

(i) Prove that the quantity $\mathrm{rad}_{\mathrm{conf}}(\Omega, z)$ is independent of the choice of the Riemann map φ.

(ii) Define

$$r_z = \sup\{r > 0 : B(z, r) \subset \Omega\}.$$

Prove that

$$r_z \le \mathrm{rad}_{\mathrm{conf}}(\Omega, z) \le 4r_z.$$

(iii) Let $\Omega' \subset \Omega$ be a simply connected set that contains z. Prove that

$$\mathrm{rad}_{\mathrm{conf}}(\Omega', z) < \mathrm{rad}_{\mathrm{conf}}(\Omega, z).$$

Exercise 6.6. Prove that there is $r > 0$ such that for every one-to-one and holomorphic map $f : \mathbb{D} \to \mathbb{C}$, the set $f(B(0, r))$ is a convex subset of $\mathbb{C}$.

Chapter 7

Quasi-conformal maps and Beltrami equation

7.1 Linear distortion

Assume that $f(x+iy) = u(x+iy) + iv(x+iy)$ be a (real) linear map from $\mathbb{C} \to \mathbb{C}$ that is orientation preserving. Let $z = x+iy$ and $w = u+iv$. The map $z \mapsto w = f(z)$ can be expressed by a matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \tag{7.1}$$

where T is the 2×2 matrix

$$T = Df(z) = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some real constants a, b, c, and d. As f is orientation preserving, the determinant of the matrix T is positive, that is, $ad - bc > 0$.

The circle $|z|^2 = x^2 + y^2 = 1$ is mapped by f to an ellipse with equation $|T^{-1}w|^2 = 1$. The *distortion* of f, denoted by K_f, is defined as the eccentricity of this ellipse, that is, K_f is the ratio of the length of the major axis of the ellipse to the length of its minor axis of the ellipse. Since f is a linear map, the distortion of f is independent of the radius of the circle $|z| = 1$ we choose to define the ellipse.

A basic calculation leads to the equation

$$K_f + 1/K_f = \frac{a^2 + b^2 + c^2 + d^2}{ad - bc}.$$

for K_f in terms of a, b, c, and d. The above simple quantity and the forthcoming relations are rather complicated when viewed in real coordinates, but find simple forms in complex notations.

Any real-linear map $T : \mathbb{C} \to \mathbb{C}$ can be expressed in the form

$$w = T(z) = Az + B\overline{z}, \tag{7.2}$$

for some complex constants A and B. If T is orientation preserving, we have $\det T = |A|^2 - |B|^2 > 0$. Then, T can be also represented as

$$T(z) = A(z + \mu\bar{z}),$$

where

$$\mu = B/A, \text{ and } |\mu| < 1.$$

That is, T may be decomposed as the stretch map $S(z) = z + \mu\bar{z}$ post-composed with the multiplication by A. The multiplication consists of rotation by the angle $\arg(A)$ and magnification by $|A|$. Thus, all of the distortion caused by T is expressed in terms of μ. From μ one can find the angles of the major axis and minor axis of the image ellipse. The number μ is called the complex dilatation of T.

The maximal magnification occurs in the direction $(\arg\mu)/2$ and the magnification factor is $1+|\mu|$. The minimal magnification occurs in the orthogonal direction $(\arg\mu-\pi)/2$ and the magnification factor is $1-|\mu|$. Thus, the distortion of T, which only depends on μ, is given by the formula

$$K_T = \frac{1+|\mu|}{1-|\mu|}.$$

A basic calculation implies

$$|\mu| = \frac{K_T - 1}{K_T + 1}.$$

If T_1 and T_2 are real-linear maps from $\mathbb{C}$ to $\mathbb{C}$ one can see that

$$K_{T_2 \circ T_1} \leq K_{T_2} \cdot K_{T_1}. \tag{7.3}$$

The equality in the above equation may occur when the major axis of $T_1(\partial\mathbb{D})$ is equal to the direction in which the maximal magnification of T_2 occurs and the minor axis of $T_1(\partial\mathbb{D})$ is equal to the direction at which the minimal magnification of T_2 occurs. Otherwise, one obtains strict inequality.

7.2 Dilatation quotient

Assume that $f : \Omega \to \mathbb{C}$ is an orientation preserving diffeomorphism. That is, f is homeomorphism, and both f and f^{-1} have continuous derivatives. Let $z = x + iy$ and $f(x+iy) = u(x,y) + iv(x,y)$. At $z_0 = x_0 + iy_0 \in \Omega$ and $z = x + iy$ close to zero we have

$$f(z_0 + z) = f(z_0) + \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + o(z). \tag{7.4}$$

In the above equation, the little o notation means any function $g(z)$ which satisfies $\lim_{z\to 0} g(z)/z = 0$.

We may write Equation (7.4) in the complex notation

$$f(z_0 + z) = f(z_0) + Az + B\overline{z} + o(z),$$

where A and B are complex numbers (which depend on z_0). Comparing the above two equations we may determine A and B in terms of the partial derivatives of f. That is, setting $z = 1$ and $z = i$ we obtain (respectively)

$$A + B = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad Ai - Bi = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

These imply that

$$A = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Big), \quad B = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Big).$$

If we introduce the notation

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \tag{7.5}$$

then the diffeomorphism f may be written in the complex notation as

$$f(z_0 + z) = f(z_0) + \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \overline{z}}(z_0) \cdot \overline{z} + o(z).$$

In this notation, the Cauchy-Riemann condition we saw in Equation (1.4) is equivalent to

$$\frac{\partial}{\partial \overline{z}} f(z) = 0, \forall z \in \Omega, \tag{7.6}$$

and when f is holomorphic,

$$f'(z) = \frac{\partial}{\partial z} f(z).$$

Fix $\theta \in [0, 2\pi]$, and define

$$D_\theta f(z_0) = \lim_{r\to 0} \frac{f(z_0 + re^{i\theta}) - f(z)}{re^{i\theta}}.$$

This is the partial derivative of f at z_0 in the direction $e^{i\theta}$. By comparing to the distortion of real-linear maps we see that

$$\max_{\theta\in[0,2\pi]} |D_\theta f(z_))| = |A|\Big(1 + \Big|\frac{B}{A}\Big|\Big) = \Big|\frac{\partial f}{\partial z}(z_0)\Big| + \Big|\frac{\partial f}{\partial \overline{z}}(z_0)\Big|$$

and

$$\min_{\theta\in[0,2\pi]} |D_\theta f(z_))| = |A|\Big(1 - \Big|\frac{B}{A}\Big|\Big) = \Big|\frac{\partial f}{\partial z}(z_0)\Big| - \Big|\frac{\partial f}{\partial \overline{z}}(z_0)\Big|$$

The quantity μ that determines the local distortion of f at z_0 is

$$\mu = \mu_f(z_0) = \frac{\partial f/\partial \overline{z}(z_0)}{\partial f/\partial z(z_0)}.$$

Here, μ is a continuous function of z_0 defined on Ω and maps into $\mathbb{D}$. The function μ_f is called the *complex dilatation* of f. The *dilatation quotient* of f at z_0 is defined as

$$K_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|} = \frac{\max_{\theta\in[0,2\pi]} |D_\theta f(z_0)|}{\min_{\theta\in[0,2\pi]} |D_\theta f(z_0)|}. \tag{7.7}$$

7.3 Absolute continuity on lines

Definition 7.1. A function $g : \mathbb{R} \to \mathbb{C}$ is called *absolutely continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every finite collection of intervals (a_1, b_1), (a_2, b_2), ..., (a_n, b_n) in $\mathbb{R}$ we have

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \implies \sum_{i=1}^{n} |g(b_i) - g(a_i)| < \varepsilon.$$

A function $g : [a, b] \to \mathbb{C}$ is called absolutely continuous, if the above condition is satisfied when all the intervals lie in $[a, b]$.

For example, any C^1 function $g : [a, b] \to \mathbb{C}$ is absolutely continuous. In general, if $g : \mathbb{R} \to \mathbb{C}$ is differentiable at every $x \in \mathbb{R}$ and $|g'|$ is uniformly bounded, then g is absolutely continuous.

On the other hand, any absolutely continuous function is uniformly continuous (use with $n = 1$). But, there are uniformly continuous functions that are not absolutely continuous (for example Cantor's function).

Definition 7.2. Let $A \subset \mathbb{R}^n$, $n \geq 1$. We say that a property holds at almost every point in A if the set of points where the property does not hold forms a set of measure zero. For example, when we say that a function $f : A \to \mathbb{R}$ is continuous at almost every point in A it means that there is a set $B \subset A$ of measure zero such that for every $x \in A \setminus B$ the function f is continuous at x.

Definition 7.3. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a continuous map. We say that $f : \Omega \to \mathbb{C}$ is *absolutely continuous on lines* (ACL) if for each closed rectangle $\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b, c \leq \operatorname{Im} z \leq d\}$ contained in Ω we have the following two properties:

(i) for almost all $y \in [c, d]$, the function $x \mapsto f(x + iy)$ is absolutely continuous on $[a, b]$,

(ii) for almost all $x \in [a, b]$, the function $y \mapsto f(x + iy)$ is absolutely continuous on $[c, d]$.

For example, if $g : \Omega \to \mathbb{C}$ is C^1, then it is ACL. If g is C^1 at all points except at a discrete set of points, it is ACL.

It is clear form the above definitions that a complex valued function is absolutely continuous iff its real and imaginary parts are absolutely continuous functions. The same statement is true for ACL property.

It follows from the standard results in real analysis that if $g : [a, b] \to \mathbb{R}$ is absolutely continuous, then it (has *bounded variation* and hence) is differentiable at almost every point. That is, at almost every $t \in [a, b]$, $g'(t)$ exists and is finite.

Proposition 7.4. *If $f : \Omega \to \mathbb{C}$ is ACL, then the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist (and are finite) at almost every $x + iy \in \Omega$.*

In particular, by the above proposition, at almost every $z \in \Omega$, the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ exist and are finite.

The proof of the above proposition may be found in any standard book on real analysis, see for example, the nice book by G. Folland [Fol99].

7.4 Quasi-conformal mappings

Definition 7.5 (Analytic quasi-conformality)**.** Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an orientation preserving homeomorphism. We say that $f : \Omega \to \mathbb{C}$ is K-quasi-conformal if we have

(i) f is absolutely continuous on lines,

(ii) for almost every $z \in \Omega$ we have $K_f(z) \leq K$.

An orientation preserving homeomorphism $f : \Omega \to \mathbb{C}$ is called quasi-conformal, if it is K-quasi-conformal for some $K \geq 1$.

Note that the condition (i) in the above definition guarantees that the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ are defined at almost every point in Ω. Hence, $\mu_f(z)$ is defined at almost every point and the condition (ii) is meaningful.

Definition 7.6. Let $f : \Omega \to \mathbb{C}$ be a quasi-conformal mapping. The quantity

$$K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$$

is called the *dilatation quotient* of f at z. The function

$$\mu_f(z) = \frac{\partial f/\partial \overline{z}}{\partial f/\partial z} \tag{7.8}$$

is called the *complex dilatation* of f. Both of these functions are defined at almost every point in Ω.

Recall that for a function $f : \Omega \to \mathbb{C}$, the *supremum norm* of f is defined as

$$\|f\|_\infty = \inf\Big\{\sup_{z\in A}|f(z)| \mid A \subseteq \Omega,\ \text{and } \Omega\setminus A \text{ has zero measure}\Big\}.$$

This is also called the *essential supremum* of f on Ω.

Note that the inequality $K_f(z) \leq K$ corresponds to

$$|\mu_f(z)| \leq \frac{K-1}{K+1}.$$

Thus, for a quasi-conformal map $f : \Omega \to \mathbb{C}$, we have

$$\|\mu_f\|_\infty < 1.$$

Theorem 7.7 (Pompeiu formula)**.** *Let Ω be a simply connected domain in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a C^1 map which is quasi-conformal. Let γ be a piece-wise C^1 simple closed curve in Ω and B denote the bounded connected component of $\mathbb{C}\setminus\gamma$. For every $z_0 \in B$ we have*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\iint_B \frac{\partial f(z)/\partial\overline{z}}{z-z_0}dz d\overline{z}.$$

Proof. Let D be a bounded domain with a piece-wise C^1 boundary, and g be a complex valued C^1 function defined on $D\cup\partial D$. With notation $z = x+iy$ we have

$$\begin{aligned}\int_{\partial D} g(z)\,dz &= \int_{\partial D} g(z)\,dx + \int_{\partial D} ig(z)dy\\ &= \iint_D \Big(i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\Big)\,dxdy = 2i\iint_D \frac{\partial g}{\partial \overline{z}}\,dxdy = \iint_D \frac{\partial g}{\partial\overline{z}}\,d\overline{z}dz.\end{aligned}$$

Using $dz = dx + idy$, we have $d\overline{z}dz = (dx - idy)(dx + idy) = idxdy - idydx = 2idxdy$. This gives us the complex version of the Green's integral formula

$$\int_{\partial D} g(z)\,dz = \iint_D \frac{\partial g}{\partial\overline{z}}\,d\overline{z}dz. \tag{7.9}$$

Let z_0 be an arbitrary point in Ω and $\delta > 0$ small enough so that the closed ball $|z-z_0| \leq \delta$ is contained in Ω. Define the open set

$$B_\delta = B \setminus \{z\in B : |z-z_0| \leq \delta\}.$$

The function $g(z) = f(z)/(z-z_0)$ is C^1 on $B\cup\partial B$, and at every $z\in B_\delta$ we have

$$\frac{\partial}{\partial\overline{z}}\Big(\frac{f(z)}{z-z_0}\Big) = \frac{\partial f}{\partial\overline{z}}\cdot\frac{1}{z-z_0} + f(z)\cdot\frac{\partial}{\partial\overline{z}}\frac{1}{z-z_0} = \frac{\partial f}{\partial\overline{z}}\cdot\frac{1}{z-z_0}.$$

In the above equation we have used the complex version of the Cauchy-Riemann condition in Equation (7.6).

We applying the complex Green's formula to g on B_α to obtain

$$\int_{\partial B_\delta} \frac{f(z)}{z-z_0}\,dz = \iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz. \tag{7.10}$$

Now we want to take limits of the above equation as δ tends to 0 from above.

$$\begin{aligned}
\lim_{\delta\to 0} \int_{\partial B_\delta} \frac{f(z)}{z-z_0}\,dz &= \lim_{\delta\to 0} \Big(\int_\gamma \frac{f(z)}{z-z_0}\,dz - \int_{|z-z_0|=\delta} \frac{f(z)}{z-z_0}\,dz \Big) \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - \lim_{\delta\to 0} \int_{|z-z_0|=\delta} \frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\,dz \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - \lim_{\delta\to 0} \int_{|z-z_0|=\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz - 2\pi i f(z_0) \\
&= \int_\gamma \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0).
\end{aligned} \tag{7.11}$$

In the last line of the above equation we have used that $|f(z)-f(z_0)/(z-z_0)|$ is uniformly bounded from above.

On the other hand,

$$\iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz = \Big(\iint_B \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz - \iint_{|z-z_0|\leq\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz \Big)$$

and since f is C^1, and $|z-z_0|\leq\delta$ is compact, there is a constant $C>0$ such that

$$\Big| \iint_{|z-z_0|\leq\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz \Big| \leq C \iint_{|z-z_0|\leq\delta} \Big| \frac{1}{z-z_0} \Big| |d\bar{z}dz|$$

We can calculate the integral on the right hand side as in

$$\begin{aligned}
\iint_{|z-z_0|\leq\delta} \Big| \frac{1}{z-z_0} \Big| \, |d\bar{z}dz| &= 2 \iint_{|z-z_0|\leq\delta} \Big| \frac{1}{z-z_0} \Big| \, dxdy \\
&= 2 \int_0^{2\pi} \int_0^\delta \Big| \frac{1}{z-z_0} \Big| \, rdrd\theta = 4\pi\delta.
\end{aligned}$$

The above relations imply that

$$\lim_{\delta\to 0} \iint_{B_\delta} \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz = \iint_B \frac{\partial f}{\partial \bar{z}} \cdot \frac{1}{z-z_0} d\bar{z}dz. \tag{7.12}$$

Combining Equations 7.10, (7.11), and (7.12), we obtain the formula in the theorem.

□

Remark 7.8. In Theorem 7.7 it is not required to assume that f is C^1. This has an important consequence we state in Lemma 7.10. Below we give a brief argument how the statement is proved without assuming C^1 condition.

As we saw in Proposition 7.4, the ACL condition in quasi-conformality implies that the first partial derivatives of f exist and are finite at almost every point. If the first order partial derivatives are defined almost everywhere, the Jacobian of f, $\det Df$, is defined almost everywhere. Then, as f maps bounded sets to bounded set (that have bounded area), we conclude that $\det Df$ is locally in L^1. On the other hand,

$$|\partial f/\partial x|^2 \leq \max_{\theta\in[0,2\pi]} |D_\alpha f|^2 \leq (K min_{\theta\in[0,2\pi]}|D_\theta f|) \cdot max_{\theta\in[0,2\pi]}|D_\theta f| \leq K \det Df(z).$$

As $\det Df(z)$ belongs to L^1 locally, we conclude that $|\partial f/\partial x|$ belongs to L^2 locally. By a similar argument we conclude that $|\partial f/\partial y|$ also belongs to L^2. These imply that the derivatives $\partial f/\partial \overline{z}$ and $\partial f/\partial z$ exist at almost every point and are integrable. So, the integrals in Theorem 7.7 are meaningful.

Corollary 7.9. *Let $f : \Omega \to \mathbb{C}$ be a C^1 map which is 1-quasi-conformal. Then, $f : \Omega \to \mathbb{C}$ is a conformal map.*

Proof. The condition 1-quasi-conformal implies that $\mu_f(z) = 0$ at almost every point in Ω. Hence, $\partial f/\partial \overline{z} = 0$ at almost every point. It follows from the formula in Theorem 7.7 that f satisfies the Cauchy integral formula, and therefore it is holomorphic. □

As we remarked in Remark 7.8, the C^1 condition is not required in Theorem 7.7. This stronger statement has an important consequence known as the Weyl's lemma. But the proof requires some standard real analysis that is not the prerequisite for this course!

Lemma 7.10 (Weyl's lemma)**.** *Any 1-quasi-conformal map $f : \Omega \to \mathbb{C}$ is conformal.*

Proposition 7.11. *If $f : \Omega_1 \to \Omega_2$ is K-quasi-conformal, $g : \Omega_0 \to \Omega_1$ is conformal, and $h : \Omega_2 \to \Omega_3$ is conformal, then $h \circ f \circ g : \Omega_0 \to \Omega_3$ is K-quasi-conformal.*

Proof. For the first part of the theorem we need to verify the two condition in Definition 7.5 for the map $h\circ f\circ g$. Let $A_1 \subset \Omega_1$ be the set of points where $K_f(z)$ is defined and bounded by K. As f is K-quasi-conformal, $\Omega_1 \setminus A_1$ has zero area. Define $A_0 = g^{-1}(A_1)$. It is easy to show that A_0 has zero area (use exhaustion of Ω_1 by compact sets, and use that $|g'|$ is bounded from above and below on each compact set).

Note that since g and h are holomorphic functions, by Equation (7.7), $K_g \equiv 1$ and $K_h \equiv 1$. Then, for every $w \in A_0$, by the inequality in Equation (7.3), we have

$$K_{h\circ f\circ g}(w) \leq K_g(w) \cdot K_f(g(w)) \cdot K_h(f \circ g(w)) = K_f(g(w)) \leq K.$$

This proves condition (ii) in the definition of quasi-conformality.

We need to prove that $h \circ f \circ g$ is ACL on Ω_0. Since g and h are C^1, they are ACL. In fact, for every rectangle bounded by horizontal and vertical sides, in their domain of definition, these maps are absolutely continuous on every horizontal and every vertical line. In fact, g and h are absolutely continuous on every piece-wise C^1 curves in their domain of definition. We also know that for every rectangle $R \subset \Omega_1$ bounded by horizontal and vertical sides, f is absolutely continuous on almost every horizontal and almost every every vertical line in R. With these properties, it is easy to see that $h \circ f$ is ACL. But the problem with $f \circ g$ is that g does not map horizontal lines to horizontal or vertical lines. And we do not a priori know that f is absolutely continuous on almost every analytic curves (these are images of a horizontal and vertical lines by g). As in Remark 7.8 we need to use some standard results from real analysis. That is, a homeomorphism f is ACL iff the first partial derivatives of f exist at almost every point in the domain of f and are locally in L^1. From this criterion it is easy to see that the composition of ACL homeomorphisms is ACL. (We skip the details as this requires material that are not the prerequisite for this course.) $\square$

Proposition 7.12. *If $f : \Omega_1 \to \Omega_2$ is K-quasi-conformal, $g : \Omega_0 \to \Omega_1$ is conformal, and $h : \Omega_2 \to \Omega_3$ is conformal, then for almost every $z \in \Omega_1$ and almost every $w \in \Omega_0$ we have*

$$\mu_{h \circ f}(z) = \mu_f(z), \quad \mu_{f \circ g}(w) = \Big(\frac{|g'(z)|}{g'(z)}\Big)^2 \cdot \mu_f(g(w)).$$

Proof. This is easy to see from the definition of μ in terms of the length of major and minor axis, and their direction. See Exercise 7.1 $\square$

Remark 7.13. Many theorems in complex analysis are valid, with some modifications, for quasi-conformal mappings. The Pompeiu formula is an example of such statements. In general, it is possible to show that the composition of quasi-conformal maps are quasi-conformal. If a sequence of K-quasi-conformal maps converges uniformly on compact sets to some function, then the limiting function is either constant or quasi-conformal. The class of K-quasi-conformal maps $f : \mathbb{C} \to \mathbb{C}$ normalized with $f(0) = 0$ and $f(1) = 1$ forms a normal family.

Quasi-conformal maps, in contrast to conformal maps, enjoy the flexibility that allows one to build such maps by hand. This makes them a powerful tool in complex analysis.

7.5 Beltrami equation

Given a diffeomorphism $f : \Omega \to \mathbb{C}$ with $\mu_f : \Omega \to \mathbb{D}$ one may look at f in Equation (7.8) as the solution of the differential equation

$$\frac{\partial f}{\partial \overline{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z), \forall z \in \Omega. \tag{7.13}$$

That is, given a function $\mu : \Omega \to \mathbb{D}$, is there a diffeomorphism $f : \Omega \to \mathbb{C}$ such that the above equation holds. The above equation is called the *Beltrami equation*, and the function μ is called the *Beltrami coefficient* of f.

There is a geometric interpretation of the Beltrami equation similar to the solutions of vector fields in the plane. The function μ specifies a field of ellipses in Ω where at each $z \in \Omega$ the major axis of the ellipse has angle $(\arg \mu(z) + \pi)/2$ and size $1/(1 - |\mu(z)|)$. The minor axis of the ellipse at z has angle $\mu(z)/2$ and has size $1/(1 + |\mu(z)|)$. The solution f of the above equation is a diffeomorphism that infinitesimally maps the field of ellipses to the field of round circles.

The Beltrami equation has a long history. It was already considered by Gauss in 1820's in connection with a seemingly different problem of finding isothermal coordinates on a surface for real analytic maps. Most of the developments in the study of the Beltrami equation took place in $1950's$. These mainly focused on reducing the regularity condition required for f; see Remark 7.15.

Theorem 7.14. *[Measurable Riemann mapping theorem-continuous version] Let $\mu : \mathbb{C} \to \mathbb{D}$ be a continuous map with $\sup_{z\in\mathbb{C}} |\mu(z)| < 1$. Then, there is a quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$ such that the Beltrami equation* (7.13) *holds on $\mathbb{C}$.*

Moreover, the solution f is unique if we assume that $f(0) = 0$ and $f(1) = 1$.

Remark 7.15. The condition of continuity of μ in the above theorem is not necessary. The sufficient condition is that μ is measurable and $\|\mu\|_\infty < 1$. This result is known as the measurable Riemann mapping theorem, and has many important consequences.

The relation between the regularity of the solution and the regularity of μ is not simple. For example, if μ is Hölder continuous, then the solution becomes a diffeomorphism. But, this condition is far from necessary. There are discontinuous functions μ where the solution is diffeomorphism.

7.6 An application of MRMT

In the theory of dynamical systems one wishes to understand the behavior of the sequences of points generated by consecutively applying a map at a given point. That is, if $g : X \to X$, and $x_0 \in X$, one studies the sequence $\{x_n\}$ defined as $x_{n+1} = g(x_n)$. This is called *the orbit* of x_0 under g. We shall look at the special case when $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Recall the homeomorphism π from the unit sphere $S \subset \mathbb{R}^3$ to the Riemann sphere $\hat{\mathbb{C}}$ we discussed in Section 3.1. There is a spherical metric d' on S which is defined as the Euclidean length of the shortest curve on S. We may use π and d' to define a metric on $\hat{\mathbb{C}}$ as $d(z, w) = d'(\pi^{-1}(z), \pi^{-1}(w))$.

We may naturally extend the notion of normal families for holomorphic maps of $\mathbb{C}$ we presented in Definition 5.9 to holomorphic maps of $\hat{\mathbb{C}}$. Let Ω be an open set in $\hat{\mathbb{C}}$ and $f_n : \Omega \to \hat{\mathbb{C}}$ be a sequence of maps. We say that f_n converges uniformly on E to $g : E \to \hat{\mathbb{C}}$, if for every $\varepsilon > 0$ there is $n_0 \geq 1$ such that for all $n \geq n_0$ and all $z \in E$ we have $d(f_n(z), g(z)) < \varepsilon$.

Definition 7.16. Let Ω be an open set in $\hat{\mathbb{C}}$ and $\mathcal{F}$ be a family (set) of maps $f : \Omega \to \hat{\mathbb{C}}$. We say that the family $\mathcal{F}$ is *normal*, if every sequence of maps in $\mathcal{F}$ has a sub-sequence which converges uniformly on compact subset of Ω to some $g : \Omega \to \hat{\mathbb{C}}$.

Given a holomorphic map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and any integer $n \geq 1$ we may compose the map R with itself n times to obtain a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. We use the notation $R^{\circ n}$ to denote this n-fold composition.

Definition 7.17. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. We say that $z \in \hat{\mathbb{C}}$ is *stable* for the iterates $R^{\circ n}$, $n \geq 1$, if there is an open set $U \subset \hat{\mathbb{C}}$ containing z such that the family $\{R^{\circ n}\}_{n=0}^{\infty}$ restricted to U forms a normal family.

By the above definition, the set of stable points of a rational function forms an open subset of $\hat{\mathbb{C}}$. The set of all stable points of a rational function R is called the *Fatou* set of R, and denoted here by $\mathcal{F}(R)$. The complement of the Fatou set, $\hat{\mathbb{C}} \setminus \mathcal{F}(R)$, which is a closed subset of $\hat{\mathbb{C}}$, is called the *Julia* set of R. This is denoted by $\mathcal{J}(R)$. These are named after the pioneering works of P. Fatou and G. Julia in 1920's on properties of these sets.

Lemma 7.18. *Let $R(z) = z^d$, for some integer $d \geq 2$. Then $\mathcal{J}(R) = \partial\mathbb{D}$.*

Proof. First we show that the open disk $\mathbb{D}$ is contained in $\mathcal{F}(R)$. To see this, let E be an arbitrary compact set in $\mathbb{D}$. There is $r \in (0, 1)$ such that $E \subset B(0, r)$. Then, for every $w \in E$ we have $|R^{\circ n}(w)| = |w^{d^n}| \leq r^{d^n} \to 0$, as n tends to infinity. That is, the iterates $R^{\circ n}$ converges uniformly on E to the constant function 0. As E was an arbitrary compact set in $\mathbb{D}$, we conclude that $R^{\circ n}$ converges uniformly on compact subsets of $\mathbb{D}$ to the constant function 0.

By a similar argument, the iterates $R^{\circ n}$ converges uniformly on compact subsets of $\hat{\mathbb{C}} \setminus (\mathbb{D} \cup \partial\mathbb{D})$ to the constant function ∞.

By the above two paragraphs, $\hat{\mathbb{C}} \setminus \partial\mathbb{D}$ is contained in $\mathcal{F}(R)$. On the other hand, let $z \in \partial\mathbb{D}$ and U be an arbitrary neighborhood of z. For w in U with $|w| > 1$ we have $R^{\circ n}(w) \to \infty$ and for $w \in U$ with $|w| < 1$ we have $R^{\circ n}(w) \to 0$. Thus, there is no subsequence of $R^{\circ n}$ that converges to some continuous function on U. As U was arbitrary, we conclude that $z \notin \mathcal{F}(R)$. Then, $z \in \mathcal{J}(F)$. $\square$

The above example is a very special case where the Julia set has a simple structure (is smooth). For a typical rational map the Julia set has a rather complicated structure, see some examples of Julia sets in Figure 7.6. The self-similarity of the figures is due to the invariance of $\mathcal{J}(R)$ under R we state below.

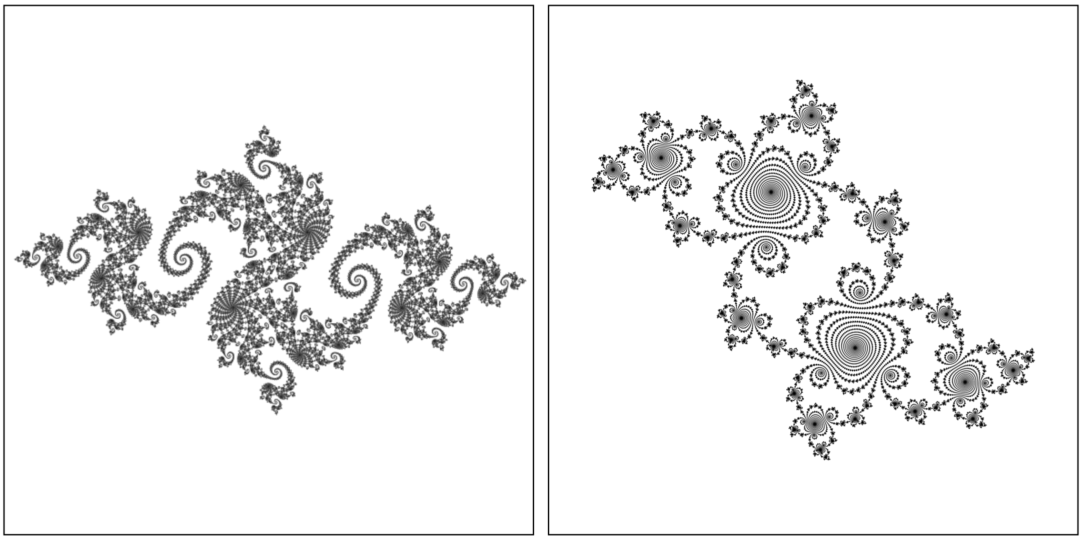

Figure 7.1: Two examples of Julia sets.

Lemma 7.19. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then, $z \in \mathcal{F}(R)$ if and only if $R(z) \in \mathcal{F}(R)$.*

Proof. See Exercise 7.5. $\square$

By the above lemma $R^{-1}(\mathcal{F}(R)) = \mathcal{F}(R)$, which implies $R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$.

Assume that the Fatou set of some $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is not empty. Let U_0 be a connected component of $\mathcal{F}(R)$. There is a sequence R^{n_k}, $k \geq 1$, that converges to some $g : U_0 \to \hat{\mathbb{C}}$. Then the function g describes the limiting behavior of the orbit $R^{\circ n_k}(z)$. But, to understand the behavior of the orbit of z one needs to know all limiting functions of convergent sub-sequences of $R^{\circ n}$.

Let U be a connected components of $\mathcal{F}(R)$. It follows from Lemma 7.19 that $R(U)$ is a components of $\mathcal{F}(R)$ which may or may not be distinct from U.

Definition 7.20. A component U of $\mathcal{F}(R)$ is called *wandering*, if $R^{\circ i}(U) \cap R^{\circ j}(U) = \emptyset$, for distinct integers i and j. A component U of $\mathcal{F}(R)$ is called *eventually periodic*, if there are positive integers $i \geq 0$ and $p \geq 1$ such that $R^{\circ(i+p)}(U) = R^{\circ i}(U)$.

By definition, if a Fatou component is not wandering, then it is eventually periodic. In 1985, D. Sullivan established the following remarkable property that settled a conjecture of Fatou from 1920's.

Theorem 7.21 (No wandering domain)**.** *Let U be a connected component of the Fatou set of a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then, U is eventually periodic.*

Remark 7.22. Theorem 7.21 is a major step towards characterizing the limiting functions of the iterates $R^{\circ n}$. When, $R^{\circ(i+p)}(U) = R^{\circ i}(U)$. The map $h = R^{\circ p}$ is a holomorphic map from $V = R^{\circ i}(U)$ to V. This allows one to study all possible limits of the iterates $R^{\circ n}$ on U. For example when V is a simply connected subset of $\hat{\mathbb{C}}$ one has Exercise 3.6.

The complete proof requires some advanced knowledge of quasi-conformal mappings. However, we present an sketch of the argument in the class, only emphasizing the use of the measurable Riemann mapping theorem.

7.7 Exercises

Exercise 7.1. Prove Proposition 7.12.

Exercise 7.2. Let Ω_1, Ω_2, and Ω_3 be open sets in $\mathbb{C}$. Assume that $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ are C^1 maps. With the notations $z \in \Omega_1$ and $w = f(z) \in \Omega_2$, prove the complex chain rules,

$$\frac{\partial(g \circ f)}{\partial z} = (\frac{\partial g}{\partial w} \circ f) \cdot \frac{\partial f}{\partial z} + (\frac{\partial g}{\partial \overline{w}} \circ f) \cdot \frac{\partial \overline{f}}{\partial z},$$

and

$$\frac{\partial(g \circ f)}{\partial \overline{z}} = (\frac{\partial g}{\partial w} \circ f) \cdot \frac{\partial f}{\partial \overline{z}} + (\frac{\partial g}{\partial \overline{w}} \circ f) \cdot \frac{\partial \overline{f}}{\partial \overline{z}}.$$

Exercise 7.3. Assume that $\mu : \mathbb{C} \to \mathbb{D}$ is a continuous map with $\sup_{z\in\mathbb{C}} |\mu(z)| < 1$, and $f, g : \mathbb{C} \to \mathbb{C}$ are diffeomorphisms with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ that satisfying the Beltrami equation. Prove that $f(z) = g(z)$ for all $z \in \mathbb{C}$. [This is a special case of the uniqueness part in Theorem 7.14.]

Exercise 7.4. We say that a function $f : [a, b] \to \mathbb{C}$ has *bounded variation*, if

$$\sup \Big\{ \sum_{i=1}^{N} |f(x_{i+1}) - f(x_i)| \;\Big|\; a = x_1 < x_2 < x_3 < \cdots < x_{N+1} = b, N \in \mathbb{N} \Big\} < \infty.$$

Prove that if $f : [a, b] \to \mathbb{C}$ is absolutely continuous, then f has bounded variation on $[a, b]$.

Exercise 7.5. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ration map. Prove that $R^{-1}(\mathcal{F}(R)) = \mathcal{F}(R)$. Then, conclude that $R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$.

Exercise 7.6. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Assume that there is $n \in \mathbb{N}$ and $z \in \hat{\mathbb{C}}$ such that $R^{\circ n}(z) = z$. Prove that

(i) if $|(R^{\circ n})'(z)| < 1$, then z belongs to $\mathcal{F}(R)$;

(ii) if $|(R^{\circ n})'(z)| > 1$, then z belongs to $\mathcal{J}(R)$;

(iii) if $|(R^{\circ n})'(z)| = e^{2\pi i p/q}$ for some $p/q \in \mathbb{Q}$, then z belongs to $\mathcal{J}(R)$. [hint: first consider the case $n = 1$ and look at $(R^{\circ n})''(z)$ as n tends to infinity.

Chapter 8

Appendix

8.1 Hints to exercises

In this appendix we provide hints to the exercises. Please note that these will be brief and do not suggest a suitable style of writing proofs in mathematics. The complete solutions to the difficult exercises are given. These have been indicated by [complete solution] at the beginning of the solution, and suggest a proper way of writing solutions.

Chapter 2

2.1: Let $\varphi_1(z) = rz + a$ and $\varphi_2(z) = sz + b$. Then, $\varphi_1 : \mathbb{D} \to B(a, r)$ and $\varphi_2 : \mathbb{D} \to B(b, s)$ are biholomorphisms. It follows that $\varphi_2^{-1} \circ f \circ \varphi_1 : \mathbb{D} \to \mathbb{D}$ is defined and holomorphic, and maps 0 to 0. By Lemma 2.1, we have $|(\varphi_2^{-1} \circ f \circ \varphi_1)'(0)| \leq 1$. This implies $|f'(a)| \leq s/r$.

2.2: We have seen that $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ belongs to $\mathrm{Aut}(\mathbb{D})$. Recall that φ_a is the inverse of φ_a.

(i) Apply Lemma 2.1-(ii) to the map $\varphi_{f(a)}^{-1} \circ f \circ \varphi_a$, and explicitly calculate the derivatives of φ_a and $\varphi_{f(a)}$.

(ii) Apply Lemma 2.1-(i) to the map $\varphi_{f(a)}^{-1} \circ f \circ \varphi_a$ at $\varphi_a^{-1}(b)$.

2.3: The map $\varphi(z) = \frac{\operatorname{Im} a}{\operatorname{Im} h(a)} z + (\operatorname{Re} a - \frac{\operatorname{Im} a}{\operatorname{Im} h(a)} \operatorname{Re} a)$ is an automorphism of $\mathbb{H}$ that maps $h(a)$ to a. Let $\psi : \mathbb{D} \to \mathbb{H}$ be a biholomorphic map with $\psi(0) = a$. Then, apply Lemma 2.1-(ii) to the map $\psi^{-1} \circ \varphi \circ h \circ \psi$. Note that $(\psi^{-1})'(a) = 1/\psi'(0)$, so $|(\varphi \circ h)'(a)| \leq 1$. You need to calculate $\varphi'(h(a))$.

2.4: First note that it is enough to show that every point in $\mathbb{D}$ can be mapped to 0. Then compose such maps to obtain an automorphism that maps z to w.

Chapter 3

3.1: (i) Solve for A and B in $f(z) = Az + B$.

(ii) First note that it is enough to show that any three distinct points can be mapped to 0, 1, and ∞. Then, compose such maps and their inverses to get the desired map.

3.2: Apply the removable singularity theorem to the map $z \mapsto 1/f(1/z)$.

3.3: [Complete solution]

(i) Since Ω is an opens set, there is $r > 0$ such that $B(z_0, r) \subset \Omega$. Then, f has a convergent power series expansion on $B(z_0, r)$, say

$$f(z) = f(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + \dots.$$

As z_0 is a zero of order k, we must have $a_i = 0$ for all $1 \leq i \leq k-1$, and $a_k \neq 0$. Then,

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots = (z - z_0)^k\big(a_k + a_{k+1}(z - z_0) + \dots\big).$$

The function $h(z) = a_k + a_{k+1}(z - z_0) + \dots$ is holomorphic on $B(z_0, r)$, and in particular it is continuous. Then, for $\varepsilon = |a_k|/2 > 0$ there is $\delta > 0$ such that if $|z - z_0| < \delta$ then $|h(z) - h(z_0)| < \varepsilon$. Here we may assume that $\delta < r$, as otherwise we may take $\min\{\delta, r\}$. The inequality means that h maps $B(z_0, \delta)$ into $B(a_k, |a_k|/2))$. On the other hand, since $B(a_k, |a_k|/2)$ does not meet the line segment $-a_k r$, for $r \in [0, \infty)$, there is a holomorphic branch of the k-th-root function defined on this ball. That is, $\sqrt[k]{h(z)}$ is defined and holomorphic on $B(z_0, \delta)$.

We have

$$f(z) = (z - z_0)^k\big(a_k + a_{k+1}(z - z_0) + \dots\big) = \big((z - z_0)\sqrt[k]{h(z)}\big)^k,$$

that is, $\psi(z) = (z - z_0)\sqrt[k]{h(z)}$.

(ii) For the map ψ obtained in part (i), we have $\psi(z_0) = 0$, and by the product rule, $\psi'(z_0) \neq 0$. By the inverse function theorem, ψ has an inverse defined on a neighborhood of $\phi(z_0) = 0$. Let g be this inverse map that is defined on $B(0, r)$, for some $r > 0$.

For every $w \in B(0, r^k)$, there are exactly k points $w_1, w_2, \dots, w_k$ in $B(0, r)$ such that $w_i^k = w$. Then the points $z_i = g(w_i)$ provide k solutions for the equation $f(z) = w$. To see that there are at most k solutions, assume that $f(z) = w$ for some $w \in B(0, r^k)$. Then $\psi(z)^k = w$, which implies that $\psi(z)$ is a k-th root of w. Thus, z_i are the only solutions.

3.4: (i) Assume that f is not constant. Let U be an open set in Ω. We need to show that $f(U)$ is open. Fix an arbitrary $w_0 \in f(U)$. There is $z_0 \in U$ with $f(z_0) = w_0$. Since, f is not constant, the function $z \mapsto f(z) - w_0$ has a zero of some finite order $k \geq 1$ at z_0. By the previous exercise, $f(z) - w_0$ is locally k to 1 near z_0. That is, for every w near w_0, there is z near z_0 such that $f(z) = w$. Since U is open, the points sufficiently close to z_0

are in U. This means that a neighborhood of w_0 is contained in $f(U)$. As $w_0 \in f(U)$ was arbitrary, we conclude that $f(U)$ is open.

(ii) Assume that f is not a constant map. As $f : \Omega \to \mathbb{C}$ is an open mapping, $f(\Omega)$ is an open set in $\mathbb{C}$. Fix an arbitrary $z \in \Omega$. As $f(\Omega)$ is open, there is $r > 0$ such that $B(f(z), r) \subset f(\Omega)$. Now, choose $w' \in B(f(z), r)$ with $|w'| > |f(z)|$. Since $B(f(z), r) \subset f(\Omega)$, there is $z' \in \Omega$ with $f(z') = w'$. Hence, $|f(z')| > |f(z)|$.

By the above argument, for every $z \in \Omega$, there is $z' \in \Omega$ such that $|f(z')| > |f(z)|$. This implies the maximum principle.

3.5: First show that the linear map $h(z) = az + b$ and the inversion $h(z) = 1/z$ map lines and circles to lines and circles. Then show that any Mobius transformation may be written as composition of these maps.

3.6: First show that there are at most finite number of points a_1, a_2, $\dots a_d$ in $\mathbb{D}$ with $g(a_i) = 0$.

Consider the function

$$h(z) = \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}$$

and show that it maps $\mathbb{D}$ into $\mathbb{D}$ and maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$.

Consider the rational function $\varphi(z) = g(z)/h(z)$. Show that there are no points in $\mathbb{D} \cup \partial\mathbb{D}$ that are mapped to 0.

Show that φ maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Conclude that for all $z \in \hat{\mathbb{C}}$ we have $\varphi(z) = 1/\overline{\varphi(1/\overline{z})}$, where $\overline{z}$ denotes the complex conjugate of z.

By the above two paragraphs, there are no points in $\hat{\mathbb{C}}$ which are mapped to 0 by φ. This implies that φ is a constant function, which must belong to $\partial\mathbb{D}$.

Chapter 4

4.1: You need to verify the three conditions for being a metric.

Property (i): This is obvious from the definition of the length of a curve. That is, the length of a curve is independent of the parametrization and the direction of the curve.

Property (ii): Since the length of any curve is non-negative, the infimum of a set of non-negative numbers is a non-negative number.

If $z = w$, then the constant curve from z to w has zero length with respect to ρ. Thus, $d_\rho(z, z) = 0$.

Now assume that $z \neq w$ and let $r = |z - w| > 0$. Since Ω is open there is $r_1 > 0$ such that $B(z, r_1) \subset \Omega$. Also, as the set of zero's of ρ is discrete, there is a positive

$\varepsilon < \min\{r, r_1\}$ such that ρ has at most one zero on $B(z, \varepsilon)$ (the only possible zero is z). Consider the compact set $A = \{\zeta \in \Omega \mid \varepsilon/4 \leq |z - \zeta| \leq \varepsilon/2\}$, which is contained in Ω. The function ρ is continuous and positive on A, and hence its minimum on A is strictly positive, say $m > 0$.

Let $\gamma : [a, b] \to \Omega$ be a piece-wise C^1 curve with $\gamma(a) = z$ and $\gamma(b) = w$. Then,

$$\begin{aligned}\ell_\rho(\gamma) = \int_{[a,b]} \rho(\gamma(s))|\gamma'(s)|ds \geq \int_{\{t\in[a,b];\gamma(t)\in A} \rho(\gamma(s))|\gamma'(s)|ds \\ \geq m \cdot \int_{\{t\in[a,b];\gamma(t)\in A} |\gamma'(s)|ds \geq m \cdot \frac{\varepsilon}{4}.\end{aligned}$$

As $m\varepsilon/4$ does not depend on γ, by the definition of infimum, $d_\rho(z, w) \geq m\varepsilon/4$. Hence, $d_\rho(z, w) > 0$.

By the above paragraphs, $d_\rho(z, w) = 0$ iff $z = w$.

Property (iii): Let η a piece-wise C^1 curve connecting x to z, and ξ a piece-wise C^1 curve connecting z to y. Then η followed by ξ is a piece-wise C^1 curve connecting x to y. By definition,

$$d_\rho(x, y) \leq \ell_\rho(\eta \cup \xi) = \ell_\rho(\eta) + \ell_\rho(\xi).$$

Now, take infimum over $\Gamma_{x,z}$, and then over $\Gamma_{z,y}$ to conclude the triangle inequality.

4.2: First note that $\rho \geq 1$, which implies that $d_\rho(z, w) \geq |z - w|$. In particular, if z_i converges to z w.r.t d_ρ, then, $|z_i - z| \to 0$.

On the other hand, if z_i converges to z w.r.t Euclidean distance, then, there is $r < 1$ such that $z_i \in B(0, r)$, for all i. Now, let M be the supremum of ρ on $B(0, r)$. M is a finite number. We have $d_\rho(z_i, z) \leq M|z_i - z|$. Hence, $d_\rho(z_i, z) \to 0$.

4.3: Let z_i be a Cauchy sequence in $(\mathbb{D}, \rho)$. First show that there is $r < 1$ such that for all $i \geq 1$, $z_i \in B(0, r)$. Then conclude that z_i is a Cauchy sequence w.r.t the Euclidean distance.

4.4: Use an isometry of the dist to map z to 0. Then use that the Mobius transformations map circles to circles in Exercise 3.5.

4.5: Show that there is a one-to-one correspondence between $\Gamma_{z,w}$ and $\Gamma_{f(z),f(w)}$.

4.6: By definition,

$$(F^*\rho)(w) = \rho(F(w)) \cdot |F'(w)| = \frac{1}{1 - |\frac{i-w}{i+w}|^2} \cdot \frac{2}{|i + w|^2} = \frac{2}{|i + w|^2 - |i - w|^2} = \frac{1}{2|\operatorname{Im} w|}.$$

Chapter 5

5.1: Write the circle of radius r as $re^{i\theta}$, and note that

$$f(re^{i\theta}) = (r + 1/r)\cos\theta + i(r - 1/r)\sin\theta,$$

and use the identity $\cos^2\theta + \sin^2\theta \equiv 1$.

5.2: From Example 5.6, replace sin and cos in terms of e^{iz} in $\tan z = \sin z/\cos z$.

5.3: On can do this by composition of a number of elementary transformations. First apply the biholomorphism $g_1(z) = i\frac{1-w}{1+w}$ (see Equation 2.1) to Ω to obtain $\mathbb{H} \setminus [0, 1/3]i$. Then apply $g_2(z) = -i\cdot z$ to get $B = \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}\setminus(0, 1/3)$. Next, apply $g_3(z) = z^2$, to obtain $\mathbb{C}\setminus(-\infty, 1/3)$, then apply $g_4(z) = z - 1/3$ to obtain $\mathbb{C}\setminus(-\infty, 0)$, and then apply $g_5(z) = \sqrt{z}$ to obtain the right half plane.

5.4: If a family of maps $\mathcal{F}$ is not uniformly bounded on compact sets, then there is a compact set $E \subset \Omega$ such that the family is not uniformly bounded on E. This means that for any $n \in \mathbb{N}$ there is $z_n \in E$ and $f_n \in \mathcal{F}$ such that $|f_n(z)| \geq n$. Since E is compact, $\{z_n\}$ has a sub-sequence, say n_k, converging to some $z \in E$. It follows that the sequence $\{f_{n_k}\}$ has no sub-sequence converging uniformly on compact subsets of Ω. That is because, if there is a sub-sequence of $\{f_{m_k}\}$ converging to some $g : \Omega \to \mathbb{C}$, then $g(z) = \lim f_{m_k}(z_{m_k}) = \infty$. This is a contradiction as g maps Ω to $\mathbb{C}$.

5.5: Properties (i) and (ii) are easy to see. For property (iii) introduce the function $h(r) = r/(1 + r)$, for $r \geq 0$. Prove $h(a + b) \leq h(a) + h(b)$ for all a and b in $(0, \infty)$.

5.6: First show that the functions

$$d_i''(f, g) = \frac{\sup_{E_i} |f(z) - g(z)|}{1 + \sup_{E_i} |f(z) - g(z)|}$$

satisfy the conditions for a metric on $C^0(E_i)$. Then prove that the sum of such metrics (multiplied by $1/2^i$ to make the sum convergent) is a metric on Ω, provided E_i form an exhaustion of Ω.

5.8: Use Proposition 5.10. That is, if $f_n \to f$ uniformly on compact subsets of Ω then $f_n' \to f'$ uniformly on compact subsets of Ω.

5.9: By Theorem 5.15, it is enough to show that the family is uniformly bounded on compact sets. Let E be a compact subsets of $\mathbb{D}$. There is $r < 1$ such that $E \subset B(0, r)$. Then, for all $z \in E$ we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n z^n| \leq r + \sum_{i=2}^{\infty} nr^n \leq \sum_{i=1}^{\infty} nr^n = M_r < \infty.$$

This means that the family is uniformly bounded from above on E.

5.10: See proof of Theorem 5.15

Chapter 6

6.1: [complete solution] By Montel's theorem from the lecture notes (Theorem 5.15) it is enough to show that the family $\mathcal{S}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Let E be an arbitrary compact set in $\mathbb{D}$. There is $\delta < 1$ such that $E \subset B(0, \delta)$. By the growth theorem, 6.9, for every $z \in E$ we have

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2} \leq \frac{\delta}{(1-\delta)^2} < \infty.$$

As the upper bound only depends on E, we conclude that the family is uniformly bounded on E.

6.2: First show that $|f'(0)| \leq 4c$. Then apply Theorem 6.9 to an appropriately normalized map.

6.3: [complete solution] (i) For every $k \geq 2$, the set Λ_k is uniformly bounded in $\mathbb{C}$. If this is not true, there is a sequence of maps f_n in $\mathcal{S}$ such that $f_n^{(k)}(0) \to \infty$. By Exercise 6.1, $\mathcal{S}$ is a normal family and there must be a sub-sequence of f_n that converges uniformly on compact sets to some holomorphic maps $g : \mathbb{D} \to \mathbb{C}$. In particular, $g^{(k)}(0)$ is defined and finite. This contradicts the convergence of $f_n^{(k)}(0) \to g^{(k)}(0)$ guaranteed in Theorem 5.10.

By the above paragraph, for every $k \geq 2$, the set

$$A_k = \{|w| \mid w \in \Lambda_k\}$$

is bounded from above. This set is also non-empty as it contains 0; the k-th derivative of the identity map in $\mathcal{S}$. It follows that the above set has a supremum which is finite. Let r_k denote the supremum of the above set. Therefore,

$$\Lambda_k \subseteq \{w \in \mathbb{C} \mid |w| \leq r_k\}.$$

Fix an arbitrary $k \geq 2$.

By the definition of supremum, either r_k belongs to A or there is a sequence of real numbers $a_i \in A_k$, for $i \geq 1$, such that $a_i \to r_k$. In the former case we conclude that there is $f \in S$ such that $|f^{(k)}(0)| = r_k$. In the latter case, let $f_i \in \mathcal{S}$ be such that $|f_i^{(k)}(0)| = a_i$. There is a sub-sequence of f_i that converges to some map g in $\mathcal{S}$. We must have $|g^{(k)}(0)| = r_k$. So, there is always an $f \in \mathcal{S}$ such that $|f^{(k)}(0)| = r_k$.

By the above paragraph there is a point on the circle $|w| = r_k$ that belongs to Λ_k. The operations of rotation and dilatation discussed in the lecture notes show that Λ_k is invariant under rotations about 0 and is invariant under multiplication by $r \in (0,1)$. We also showed earlier that 0 belongs to Λ_k. This proves that the above inclusion is equality.

(ii) By the Cauchy integral formula for the derivatives, for every $r \in (0,1)$ we have

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz.$$

Then by the growth theorem, Theorem 6.9, we obtain

$$\begin{aligned} |f^{(k)}(0)| = \frac{k!}{2\pi} \Big| \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz \Big| &\leq \frac{k!}{2\pi} \int_{|z|=r} \frac{|f(z)|}{r^{k+1}}\, |dz| \\ &\leq \frac{k!}{2\pi} \int_0^{2\pi} \frac{r}{(1-r)^2 r^{k+1}}\, r d\theta \leq \frac{k! r}{(1-r)^2 r^k}. \end{aligned}$$

The above bound holds for all $r \in (0,1)$. We may find the minimum of the function $\frac{k!r}{(1-r)^2 r^k}$ on $(0,1)$, by differentiating the function. The minimum occurs at $r = 1 - 1/k$, and the minimum value is

$$\frac{k! k^2}{(1-1/k)^{k-1}}.$$

The denominator of the above expression tends to the constant e as $k \to \infty$. Hence, the denominator is uniformly bounded away from 0, independent of k.

6.5: (i) Let φ_1 and φ_2 be two such maps. Apply the Schwarz lemma, 2.1, to the maps $\varphi_2^{-1} \circ \varphi_1$ and $\varphi_1^{-1} \circ \varphi_2$.

(ii) The upper bound follows from the 1/4-theorem, the lower bound follows from the Schwarz lemma.

6.6:[complete solution] Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth simple closed curve. Then, γ bounds a convex region if the slope of the tangent to γ is increasing. This is equivalent to saying that

$$\frac{\partial}{\partial t}\big(\arg \gamma'(t)\big) > 0, \forall t \in [0,1]. \tag{8.1}$$

For instance, for the curve $\gamma_0(t) = e^{2\pi i t}$, for $t \in [0,1]$, we have $\gamma_0'(t) = 2\pi i e^{2\pi i t}$. Thus,

$$\frac{\partial}{\partial t}\big(\arg \gamma_0'(t)\big) = \frac{\partial}{\partial t}(\pi/2 + 2\pi t) = 2\pi > 0.$$

So, for the inequality in Equation (8.1) to hold, it is enough to have

$$\Big| \frac{\partial}{\partial t}\big(\arg \gamma'(t)\big) - \frac{\partial}{\partial t}\big(\arg \gamma_0'(t)\big) \Big| \leq \pi. \tag{8.2}$$

Next we note that $\frac{\partial}{\partial t}(\arg \gamma'(t)$ is given in terms of γ' and γ''. This implies that, there is $\delta > 0$ such that if for all $t \in [0, 1,$ if we have

$$|\gamma_0'(t) - \gamma'(t)| \leq \delta, \quad |\gamma_0''(t) - \gamma''(t)| \leq \delta, \tag{8.3}$$

then Equation (8.2) holds. (In other words, if a closed curve γ, is close enough to γ_0 in C^0, C^1, and C^2 metrics, then it bounds a convex region containing 0.)

For an arbitrary $f \in \mathcal{S}$ and $r \in (0, 1)$ let

$$f_r(z) = \frac{1}{r} \cdot f(r \cdot z), \forall z \in \mathbb{D}.$$

We have, $f(B(0, r)) = r \cdot f_r(\mathbb{D})$. In particular, $f(B(0, r))$ is a convex region, iff $f_r(\mathbb{D})$ is a convex region. We aim to show that for small enough r, independent of $f \in \mathcal{S}$, $f_r(\mathbb{D})$ is convex. As γ_0 is the image of the circle $|z| = 1$ under the identity map, by virtue of Equation 8.3, it is enough to show that for all $z \in \mathbb{D}$, we have

$$|f_r'(z) - 1| \leq \delta, \quad |f_r''(z) - 0| \leq \delta. \tag{8.4}$$

However, $f_r'(z) = f'(rz)$, and $f_r''(z) = f''(rz) \cdot r$. It follows from the distortion theorems 6.7 and 6.6, that for small enough r, independent of f, one may guarantee the above inequalities. This completes the proof.

Chapter 7

7.2: These may be reduced to the usual derivatives with respect to x and y using the formulas in Equation (7.5).

7.3: Define the map $h(z) = g^{-1} \circ f$ from $\mathbb{C}$ to $\mathbb{C}$. Show that $\partial h / \partial \overline{z} \equiv 0$, that is, h is 1-quasi-conformal. Then apply Corollary 7.9 to h to conclude that $h : \mathbb{C} \to \mathbb{C}$ is holomorphic and one-to-one. As $h(0) = 0$ and $h(1) = 1$, by 3.15, h must be the identity map.

7.4: Use the definition of absolute continuity with $\varepsilon = 1$ to obtain some δ. Then, $[a, b]$ is covered by at most $N = \lfloor |b - a| / \delta \rfloor + 1$ number of intervals of length bounded by δ.

7.5: Assume that a sequence $f^{\circ n_k}$ converges uniformly on compact subsets of U. By the open mapping property of f, $V = f(U)$ is open, and one can show that the sequence of functions $f^{\circ n_k - 1}$ converges uniformly on compact subsets of V. This shows that $R(\mathcal{F}(R)) \subseteq \mathcal{F}(R)$. The argument in the other direction is similar, and uses the $f^{-1}(U)$ is open.

7.6: [complete solution]

(i): Let us define the function $g(z) = R^{\circ n}(z)$. Let $\delta = |g'(z_0)| < 1$ and choose $\delta' \in (\delta, 1)$. By the continuity of $z \mapsto g'(z)$ there is $r > 0$ such that for all $z \in \hat{\mathbb{C}}$ with $d(z, z_0) < r$ we have $|g'(z)| \leq \delta'$. Let $U = \{z \in \hat{\mathbb{C}} : d(z, z_0) < r\}$. Now, for $z \in U$ we have

$$d(g(z), z_0) = d(g(z), g(z_0)) \leq \sup_{c \in U} |g'(c)| \cdot d(z, z_0) \leq \delta' r < r.$$

This implies that g maps U into U. In particular, for $z \in U$ the iterates $g^{\circ n}(z)$, for $n \geq 1$, are all defined and belong to U.

For $z \in U$ we have

$$d(g^{\circ k}(z), z_0) = d(g^{\circ k}(z), g^{\circ k}(z_0)) \leq \sup_{c \in U} |(g^{\circ k})'(c)| \cdot d(z, z_0) \leq (\delta')^k \cdot r.$$

Since $\delta' < 1$, $(\delta')^k \cdot r$ tends to zero as n tends to infinity. Hence, the iterates $g^{\circ k}$ converge uniformly on U to the constant map z_0. In particular, the iterates $g^{\circ k} = R^{\circ nk}$, for $k \geq 1$, converges uniformly on compact sets in U to the constant function z_0.

(ii): Let us in the contrary assume that there is an open neighborhood U of z_0 and a sequence of iterates $R^{\circ k_m}$, for $m \geq 1$, which converges on compact subsets of U to some holomorphic map $g : U \to \hat{\mathbb{C}}$. Consider the integers k_m modulo n, and observe that there must be a sub-sequence of k_m, denoted by j_m, that are the same modulo n. That is, there are integers $t_m \in \mathbb{N}$, and an integer $r \geq 0$ such that $j_m = t_m n + r$. Then,

$$(R^{\circ j_m})'(z_0) = (R^{\circ r} \circ (R^{\circ n})^{\circ t_m})'(z_0) = (R^{\circ r})'(z_0) \cdot \delta^{t_m}$$

As $\delta > 1$, we conclude that $(R^{\circ j_m})'(z_0)$ tends to infinity. But, by Theorem 5.10, we must have $g'(z_0) = \lim_{m \to \infty} (R^{\circ j_m})'(z_0) = \infty$. This contradiction shows that there is no convergent sub-sequence on any neighborhood of z_0.

(iii): Let $g = R^{\circ qn}(z)$. We have $g(z_0) = z_0$ and $g'(z_0) = 1$. There is a neighborhood of z_0 on which g has a convergent power series $g(z) = z_0 + (z - z_0) + a_d (z - z_0)^d + \dots$ with $a_d \neq 0$. A basic calculation shows that $g^{\circ k}(z) = z_0 + (z - z_0) + k a_d (z - z_0)^d + \dots$. This implies that the d-th derivatives $(g^{\circ k})^{(d)}(z_0)$ tend to ∞. As in part (ii), this implies that $g^{\circ k}$ has no sub-sequence that converges uniformly on compact sets on a neighborhood of z_0.

Assume that there is a sub-sequence $R^{\circ k_m}$ that converges on some open set U containing z_0. Let $k_m = (qn) t_m + r_m$ with integers t_m and $0 \leq r_m \leq qn - 1$. There is a further sub-sequence of k_m such that r_m are equal for different values of m. Let $r = r_m$ be this constant. It follows that $R^{\circ (qn) t_{m+1}} = R^{\circ (qn - r)} \circ R^{\circ k_m}$ converges uniformly on compact subsets of U. This contradicts the above paragraph.

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